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Targeted Training Activity: Predictability of Weather and Climate: Theory and Applications to Intraseasonal Variability

27 July - 7 August, 2009

Predictability Theory

2. Nonlinear Dynamical Systems

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Dynamical Systems

Dynamical System

A system that evolves in time according to some governing laws is a *dynamical system*. Each successive state is a function of the preceding state.

Examples: oscillators, planets in gravitation, atmospheric system

Types of dynamical systems

Conservative: The system possesses a conserved quantity, such as energy, a constant of motion.

Dissipative: Energy is not conserved because of dissipation and forcing.

Linear: The system contains only first power in the dynamical equations.

Nonlinear: The system contains nonlinear terms in the dynamical equations.

Difference equations

Difference equations describe the evolution in discrete time

$$X_{i,n+1} = G_i(X_{1,n}, \cdots, X_{M,n}), \quad i = 1, \cdots M$$

e.g.,

Quadratic map (logistic map),

$$X_{n+1} = X_n^2 - c$$

Hénon map

$$X_{n+1} = Y_n + 1 - aX_n^2$$
$$Y_{n+1} = bX_n$$

Differential equations

Differential equations describe the evolution of the dynamical system in continuous time.

A coupled set of ordinary differential equations, such as Lorenz model, can be written as

$$\frac{dX_i}{dt} = F_i(X_1, \cdots, X_M), \quad i = 1, \cdots, M$$

e.g., $$\frac{dX_i}{dt} = \sum_{j,k} a_{ijk} X_j X_k - \sum_j b_{ij} X_j + c_i$$

where $\sum_{j,k} a_{ijk} X_j X_k$ vanishes identically, $\sum_j b_{ij} X_j$ is positive definite and

$c_1, c_2, \cdots, c_M$ are constants.

The set of differential equations ensures that the solutions exist and that they are unique and continuous. Because of the existence and uniqueness, different trajectories never intersect.

Phase Space is an M- dimensional Euclidean space whose coordinates are $X_1,\ldots X_M$.

Each point $(X_1,\ldots X_M)$ in the phase space represents an instantaneous state of the system.

A state varying according to the dynamical equations is represented by a *trajectory* or *orbit* in the phase space.

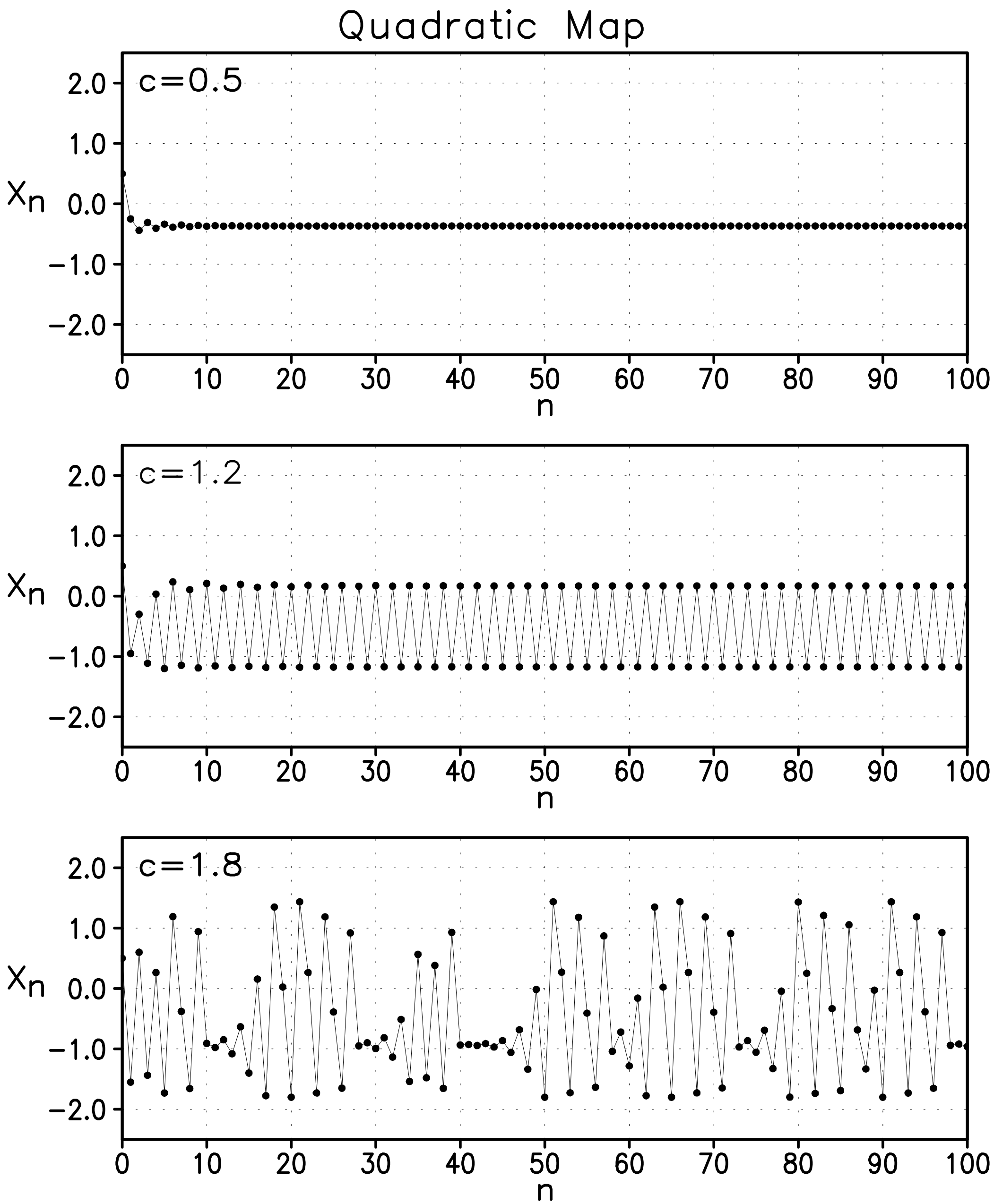
Uniqueness: Through each point, there is a unique orbit

