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From Core to Crust: Towards an Integrated Vision of Earth's Interior

20 - 24 July 2009

New appraisals on the Earth's interior from relaxation normal modes and long-term rotation

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New Appraisals on our Understanding Of the Earth's Interior from Relaxation Normal Modes and long-term Rotation

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From Core to Crust:
Towards an integrated Vision of Earth's Interior
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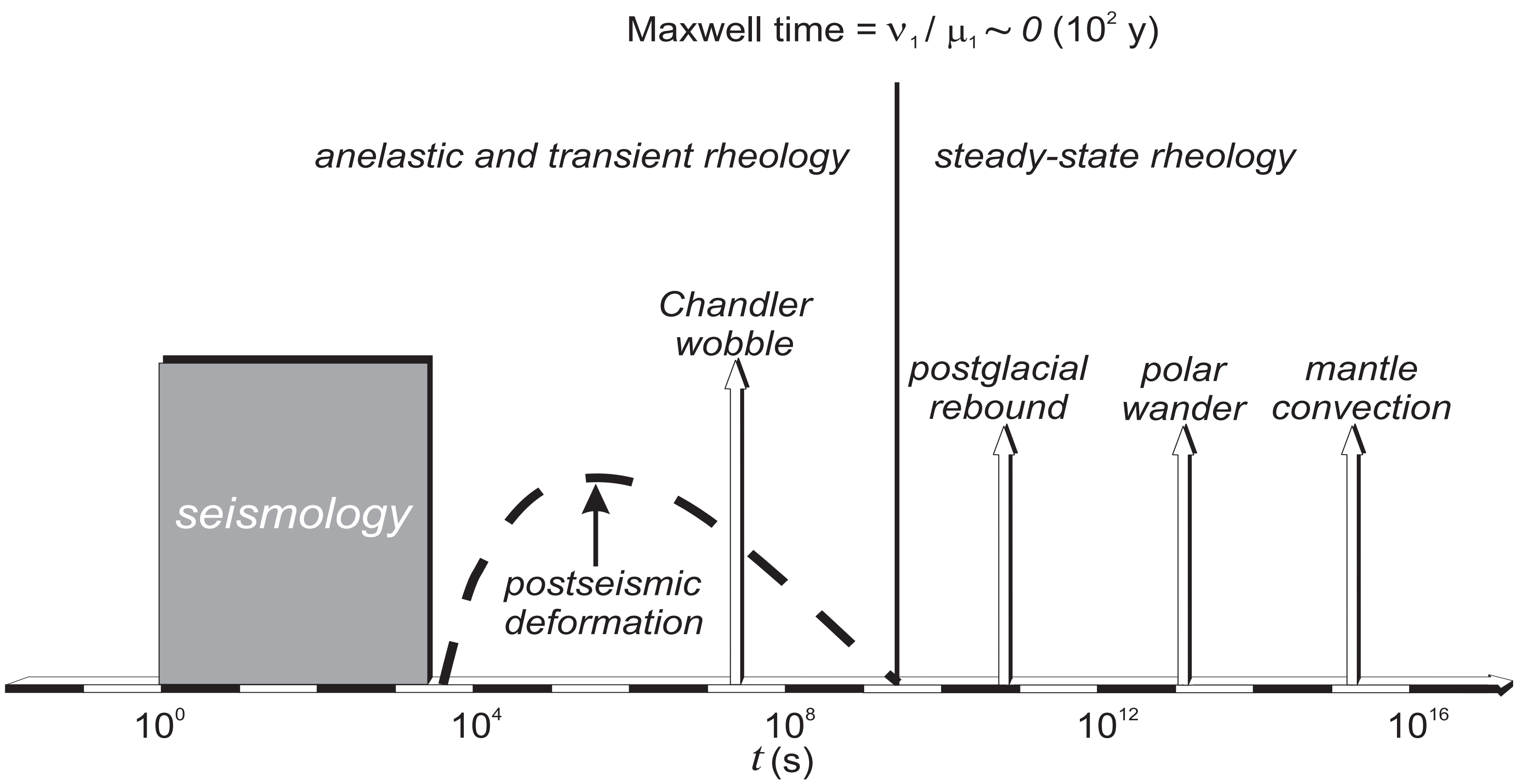
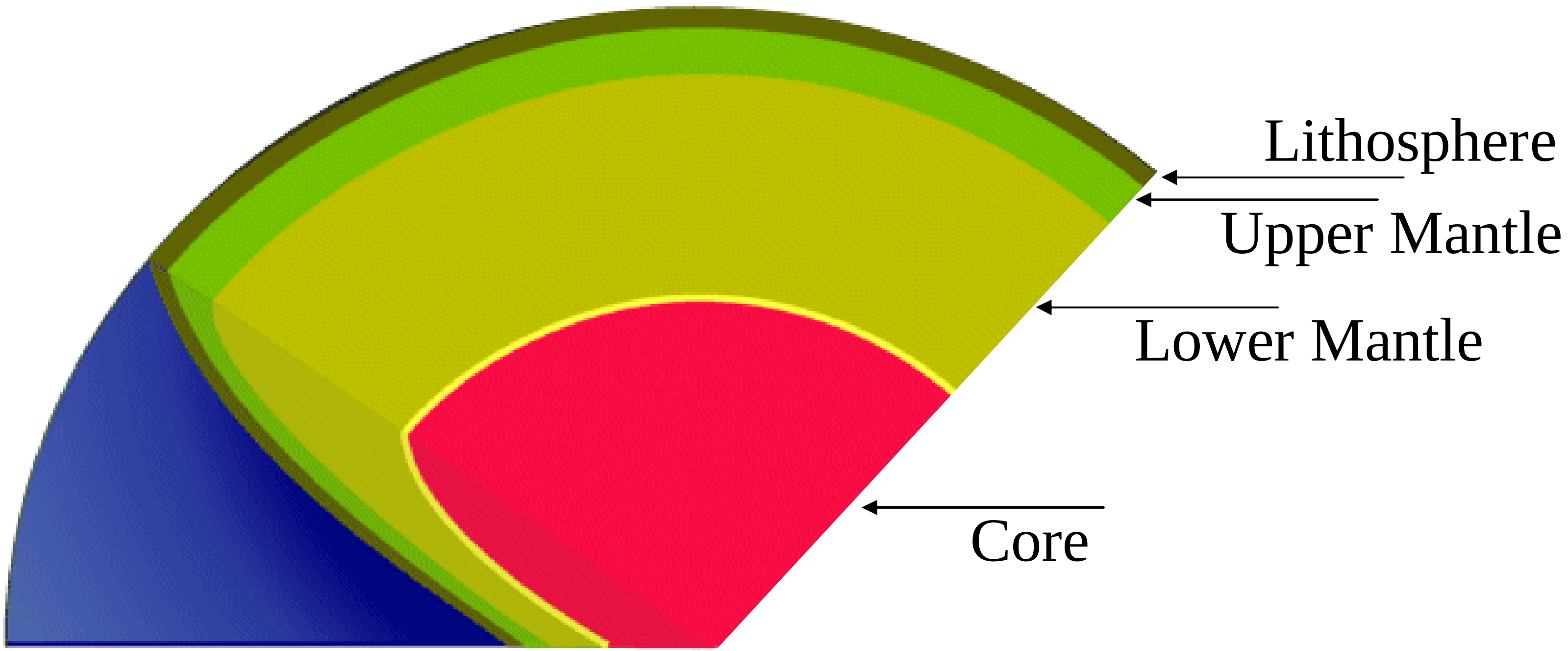
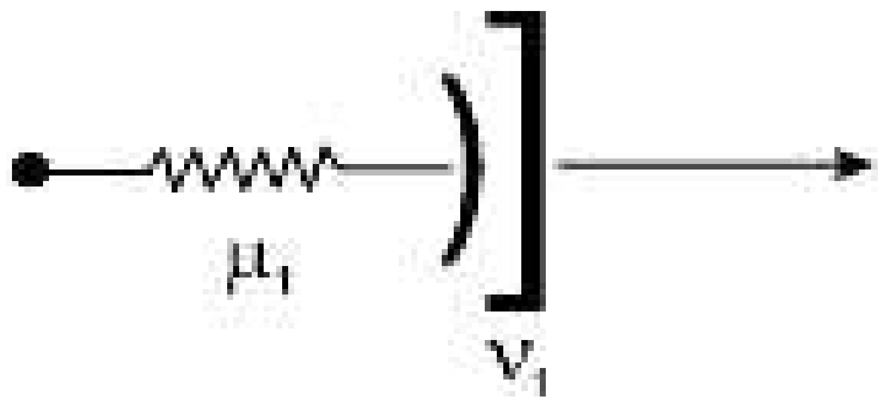


Maxwell time = $\nu_1 / \mu_1 \sim O\ (10^2\ y)$
anelastic and transient rheology
steady-state rheology
Chandler wobble
postglacial rebound
polar wander
mantle convection
seismology
postseismic deformation
10^0
10^4
10^8
10^{12}
10^{16}
t (s)



Lithosphere
Upper Mantle
Lower Mantle
Core


μ_1
ν_1

$$\begin{cases} \boldsymbol{\nabla} \cdot \boldsymbol{\sigma}' - \boldsymbol{\nabla}(\rho\, g\, \mathbf{u} \cdot \hat{\mathbf{r}}) - \rho \boldsymbol{\nabla} \phi' - \rho' g\, \hat{\mathbf{r}} + \mathbf{f} = 0 \\ \nabla^2 \phi' = 4\pi G\, (\rho' + \rho_f) \end{cases}$$

$$\dot{\sigma}_{ij} + \frac{\mu}{\nu} \left(\sigma_{ij} - \frac{1}{3} \sigma_{kk} \delta_{ij} \right) = 2\, \mu\, \dot{\epsilon}_{ij} + \lambda\, \dot{\epsilon}_{kk} \delta_{ij}$$

$$\mathcal{L}[\sigma_{ij}] = 2\, \hat{\mu}(s)\, \mathcal{L}[\epsilon_{ij}] + \hat{\lambda}(s)\, \mathcal{L}[\epsilon_{kk}]\, \delta_{ij}$$

$$\hat{\mu}(s) = \frac{\mu s}{s + \tau} \qquad \hat{\lambda}(s) = \frac{\lambda s + \kappa \tau}{s + \tau} \qquad \tau = \frac{\mu}{\nu} \qquad \kappa = \lambda + \frac{2}{3} \mu$$

$$u(\mathbf{r}) = \sum_{n=2}^{\infty} U_n(r)\, P_n(\cos\theta)$$

$$v(\mathbf{r}) = \sum_{n=2}^{\infty} V_n(r)\, \partial_\theta P_n(\cos\theta)$$

$$\phi'(\mathbf{r}) = -\sum_{n=2}^{\infty} \phi_n(r)\, P_n(\cos\theta)$$

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n (x^2 - 1)^n}{dx^n}$$

$$
\mathbf{y}(r,n,s) = \begin{pmatrix} \tilde{U}_n \\ \tilde{V}_n \\ \hat{\lambda}\,\tilde{\chi}_n + 2\,\hat{\mu}\,\partial_r\,\tilde{U}_n \\ \hat{\mu}\left(\partial_r \tilde{V}_n + \frac{1}{r}\tilde{U}_n - \frac{1}{r}\tilde{V}_n\right) \\ -\tilde{\phi}_n \\ -\partial_r\tilde{\phi}_n - \frac{n+1}{r}\tilde{\phi}_n + 4\,\pi\,G\,\rho\tilde{U}_n \end{pmatrix}
$$

$$
\boldsymbol{\nabla}\cdot\mathbf{u} = \sum_{n=2}^{\infty}\chi_n(r)\,P_n(\cos\theta)
$$

$$
\chi_n(r) = \partial_r U_n + \frac{2}{r}U_n - \frac{n(n+1)}{r}V_n
$$

$$\partial_r \mathbf{y}(r,s,n) = \mathbf{A}(r,s,n)\mathbf{y}(r,s,n) + \delta(r - r_S)\mathbf{f}(n)$$

$$\mathbf{A}(r,s,n) = \begin{pmatrix} \frac{-2\,\hat{\lambda}}{r\,\beta} & \frac{N\,\hat{\lambda}}{r\,\beta} & \frac{1}{\beta} & 0 & 0 & 0 \\ -\frac{1}{r} & \frac{1}{r} & 0 & \frac{1}{\hat{\mu}} & 0 & 0 \\ -\frac{4\,g\,\rho}{r} + \frac{4\,\gamma}{r^2\,\beta} & \frac{N\,g\,\rho}{r} - \frac{2\,N\,\gamma}{r^2\,\beta} & -\frac{4\,\hat{\mu}}{r\,\beta} & \frac{N}{r} & -\frac{(1+n)\,\rho}{r} & \rho \\ \frac{g\,\rho}{r} - \frac{2\,\gamma}{r^2\,\beta} & \frac{4\,N\,\hat{\mu}\left(\hat{\lambda}+\hat{\mu}\right)}{r^2\,\beta} - \frac{2\,\hat{\mu}}{r^2} & -\frac{\hat{\lambda}}{r\,\beta} & -\frac{3}{r} & \frac{\rho}{r} & 0 \\ -4\,\pi\,G\,\rho & 0 & 0 & 0 & -\frac{n+1}{r} & 1 \\ -\frac{4\,\pi\,G(n+1)\rho}{r} & \frac{4\,\pi\,G\,N\,\rho}{r} & 0 & 0 & 0 & \frac{n-1}{r} \end{pmatrix}$$