Solutions of Quadratic map $$X_{n+1} = X_n^2 - c$$

n	c = 0.5 X_n	c = 1.2 X_n	c = 1.8 X_n
0	0.50000	0.50000	0.50000
1	-0.25000	-0.95000	-1.55000
2	-0.43750	-0.29750	0.60250
3	-0.30859	-1.11149	-1.43699
4	-0.40477	0.03542	0.26495
5	-0.33616	-1.19875	-1.72980
6	-0.38700	0.23699	1.19221
7	-0.35023	-1.14384	-0.37863
8	-0.37734	0.10836	-1.65664
9	-0.35762	-1.18826	0.94445
10	-0.37211	0.21196	-0.90802
11	-0.36153	-1.15507	-0.97550
12	-0.36929	0.13420	-0.84839
13	-0.36362	-1.18199	-1.08023
14	-0.36778	0.19710	-0.63310
15	-0.36474	-1.16115	-1.39919
.	.	.	.
.	.	.	.

Solutions of Quadratic map

n	$c = 0.5$ X_n	$c = 1.2$ X_n	$c = 1.8$ X_n
.	.	.	.
.	.	.	.
85	-0.36603	-1.17082	-1.69006
86	-0.36603	0.17083	1.05630
87	-0.36603	-1.17082	-0.68423
88	-0.36603	0.17081	-1.33182
89	-0.36603	-1.17082	-0.02624
90	-0.36603	0.17083	-1.79931
91	-0.36603	-1.17082	1.43752
92	-0.36603	0.17082	0.26647
93	-0.36603	-1.17082	-1.72900
94	-0.36603	0.17082	1.18943
95	-0.36603	-1.17082	-0.38527
96	-0.36603	0.17082	-1.65157
97	-0.36603	-1.17082	0.92769
98	-0.36603	0.17082	-0.93940
99	-0.36603	-1.17082	-0.91753
100	-0.36603	0.17082	-0.95814
	Steady	**Periodic**	**Chaotic**


Quadratic Map
c=0.5
c=1.2
c=1.8
Xn
n
2.0
1.0
0.0
−1.0
−2.0
0
10
20
30
40
50
60
70
80
90
100

Lorenz Model
(3-variable convection model)

Lorenz, E. N., 1963: Deterministic nonperiodic flow. *J. Atmos. Sci.*, **20**, 130-141.

The behavior of a simple continuous nonlinear system is discussed.

The system is the famous Lorenz model which is a very low-order model of Rayleigh-Bénard convection.