$$N = n(n+1)$$

Green functions - Incompressible

$$\mathbf{Y}(r,s,n) = [\mathbf{Y}_R\ \mathbf{Y}_I]$$

$$\mathbf{Y}_R(r,s,n) = r^n \begin{pmatrix} \frac{n\,r}{2(2\,n+3)} & \frac{1}{r} & 0 \\ \frac{(n+3)\,r}{2(2n^2+5\,n+3} & \frac{1}{n\,r} & 0 \\ \frac{2\,((n-1)\,n-3)\,\mu+g\,n\,r\,\rho}{2(2\,n+3)} & \frac{2\,(n-1)\,\mu+g\,r\,\rho}{r^2} & -\rho \\ \frac{n\,(n+2)\,\mu}{(n+1)\,(2\,n+3)} & \frac{2\,(n-1)\,\mu}{n\,r^2} & 0 \\ 0 & 0 & -1 \\ \frac{2\,G\,n\,\pi\,r\,\rho}{3+2\,n} & \frac{4\,G\,\pi\,\rho}{r} & -\frac{1+2\,n}{r} \end{pmatrix} \quad \mathbf{Y}_I(r,s,n) = \frac{1}{r^n} \begin{pmatrix} \frac{n+1}{2(2\,n-1)} & r^{-2} & 0 \\ -\frac{n-2}{2n(2n-1)} & -\frac{1}{(n+1)\,r^2} & 0 \\ \frac{g\,(n+1)\,r\,\rho-2\,(n\,(3+n)-1)\,\mu}{2\,(2\,n-1)\,r} & \frac{g\,r\,\rho-2\,(2+n)\,\mu}{r^3} & -\frac{\rho}{r} \\ \frac{\mu(n^2-1)}{rn(2n-1)} & \frac{2\,(2+n)\,\mu}{(n+1)\,r^3} & 0 \\ 0 & 0 & -\frac{1}{r} \\ \frac{2\,G\,\pi(n+1)\,\rho}{2\,n-1} & \frac{4\,G\,\pi\,\rho}{r^2} & 0 \end{pmatrix}$$

$$\mathbf{Y}_j(R_{j+1}, s, n)\mathbf{C}_j = \mathbf{Y}_{j+1}(R_{j+1}, s, n)\mathbf{C}_{j+1}$$

$$\mathbf{y}_{omo}(r, s, n) = \mathbf{D}(r, s, n)\mathbf{y}_C$$

$$\tilde{\mathbf{X}}(r, s, n) = \frac{[\mathbf{P}_2\mathbf{D}(a, s, n)\mathbf{I}_C(n)]\ [\mathbf{P}_1\mathbf{D}(a, s, n)\mathbf{I}_C(n)]^{\dagger}\,\mathbf{b}(s, n)}{\Delta_{sec}(s, n)}$$

$$\mathbf{P}_2\mathbf{y}(r, s, n) = \begin{pmatrix} \tilde{U}(r, s, n) \\ \tilde{V}(r, s, n) \\ -\tilde{\Phi}(r, s, n) \end{pmatrix} = \tilde{\mathbf{X}}(r, s, n)$$

$$\mathrm{I}_C(n) = \begin{pmatrix} -\frac{3}{4\pi G\rho_C} r_C^{n-1} & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{4\pi G\rho_C^2}{3} r_C \\ 0 & 0 & 0 \\ r_C^n & 0 & 0 \\ 2(n-1)r_C^{n-1} & 0 & 4\pi G\rho_C \end{pmatrix}$$

$$\Delta_{sec}(s, n) = \det\left[\mathbf{P}_1\mathbf{D}(a, s, n)\mathbf{I}_C(n)\right]$$

$$\begin{pmatrix} U(r,t,n) \\ V(r,t,n) \\ -\Phi(r,t,n) \end{pmatrix} = \mathbf{X}(r,t,n) = \int_{s_0-i\infty}^{s_0+i\infty} \tilde{\mathbf{X}}(r,s,n) e^{st} ds = \mathbf{k}_E\,\delta(t) + \sum \mathbf{k}_j\, e^{s_j\,t}$$

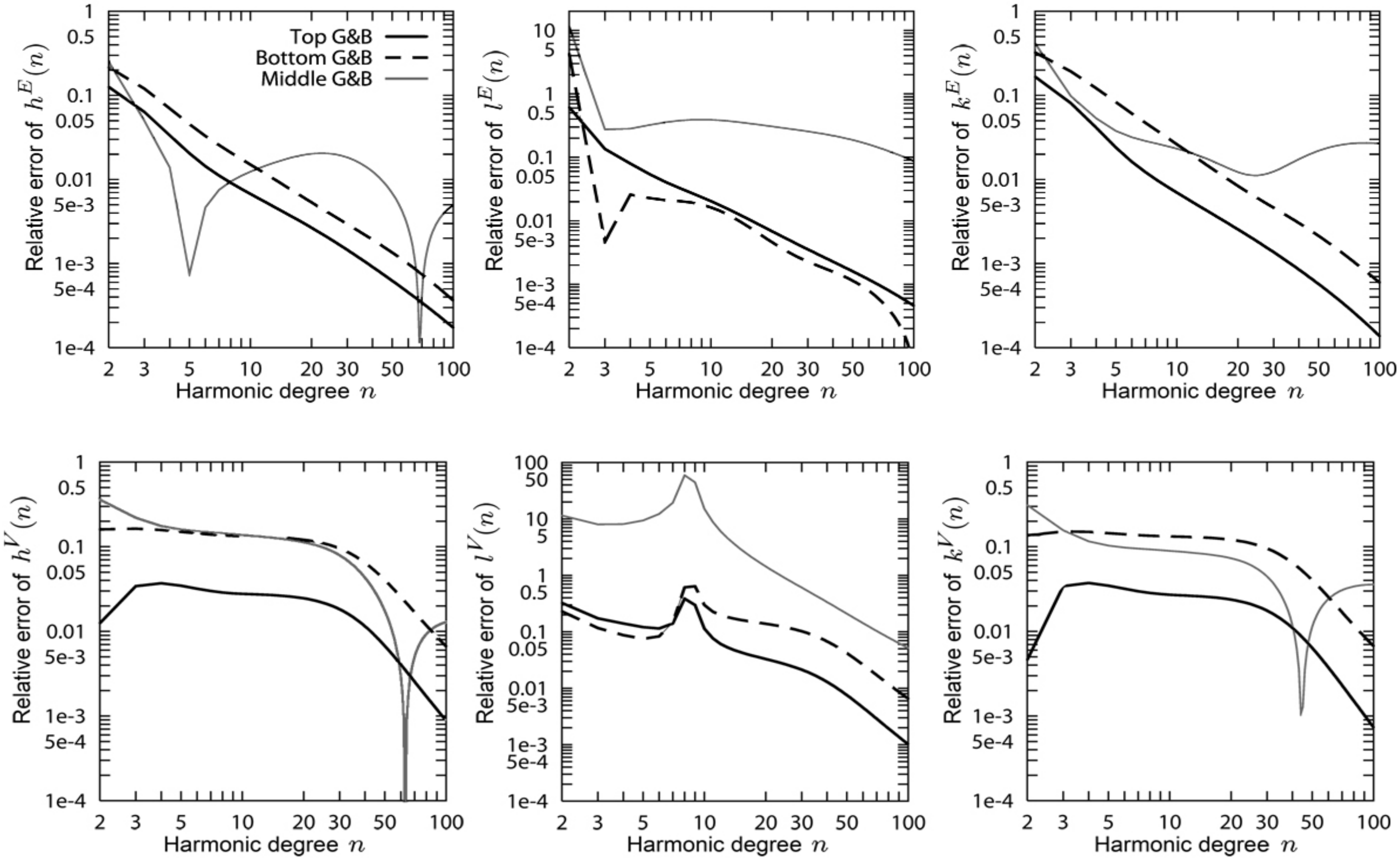
$$\mathbf{k}_E = \lim_{s\to-\infty} \tilde{\mathbf{X}}(r,s,n)$$

$$\mathbf{k}_j = \lim_{s\to s_j} (s - s_j)\tilde{\mathbf{X}}(r,s,n)$$

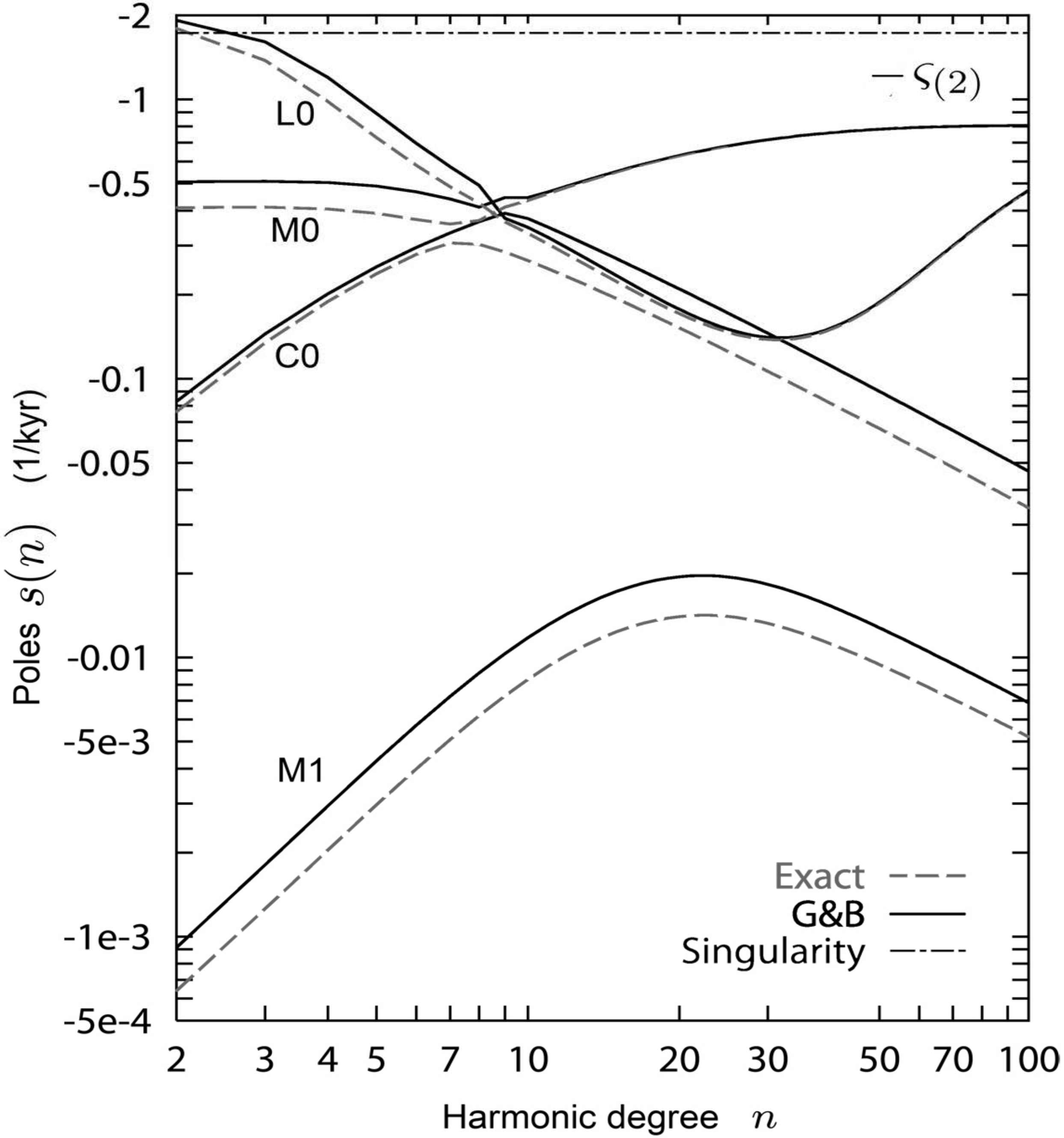
Compressible (approximated) model
Helmholtz equation

$$\mathbf{y}_k(r,s,n) = \begin{pmatrix} -\frac{N\,C}{k^2\,r} J(k\,r) - \partial_r J(k\,r) \\ -\frac{1+C}{k^2\,r} J(k\,r) - C\,\partial_r J'(k\,r) \\ \hat{\mu}\frac{2\,N\,(1+C)+k^2\,r^2\,\beta}{k^2\,r^2} J(k\,r) - \hat{\mu}\frac{2\,(N\,C-2)}{r}\partial_r J(k\,r) \\ \hat{\mu}\frac{2+\left(k^2\,r^2-2\,N+2\right)C}{k^2\,r^2}\,J(k\,r) + \hat{\mu}\frac{2(C-1)}{r}\partial_r J(k\,r) \\ \frac{\zeta}{k^2} J(k\,r) \\ -\frac{(n+1)\,\zeta\,(n\,C-1)}{k^2\,r} J(k\,r) \end{pmatrix}$$