When the motion of the fluid is caused by a difference in density created by a temperature difference, it is called convection. The fluid absorbs heat at one place, moves to another place and dissipates heat by mixing with the colder fluid. Atmospheric motions are mostly convective in nature because of the thermal inequalities set up by solar heating.

In a well-known laboratory experiment of convection, called Rayleigh-Bénard convection, a fluid is subjected to controlled temperature difference. The motion of the fluid is studied by varying the temperature difference.

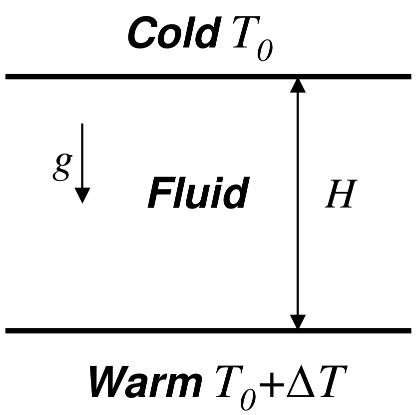
Experiment

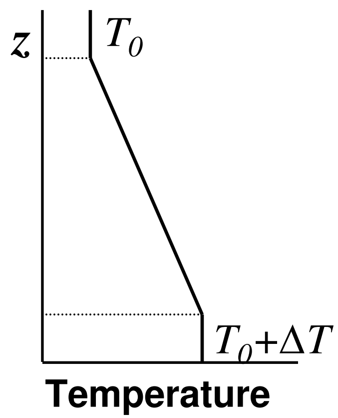
(Bergé, Pomeau and Vidal, 1984: *Order within chaos*, John Wiley & Sons, Section V.3)

A layer of fluid of uniform thickness H is confined between two horizontal plates. The upper plate is at a temperature T_0 and the lower plate at temperature $T_0+\Delta T$.

With such a temperature difference, the fluid in the upper part of the layer is cold and dense whereas the fluid in the lower part is warmer and less dense. The warm fluid tends to rise while the colder fluid tends to fall.

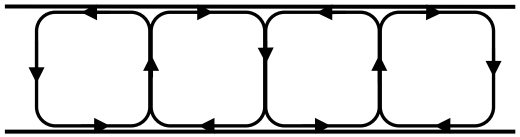
When ΔT is small, there is no convective motion because of the stabilizing effect of friction. The heat transfer is through conduction. In this steady state of the fluid in which there is no motion, the temperature varies linearly with height.





Convection

When ΔT is increased above a certain critical value, the steady state of no motion becomes unstable, and sustained convection begins. Above the convection threshold, a regular structure of rolls with parallel horizontal axes is formed.



The structure consists of alternating rising and descending currents.
The currents are equidistant from one another.
Two adjacent rolls rotate in opposite directions.
The convection is stationary (steady state).

Convective instabilities were first observed experimentally by Bénard in 1900. The theoretical explanation was first given by Rayleigh in 1916. Hence this phenomenon is called Rayleigh-Bénard convection.

When ΔT is increased, the convection pattern first becomes more complicated but retains certain regularity.

When ΔT is further increased, the pattern is completely destroyed and replaced by a disordered configuration in perpetual motion. The fluid motion is turbulent.

Governing Equations

(Saltzman, B., 1962: Finite amplitude free convection as an initial value problem. *J. Atmos. Sci.*, **19**, 329-341)

There is no variation along the y-axis.

All motions are parallel to the x-z plane.

In Boussinesq approximation, the equations governing convection are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{\partial P}{\partial x} + \nu\nabla^2 u$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} = -\frac{\partial P}{\partial z} + \nu\nabla^2 w + g\alpha T$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + w\frac{\partial T}{\partial z} = \kappa\nabla^2 T$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} = 0$$

(u,w) = velocity along (x,z)

P = pressure

T = temperature departure
(convection – no convection)

α = coefficient of thermal expansion

κ = thermal conductivity

ν = kinematic viscosity

g = acceleration due to gravity

Stream function and temperature departure:

$$u = -\frac{\partial \psi}{\partial z}, \quad w = \frac{\partial \psi}{\partial x}$$

$$T(x,z,t) = \left(\overline{T}(0,t) - \frac{\Delta T}{H} z \right) + \theta(x,z,t)$$

T is expressed as a sum of linear variation between upper and lower boundary and a departure θ from the linear variation.

Vorticity and temperature equations:

$$\frac{\partial \nabla^2 \psi}{\partial t} = -\frac{\partial(\psi, \nabla^2 \psi)}{\partial(x,z)} + \nu \nabla^4 \psi + g\alpha \frac{\partial \theta}{\partial x}$$

$$\frac{\partial \theta}{\partial t} = -\frac{\partial(\psi, \theta)}{\partial(x,z)} + \frac{\Delta T}{H} \frac{\partial \psi}{\partial x} + \kappa \nabla^2 \theta$$

Boundary conditions assuming free (no-stress) boundaries:

$$\psi = 0, \qquad \nabla^2 \psi = 0$$

Rayleigh's finding

If Rayleigh *number* $R_a = \dfrac{g\alpha H^3 \Delta T}{\kappa \nu}$ exceeds a critical value $R_c = \dfrac{\pi^4 (1+a^2)^3}{a^2}$,

motion of the following form develops.

$$\psi = \psi_0 \sin\left(\pi a H^{-1} x\right) \sin\left(\pi H^{-1} z\right)$$
$$\theta = \theta_0 \cos\left(\pi a H^{-1} x\right) \sin\left(\pi H^{-1} z\right)$$

Here a is the aspect ratio of vertical to horizontal length scales. The minimum value of R_c = $27\pi^4/4$ occurs when $a^2 = ½$.

Spectral expansion

Introduce the following highly truncated spectral expansions in vorticity and temperature equations (4.3)

$$\psi = \frac{\sqrt{2}\,\kappa\left(1+a^2\right)}{a} X \sin\left(\pi a H^{-1} x\right) \sin\left(\pi H^{-1} z\right)$$
$$\theta = \frac{\sqrt{2} R_c \Delta T}{\pi R_a} Y \cos\left(\pi a H^{-1} x\right) \sin\left(\pi H^{-1} z\right) - Z \sin\left(2\pi H^{-1} z\right)$$

Simplify to obtain a set of three ordinary differential equations.

Lorenz Model (3-variable)

The simplification leads to

$$\frac{dX}{dt} = -\sigma X + \sigma Y$$

$$\frac{dY}{dt} = -XZ + rX - Y$$

$$\frac{dZ}{dt} = XY - bZ$$

X: Intensity of convective motion
Y: Temperature difference between ascending and descending currents
Z: Distortion of vertical temperature profile from linearity

$\sigma = \nu/\kappa$ Prandtl number (determines hydrodynamic versus thermal instability)
$b = 4/(1+a2)$
$r = R_a/R_c$ (Forcing proportional to Rayleigh number)

Lorenz model is a low-order convection model described by just three ordinary differential equations. It is the simplest forced dissipative nonlinear system.

Solutions of Lorenz Model

Symmetry of the model

The equations are invariant under the transformation

$(X, Y, Z) \rightarrow (-X, -Y, Z)$ for all values of r.

i.e., if (X, Y, Z) is a solution, then $(-X, -Y, Z)$ is also a solution.

Solutions

To determine the solutions of the model, it is necessary to numerically integrate the model for a given set of parameters.

However, since the model is simple, the steady state solutions can be determined analytically. When the steady states are stable, such solutions can also be found by numerical integration starting with some arbitrary initial conditions.

Steady States

Steady states are easily found by solving

$$\frac{dX}{dt}=0,\quad \frac{dY}{dt}=0,\quad \frac{dZ}{dt}=0$$

Solve

$$\begin{aligned} -\sigma X+\sigma Y&=0\\ -XZ+rX-Y&=0\\ XY-bZ&=0 \end{aligned} \quad \Longrightarrow \quad \begin{aligned} X&=Y\\ Z&=Y^2/b\\ Y^3-b(r-1)Y&=0 \end{aligned}$$

Steady state solutions:

(1) $X=Y=Z=0$ Steady State O

(2) $X=Y=[b(r-1)]^{1/2},\quad Z=r-1$ Steady State C

(3) $X=Y=-[b(r-1)]^{1/2},\quad Z=r-1$ Steady State C'

When $\underline{r < 1}$, there is only one steady state:

$X = 0, \quad Y = 0, \quad Z = 0$ State of no convection (O)

When $\underline{r > 1}$, there are three steady states:

$X = 0, \quad Y = 0, \quad Z = 0$ No convection (O)

$X = Y = \pm[b(r-1)]1/2, \quad Z = r-1$ Steady convection $(C,\ C')$

Numerical integrations

Integrate the model for the following parameter values

$$\sigma = 10, \quad a^2 = 1/2, \quad b = 8/3$$
$$r = 0.5, 10.0$$

The numerical integration is carried out with a time increment of

$$\Delta\tau = 0.01$$

Steady State Solutions

$r = 0.5$

Rest, No Convection

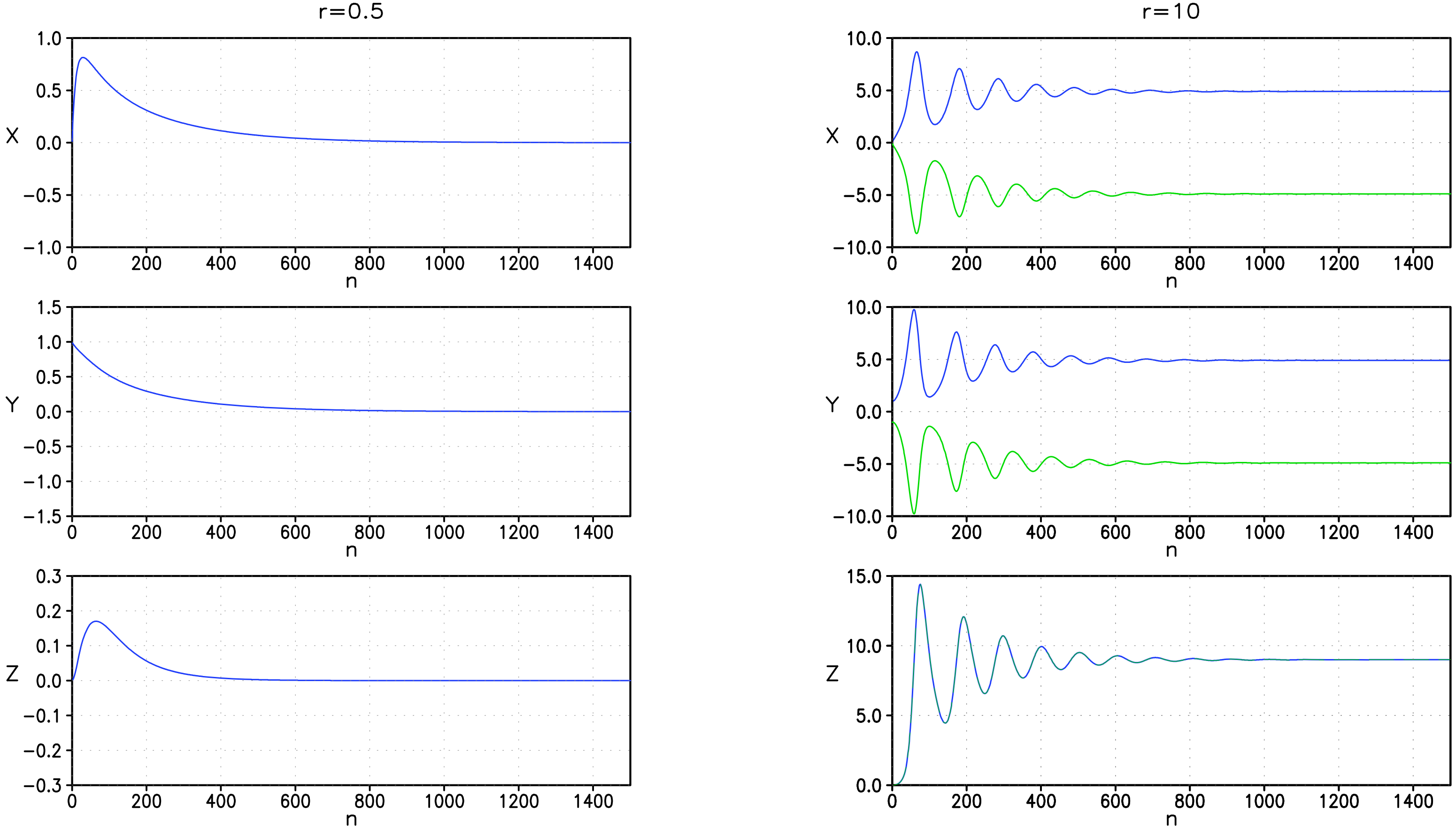
$O: X = 0, \quad Y = 0, \quad Z = 0$

$r = 10.0$

Steady Convection

$C: X = 2\sqrt{6}, \quad Y = 2\sqrt{6}, \quad Z = 9$

$C': X = -2\sqrt{6}, \quad Y = -2\sqrt{6}, \quad Z = 9$



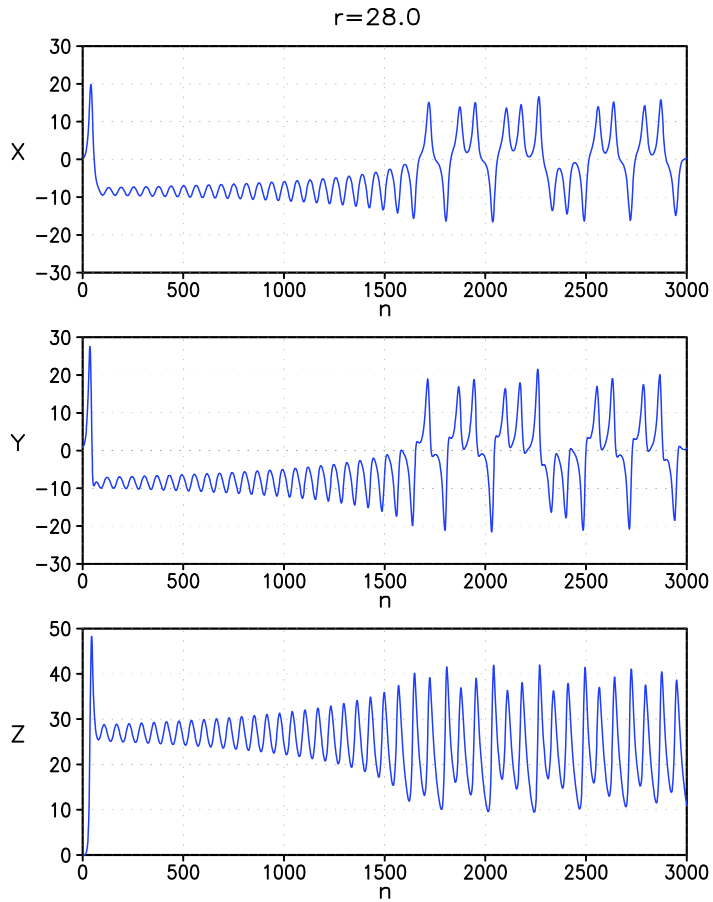
Time-dependent Solutions

As r is increased, the steady convection becomes unstable at a critical value of r. The critical Rayleigh number for instability of steady convection $(C,\ C')$ occurs when $r = 24.74$.

The solutions of Lorenz model at a slightly supercritical value $r = 28$ is studied.

The model is numerically integrated starting with a small perturbation over the state of no convection.

The state of rest is clearly unstable with all three variables growing rapidly. In less than 50 steps, the strength of convection exceeds that of steady convection and the system reaches a state close to the steady convection. The motion then undergoes systematic amplified oscillation until about step 1650. The subsequent behavior of the system is irregular or nonperiodic.