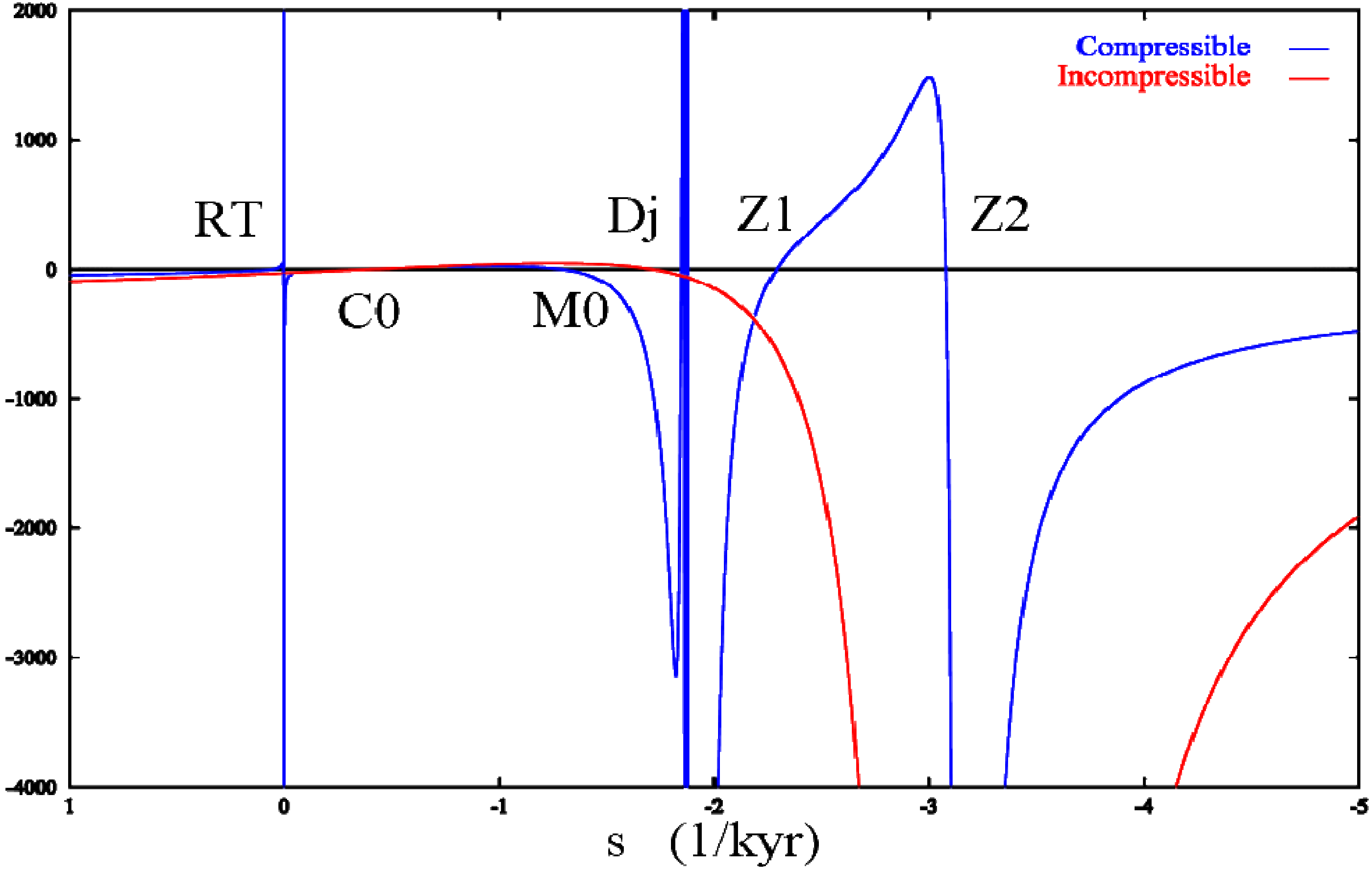
$$\xi(r) = \frac{g(r)}{r} \quad \longrightarrow \quad \bar{\xi}(r) = \xi_j$$



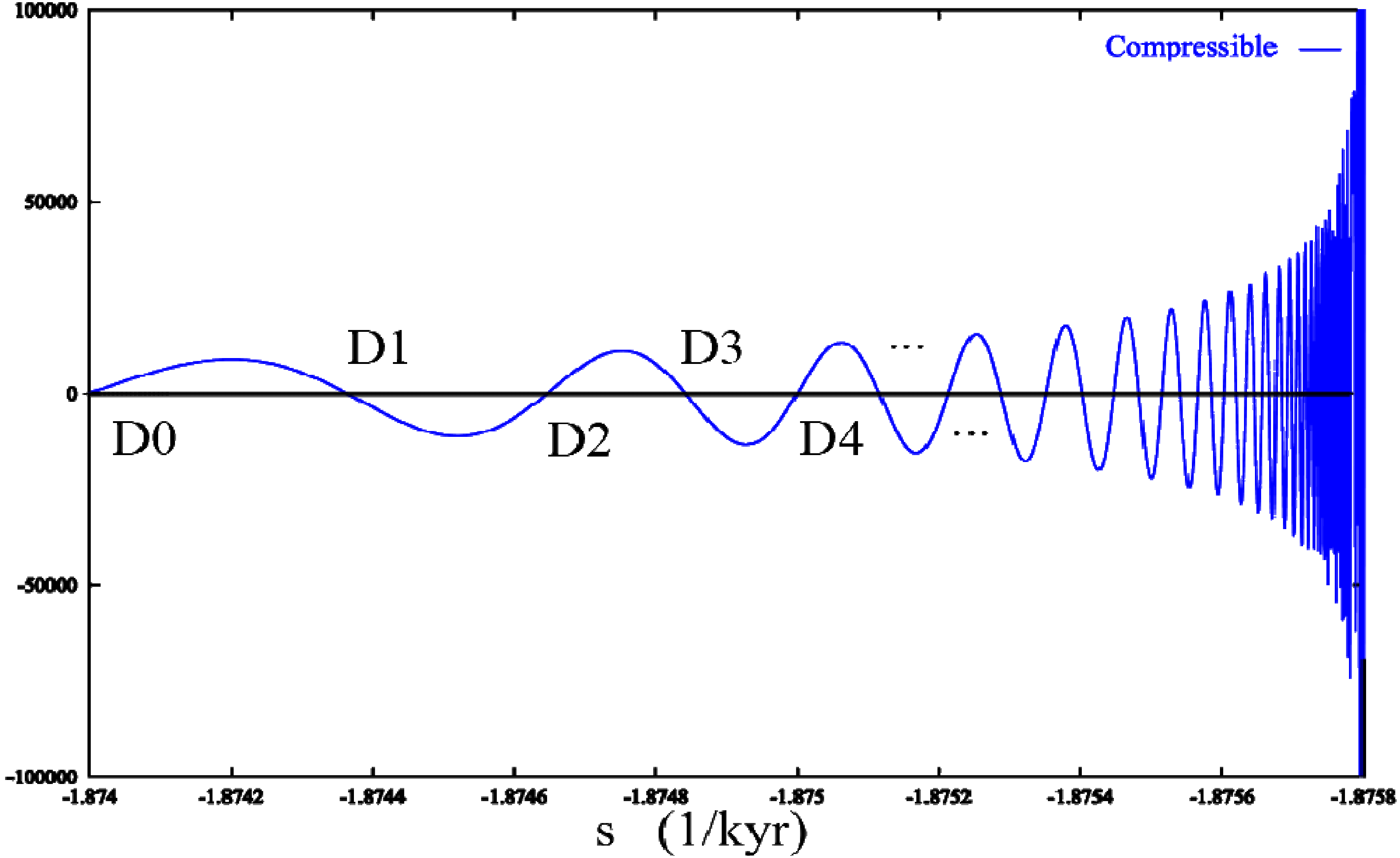
Top G&B
Bottom G&B
Middle G&B
Relative error of $h^E(n)$
Relative error of $l^E(n)$
Relative error of $k^E(n)$
Relative error of $h^V(n)$
Relative error of $l^V(n)$
Relative error of $k^V(n)$
Harmonic degree n
Harmonic degree n
Harmonic degree n
Harmonic degree n
Harmonic degree n
Harmonic degree n



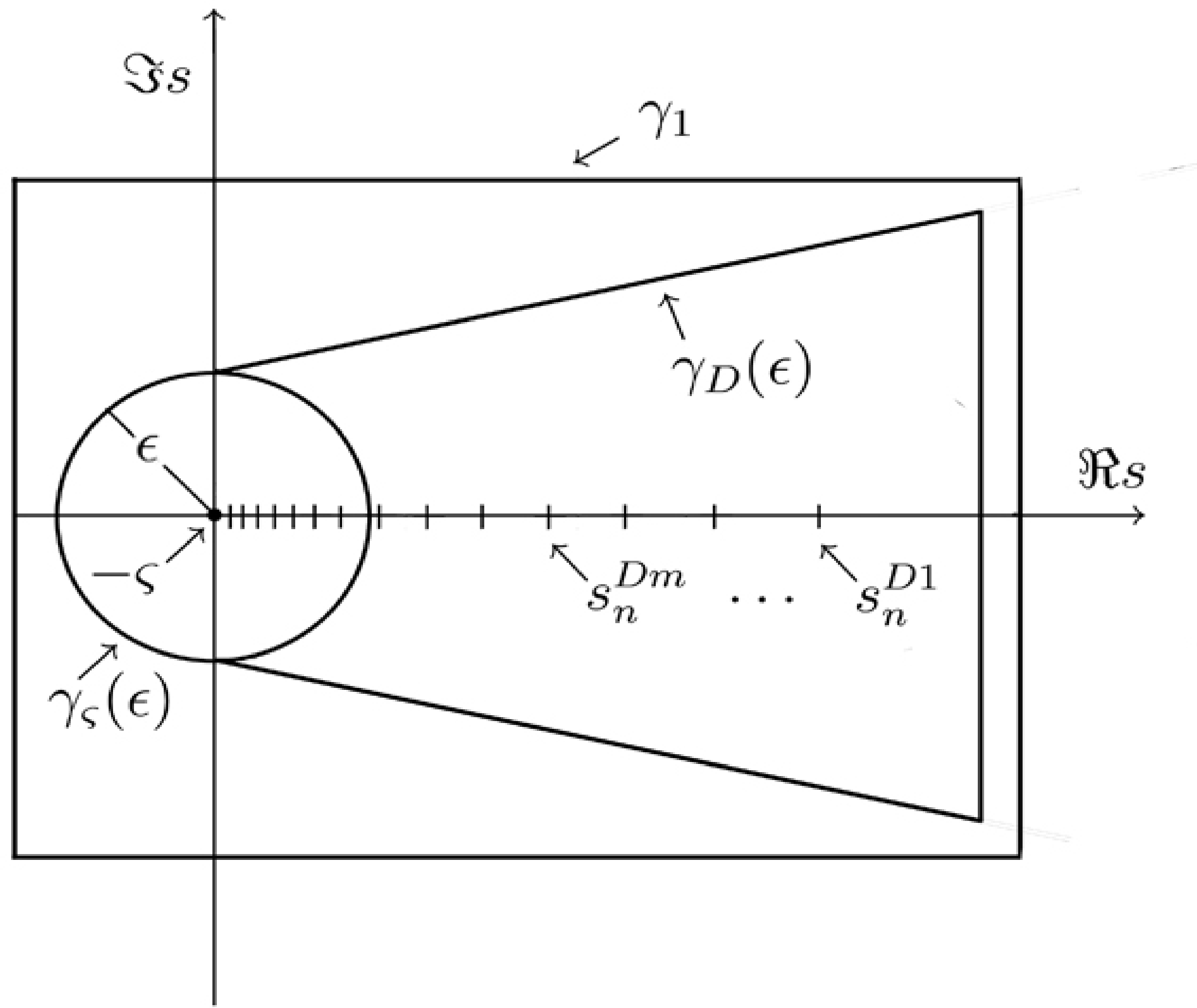
-2
-1
-0.5
-0.1
-0.05
-0.01
-5e-3
-1e-3
-5e-4
2
3
4
5
7
10
20
30
50
70
100
Harmonic degree n
Poles $s(n)$ (1/kyr)
$-\varsigma(2)$
L0
M0
C0
M1
Exact
G&B
Singularity



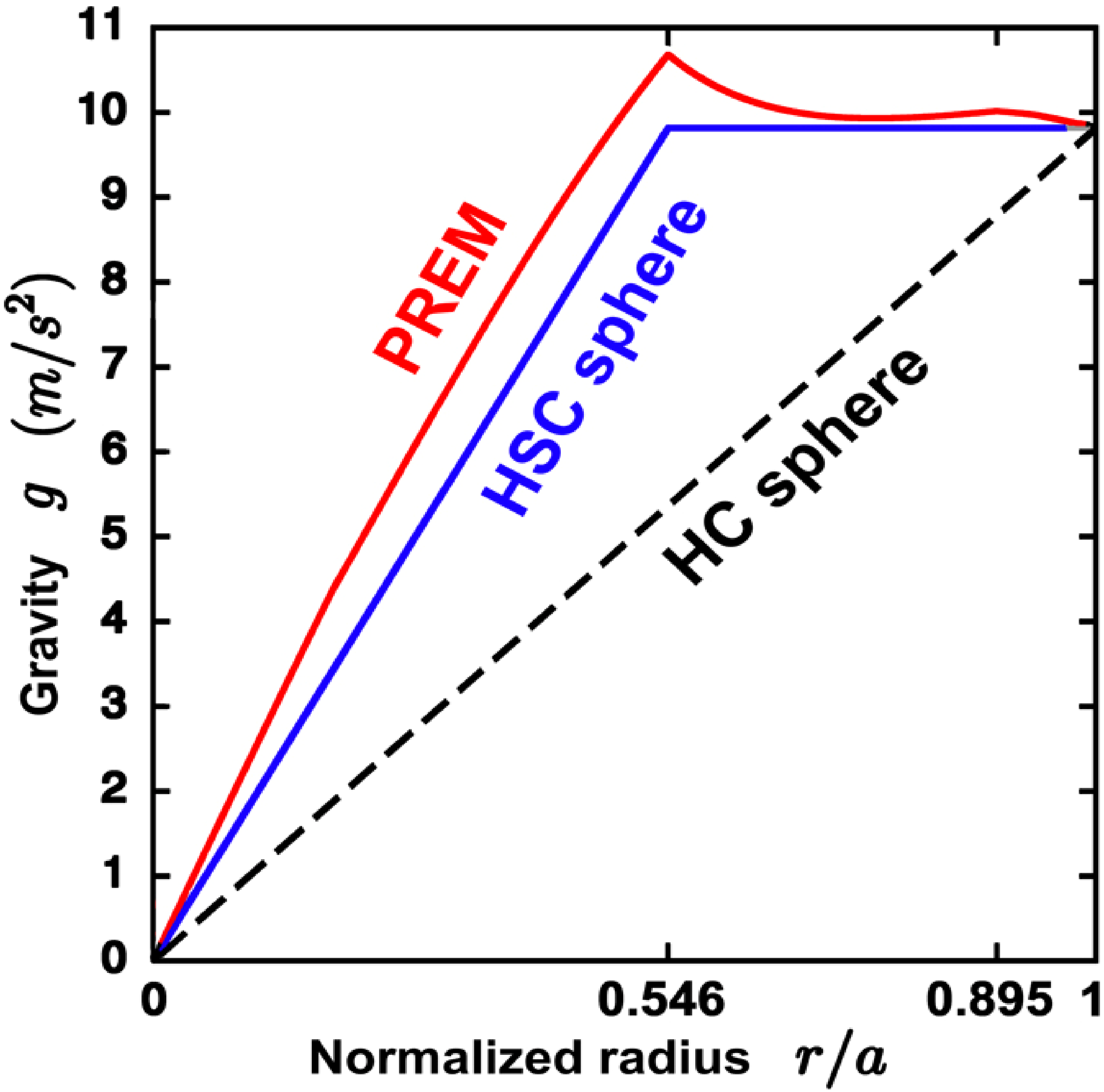
Compressible
Incompressible
RT
Dj
Z1
Z2
C0
M0
2000
1000
0
-1000
-2000
-3000
-4000
1
0
-1
-2
-3
-4
-5
s (1/kyr)



Compressible
100000
50000
0
-50000
-100000
D0
D1
D2
D3
D4
-1.874
-1.8742
-1.8744
-1.8746
-1.8748
-1.875
-1.8752
-1.8754
-1.8756
-1.8758
s (1/kyr)



$\Im s$
γ_1
$\gamma_D(\epsilon)$
ϵ
$\Re s$
$-\varsigma$
s_n^{Dm}
$\cdots$
s_n^{D1}
$\gamma_\varsigma(\epsilon)$



Williamson-Adams equation

$$d_r\rho + \frac{g\,\rho^2}{\kappa} - \lambda = 0$$

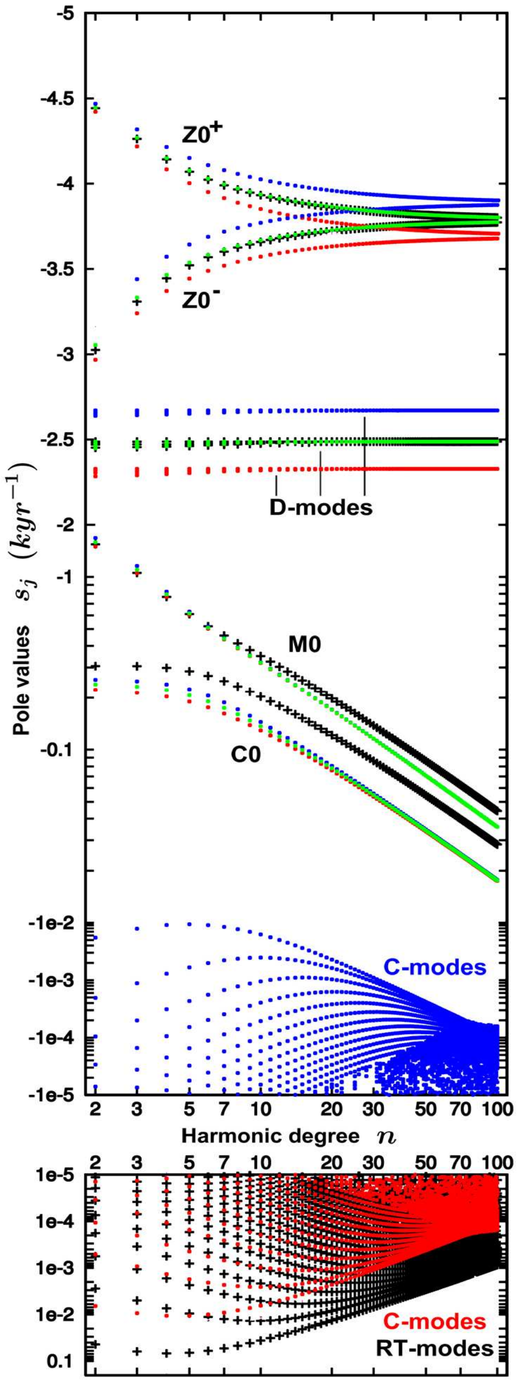
$$\bar{\lambda} \approx \frac{g\,\rho^2}{\kappa} > 0$$

Continuous density profile

$$\rho = \frac{\alpha}{r}$$

$$g = 2\pi G\,\alpha$$

$$\kappa_{SC} = 2\pi G\,\alpha^2$$

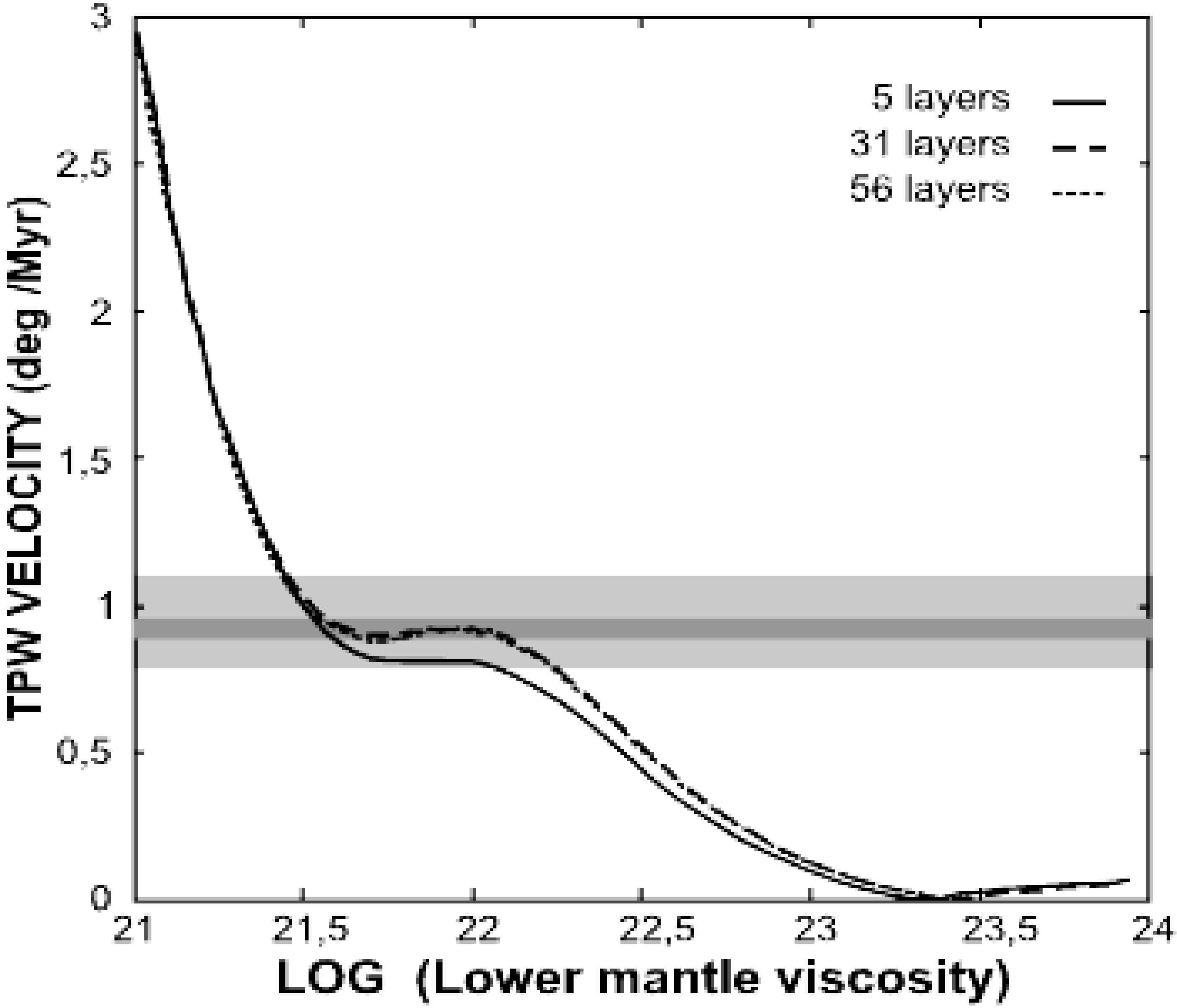


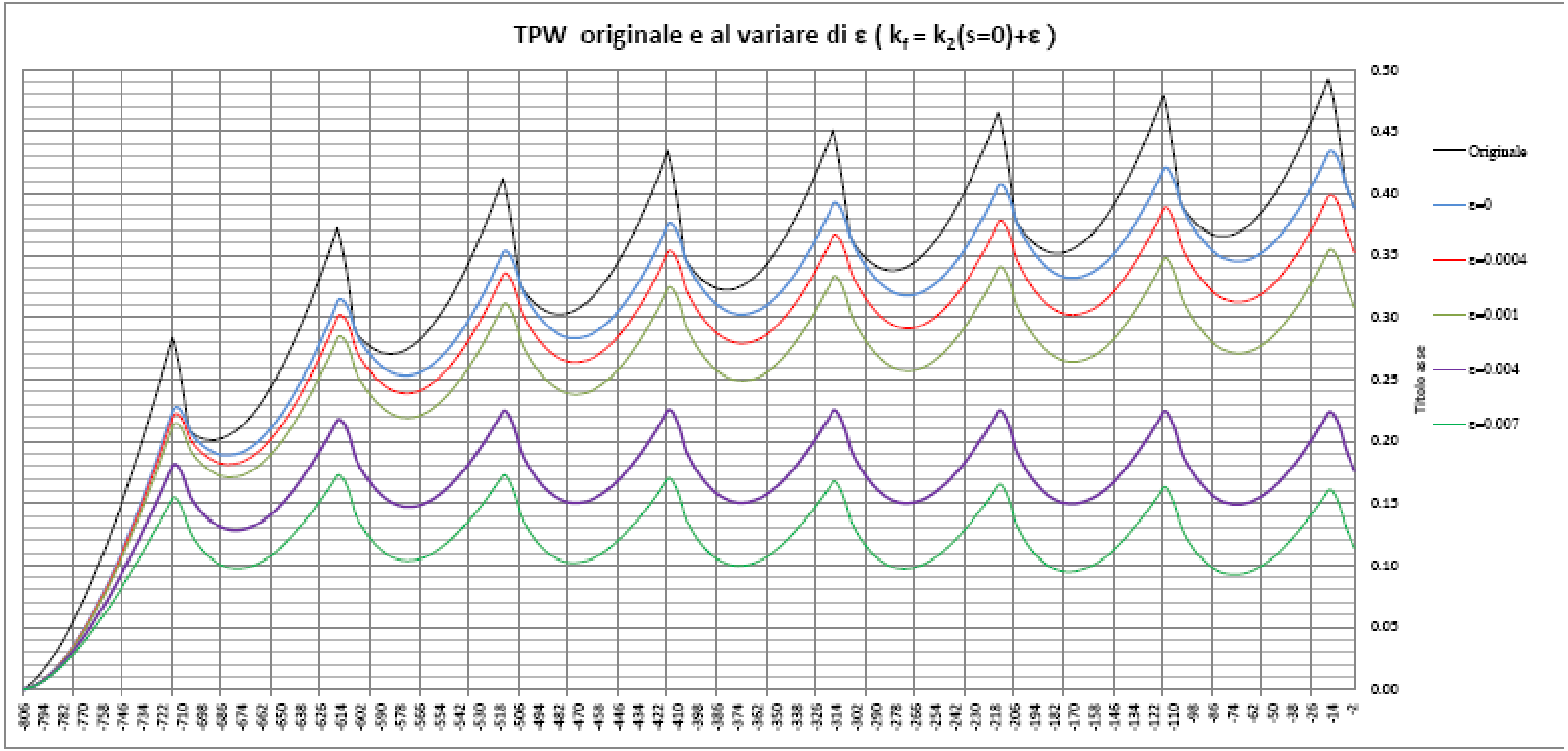
A new class of modes
Compositional C-modes

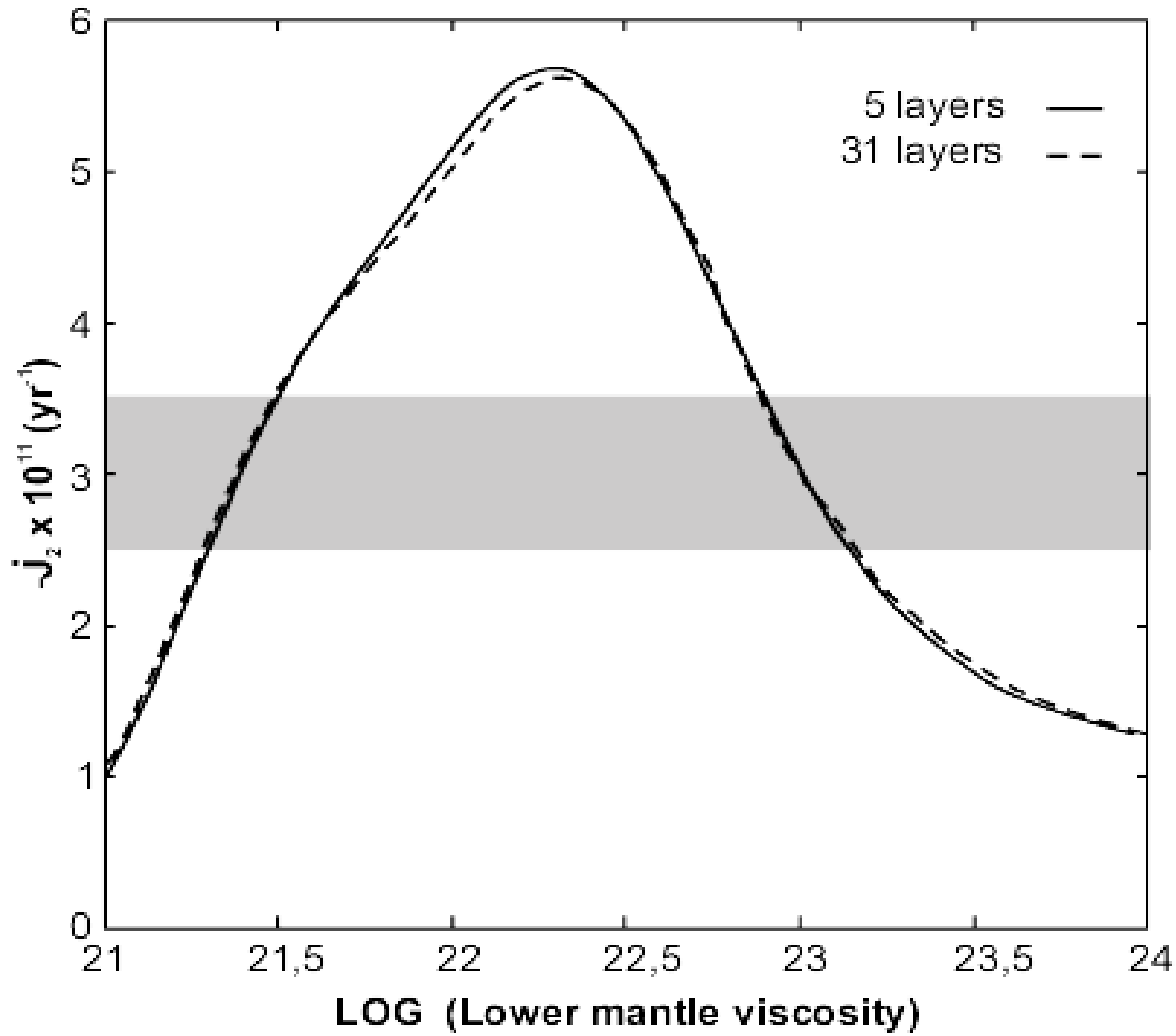
Fluid limit

$$\begin{pmatrix} U(r,t,n) \\ V(r,t,n) \\ -\Phi(r,t,n) \end{pmatrix} = \mathbf{X}(r,t,n) = \int_{s_0-i\infty}^{s_0+i\infty} \tilde{\mathbf{X}}(r,s,n) e^{st} ds = \mathbf{k}_E\, \delta(t) + \sum \mathbf{k}_j\, e^{s_j\, t}$$

$$\bar{k}_n^{\infty} = \lim_{t\to\infty} \bar{k}_n(t) = \bar{k}_E - \sum \frac{\bar{k}_j}{s_j} = \bar{k}_n^{ISO}$$

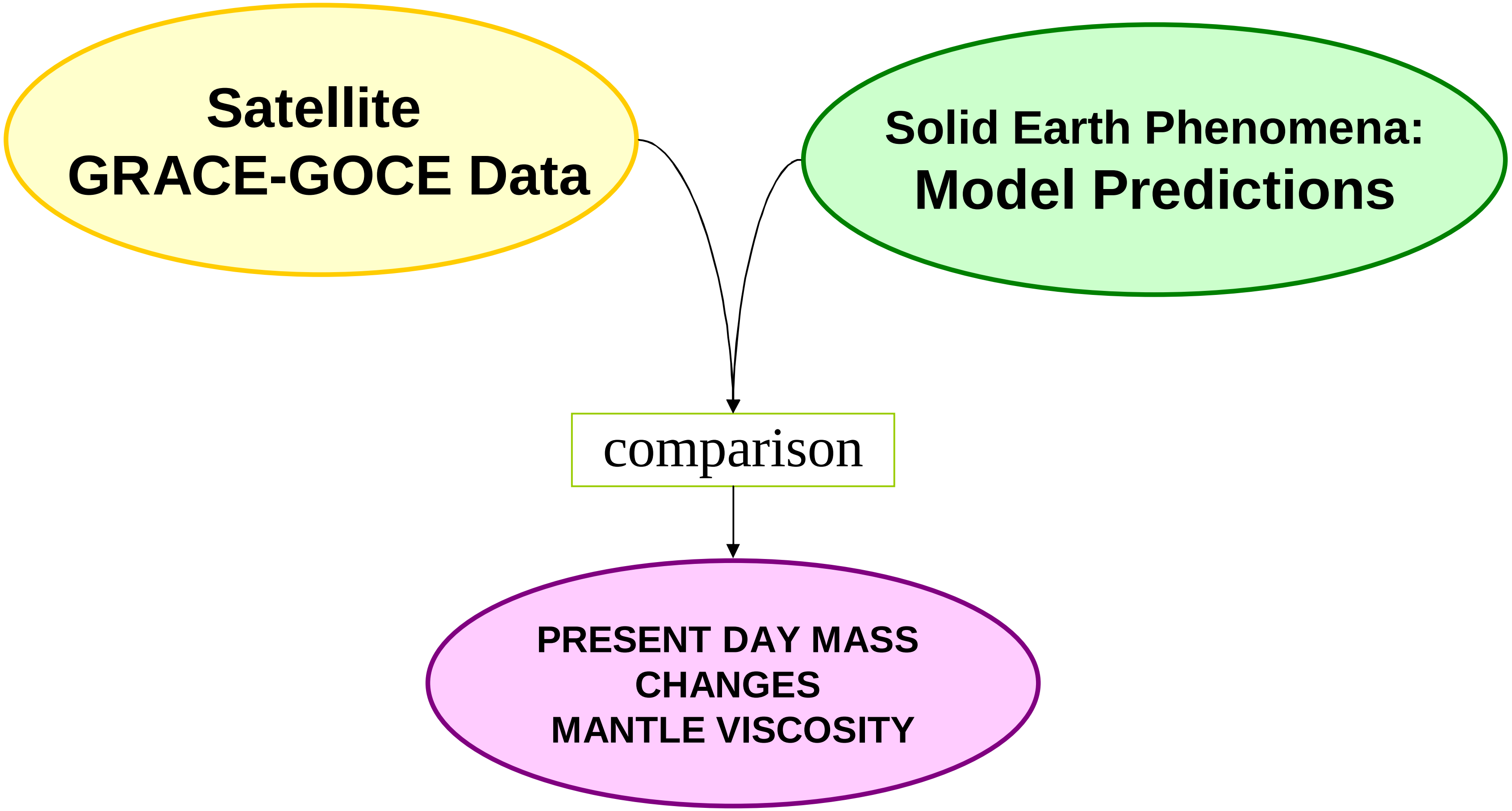

TPW VELOCITY (deg /Myr)
LOG (Lower mantle viscosity)
5 layers
31 layers
56 layers
3
2,5
2
1,5
1
0,5
0
21
21,5
22
22,5
23
23,5
24


TPW originale e al variare di ε ($k_f = k_2(s=0)+\varepsilon$)
0.50
0.45
0.40
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
Titolo asse
Originale
ε=0
ε=0.0004
ε=0.001
ε=0.004
ε=0.007
-806 -794 -782 -770 -758 -746 -734 -722 -710 -698 -686 -674 -662 -650 -638 -626 -614 -602 -590 -578 -566 -554 -542 -530 -518 -506 -494 -482 -470 -458 -446 -434 -422 -410 -398 -386 -374 -362 -350 -338 -326 -314 -302 -290 -278 -266 -254 -242 -230 -218 -206 -194 -182 -170 -158 -146 -134 -122 -110 -98 -86 -74 -62 -50 -38 -26 -14 -2



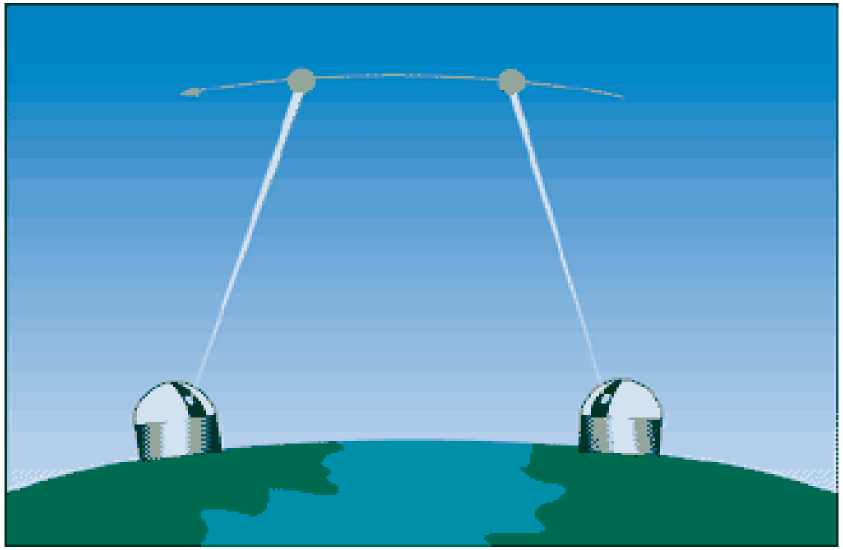
5 layers
31 layers
$\dot{J}_2 \times 10^{11}$ (yr^{-1})
LOG (Lower mantle viscosity)
0
1
2
3
4
5
6
21
21,5
22
22,5
23
23,5
24

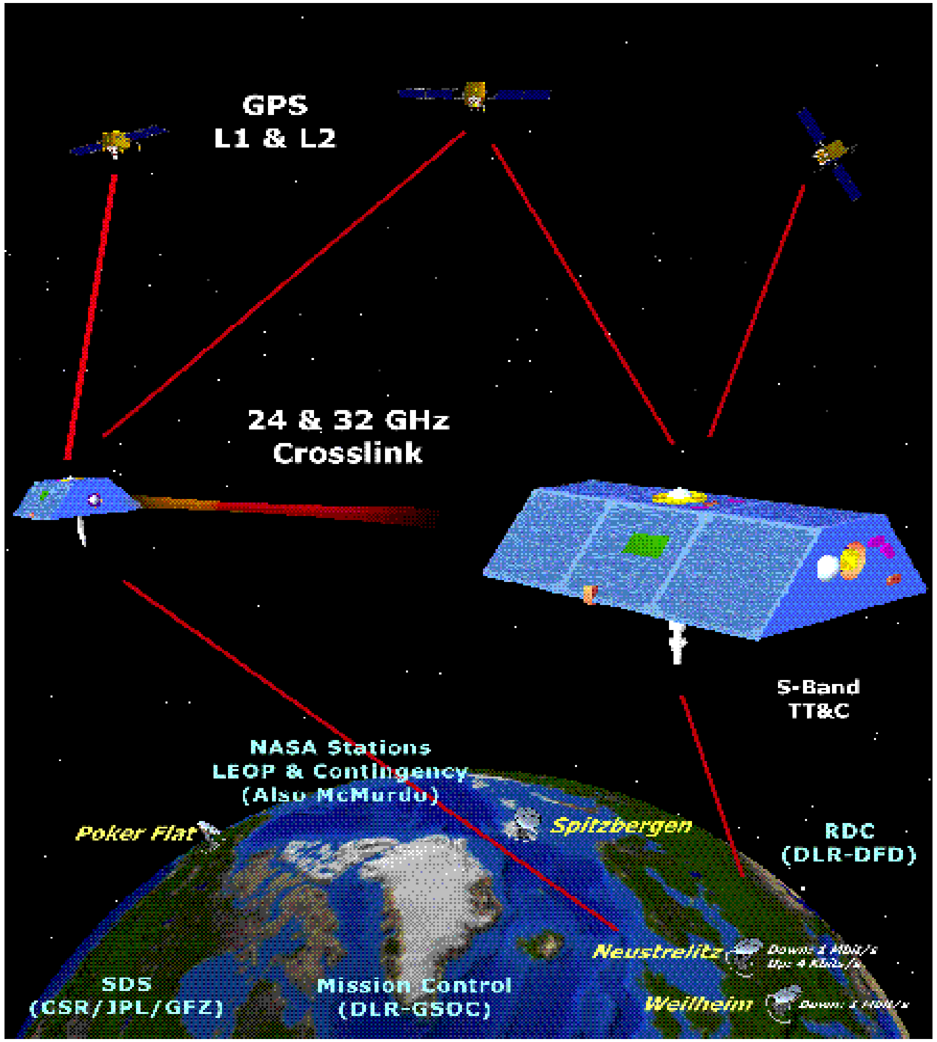
General Scheme

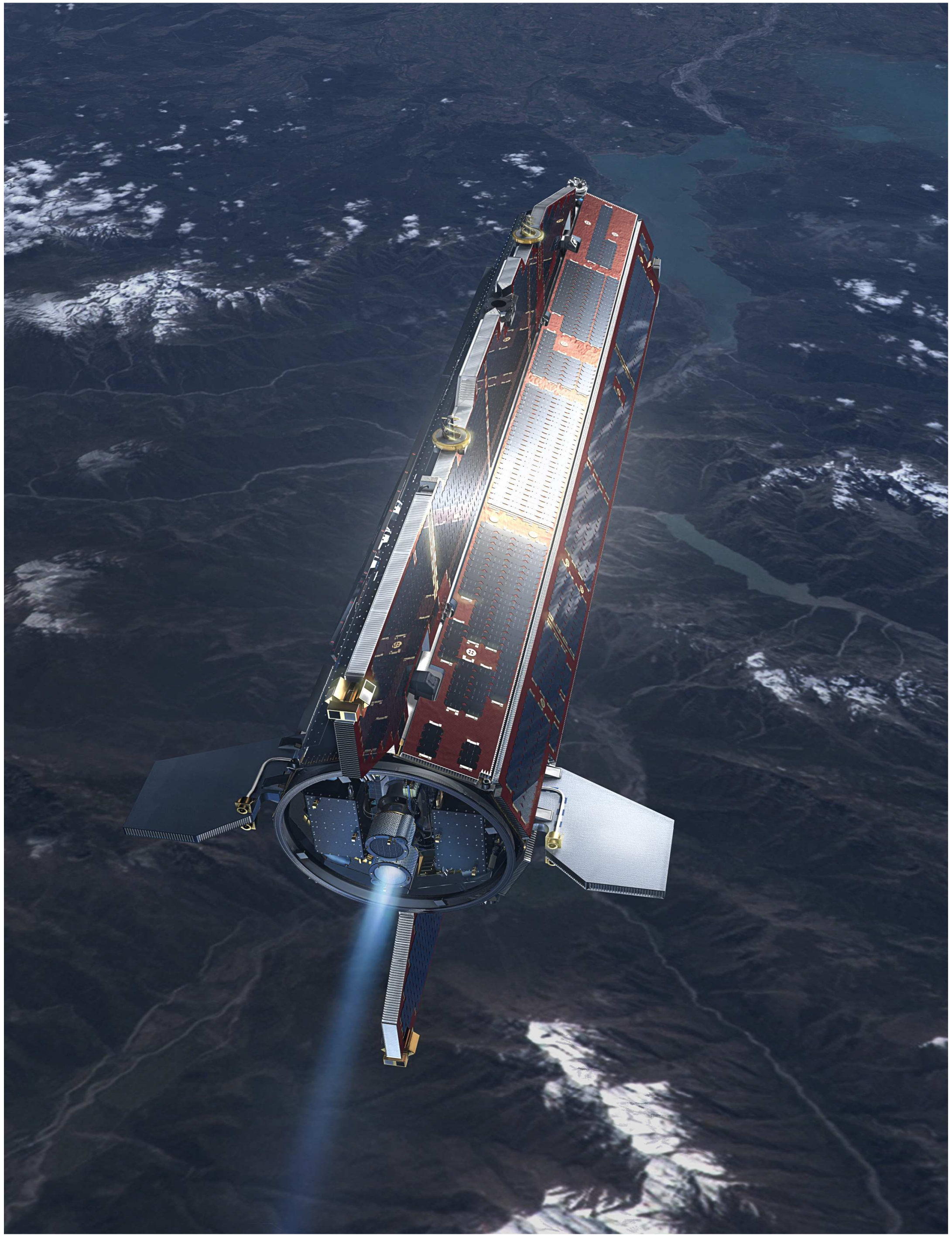


SLR and GRACE

Satellite Laser Ranging

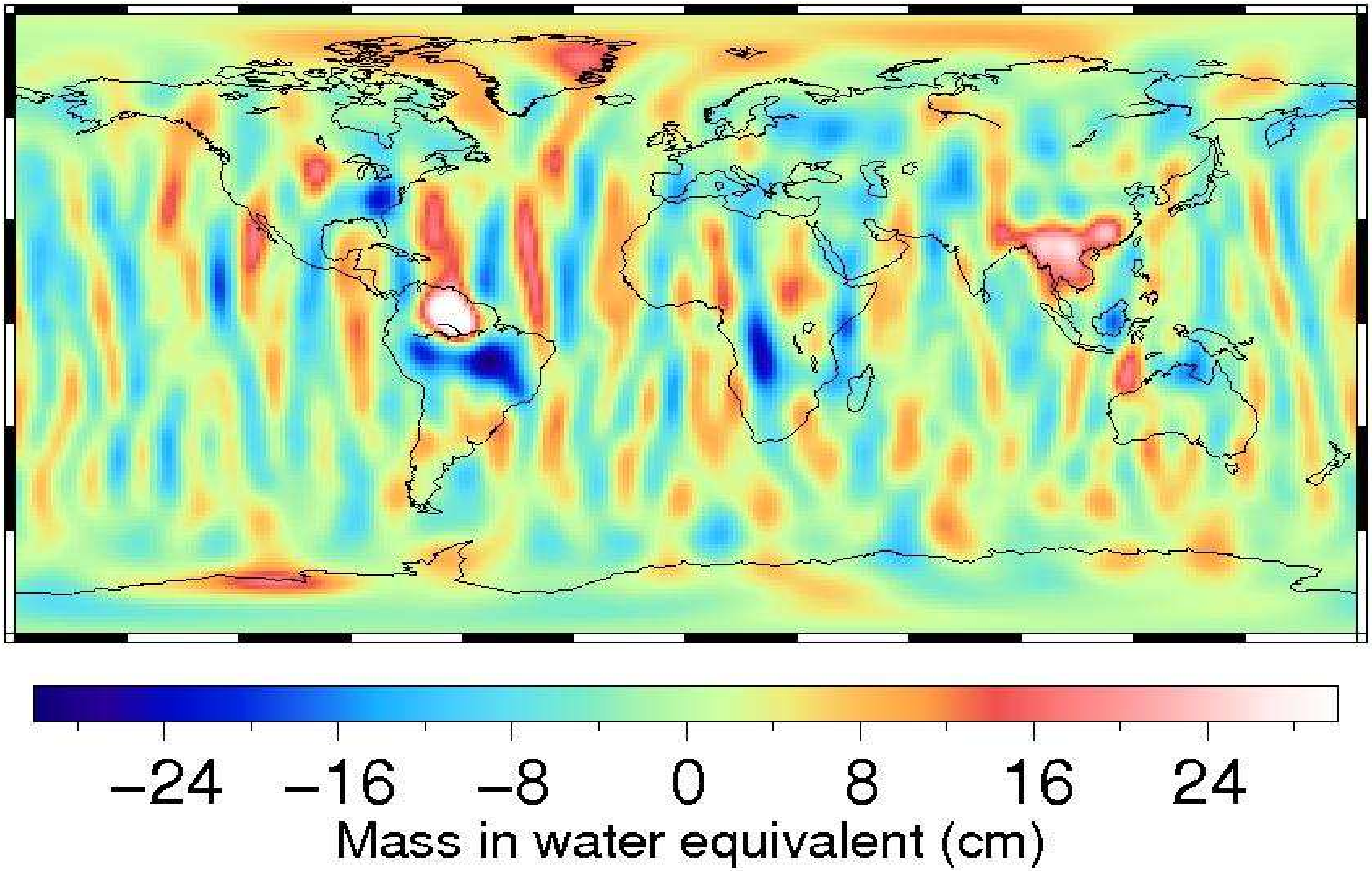




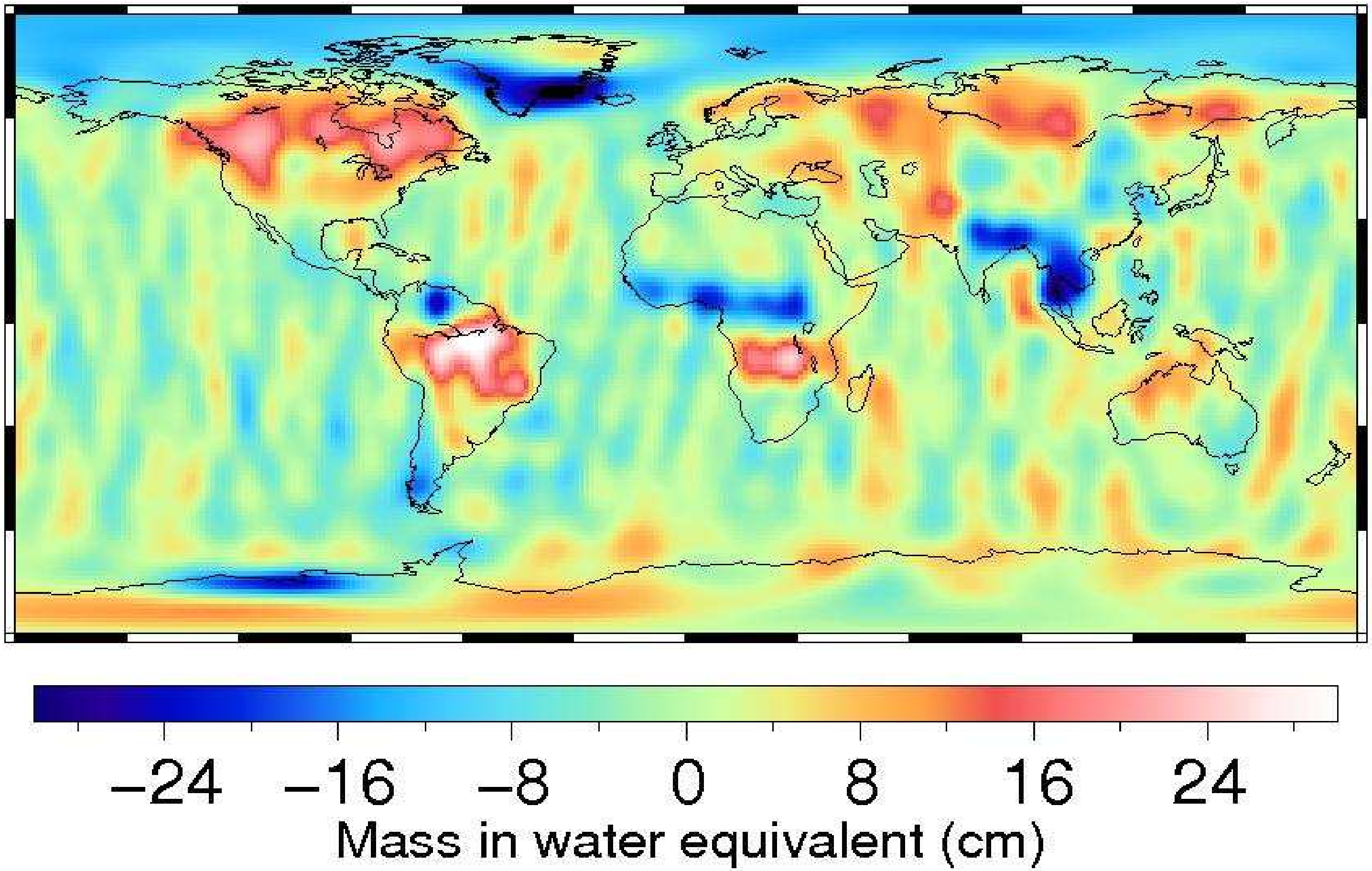


G
O
C
E

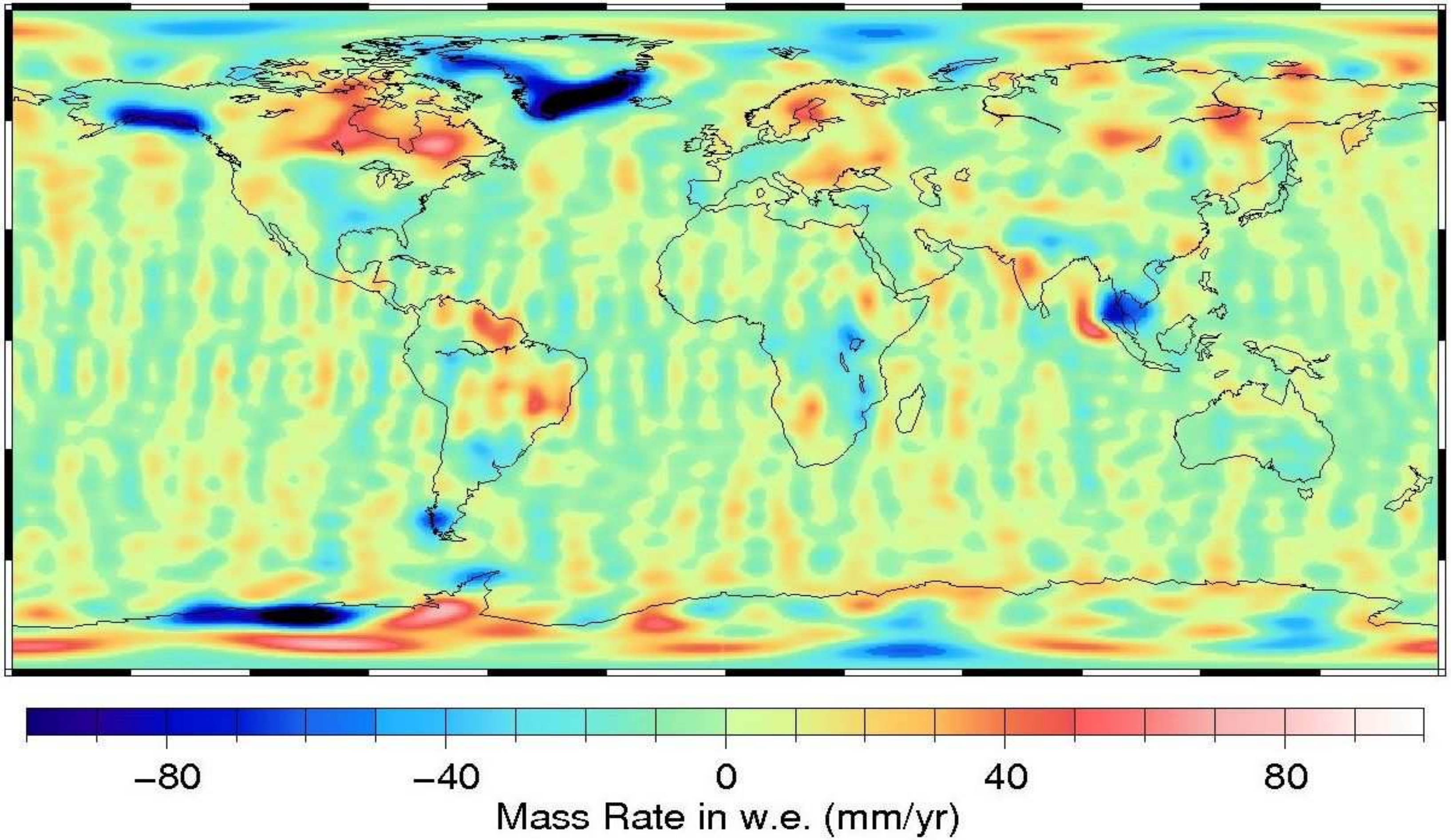
August 2002



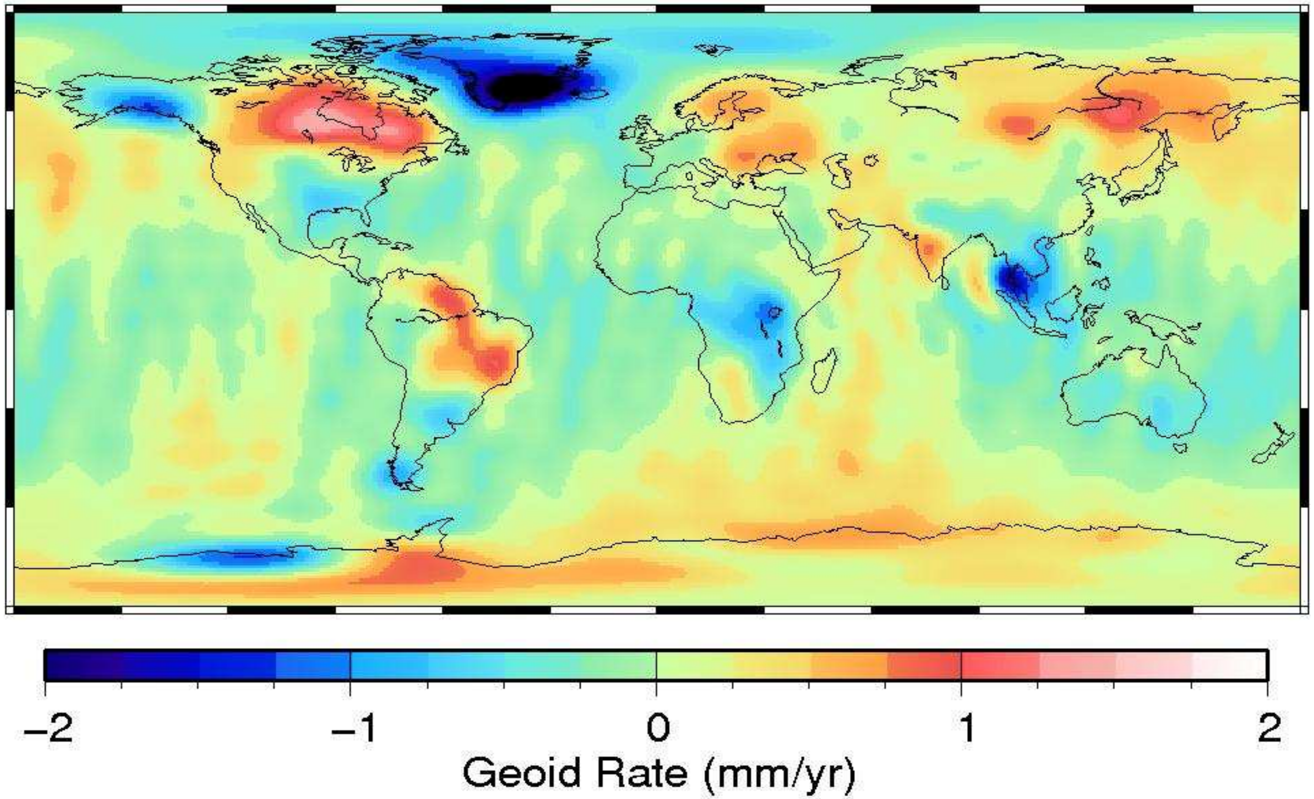
April 2007



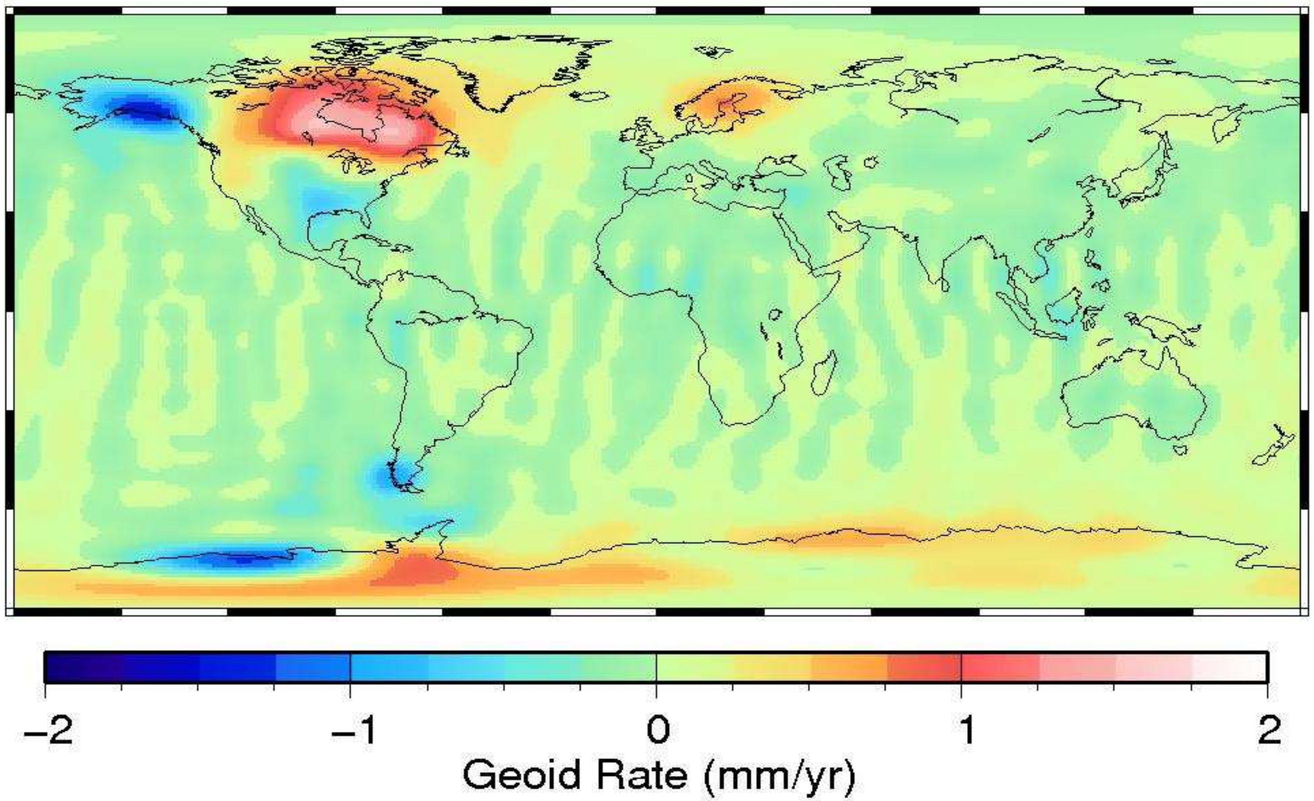
The Map of Mass Variation Trend - Filtered



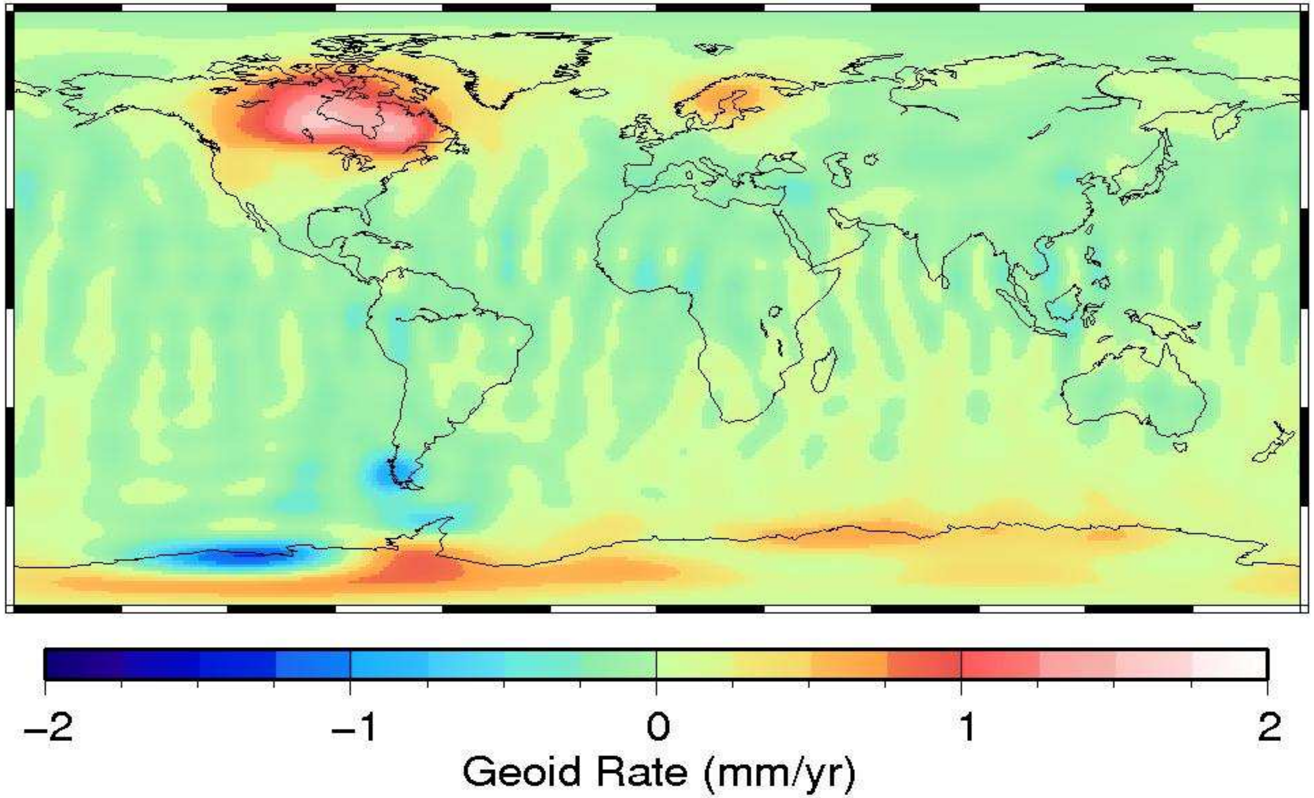
Geoid from GRACE



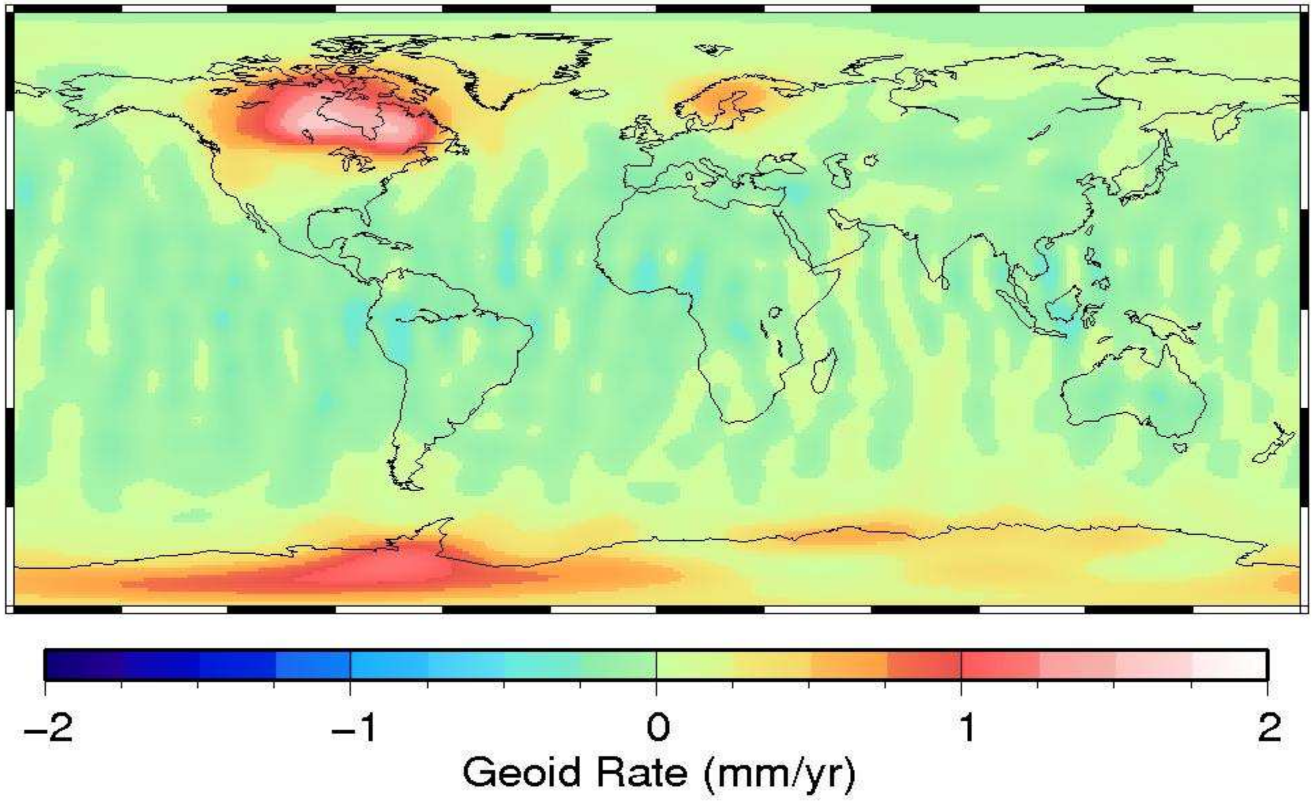
GRACE up 30 - Nearby Fennoscandia *Removed*



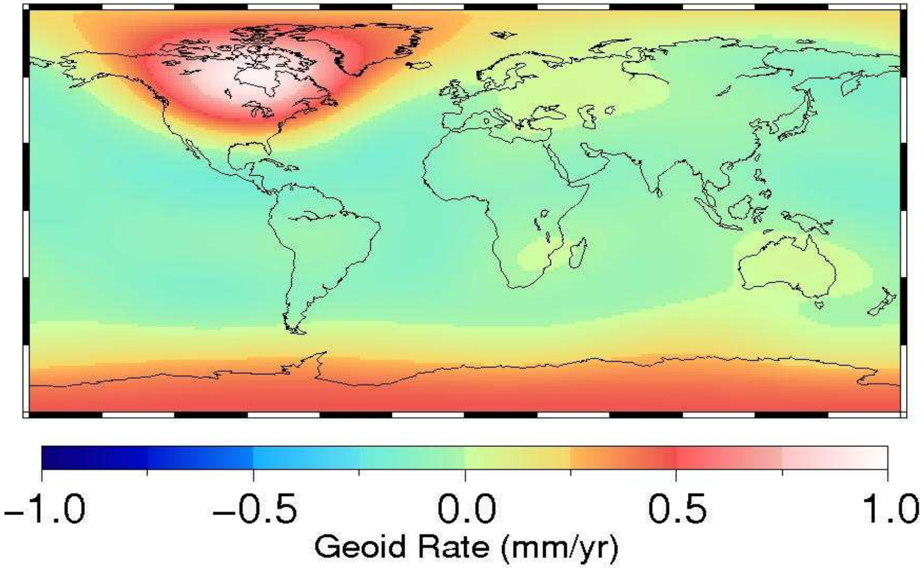
GRACE up 30 - Nearby Hudson Bay *Removed*



GRACE up 30 - West Antarctica *Removed*

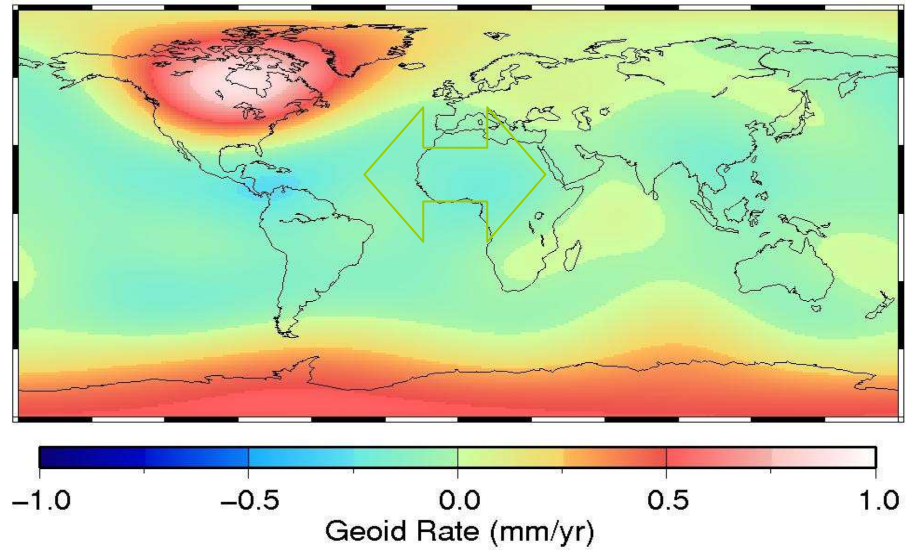


Global Problem - Search for best viscosity

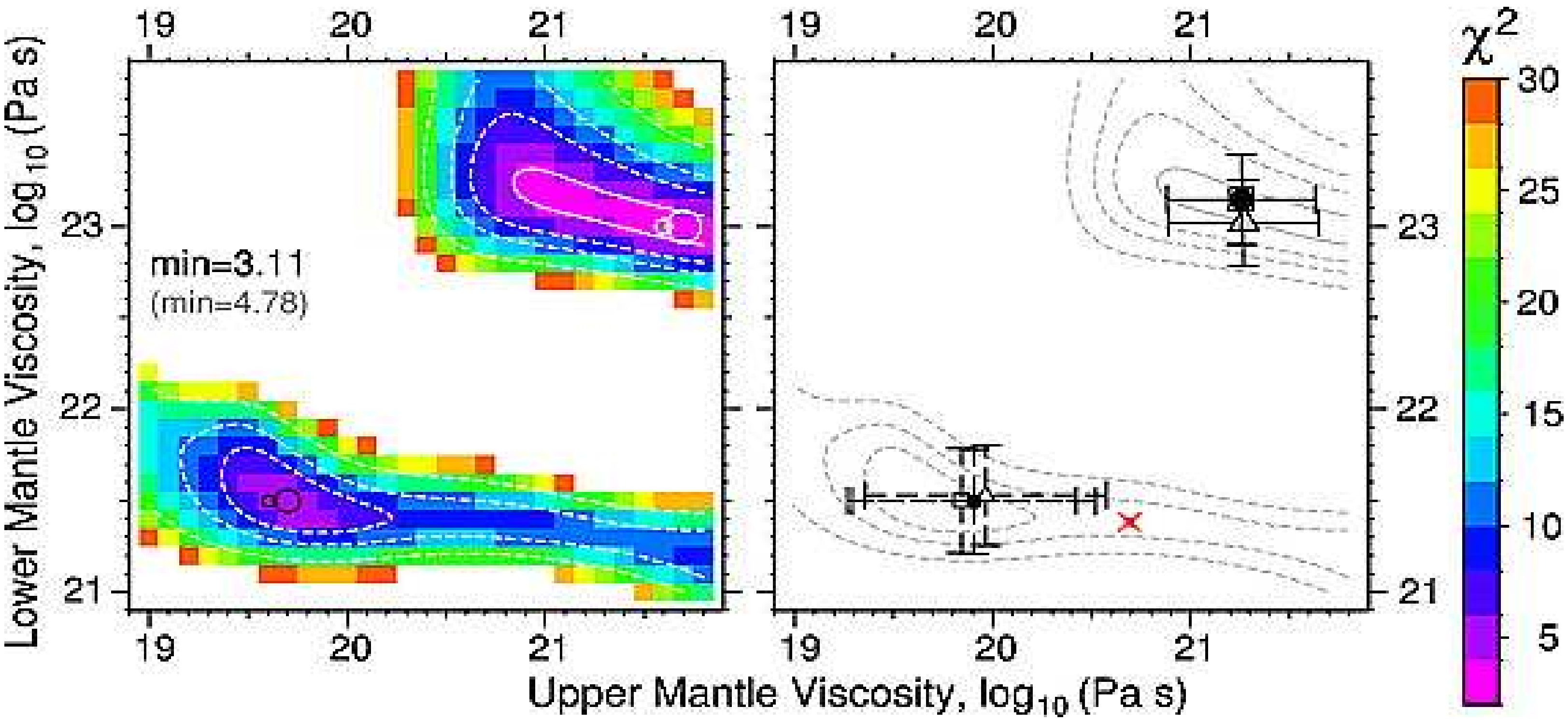


ν_{UP} = 3.1x10^{20} Pa s

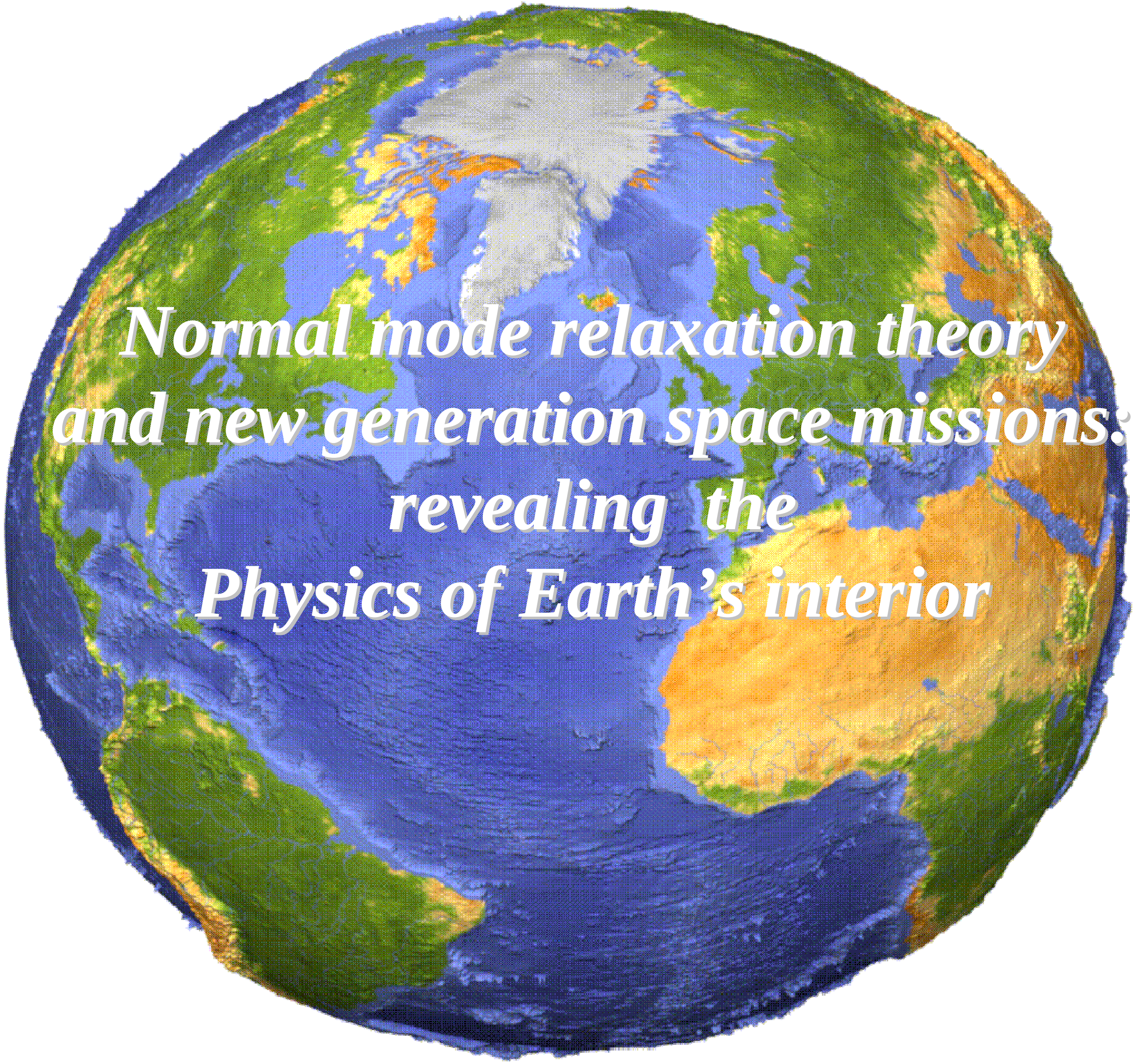
ν_{LW} = 1.5x10^{23} Pa s



Cleaned **GRACE**



min=3.11
(min=4.78)
Lower Mantle Viscosity, log10 (Pa s)
Upper Mantle Viscosity, log10 (Pa s)
χ2


Normal mode relaxation theory
and new generation space missions:
revealing the
Physics of Earth's interior