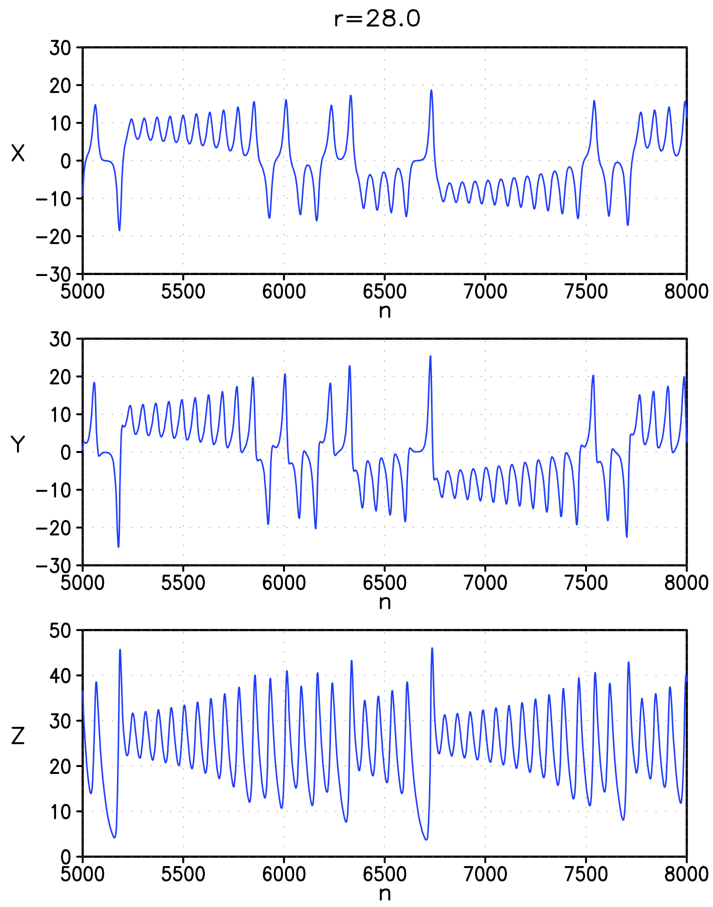
Time-dependent Solutions

$r = 28$

The irregular motion of the system reached at about step 1650 continues for subsequent time steps. The variables X and Y change sign at irregular intervals, reaching sometimes one or more extremes of one sign before changing sign again.

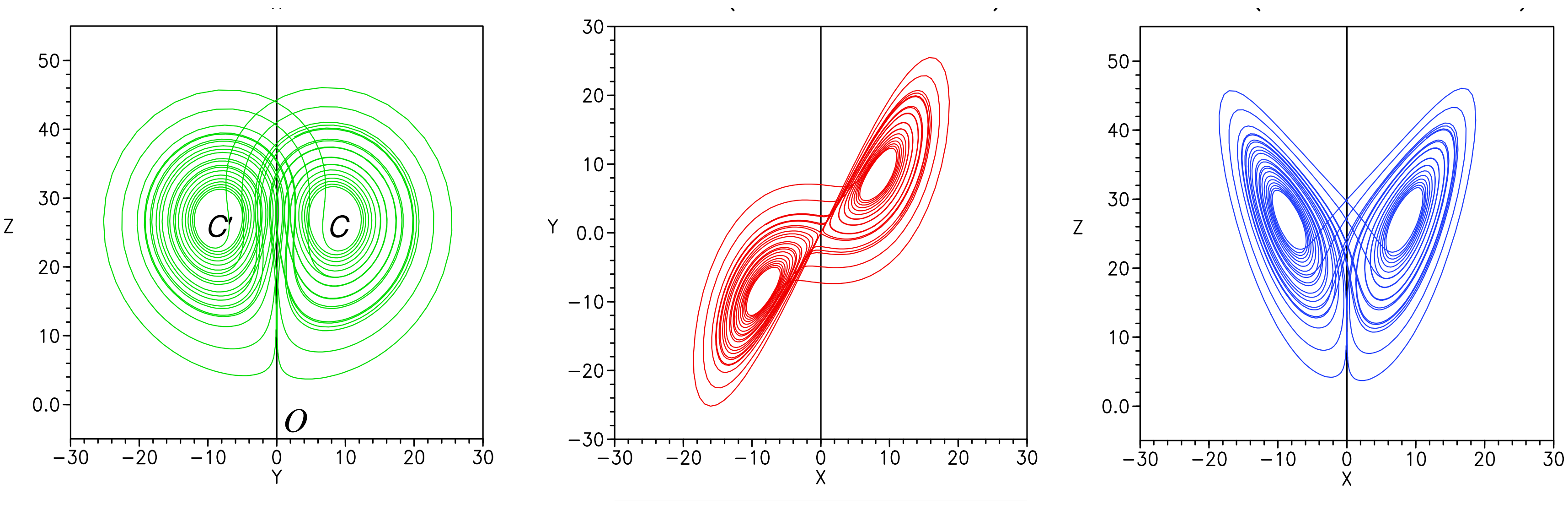
When X and Y have same sign, warm fluid is ascending and cold fluid is descending. When X and Y are of opposite signs, the warm fluid is descending and cold fluid is ascending.

The time variation of (X,Y,Z) is nonperiodic. The fluid motion is turbulent or chaotic.



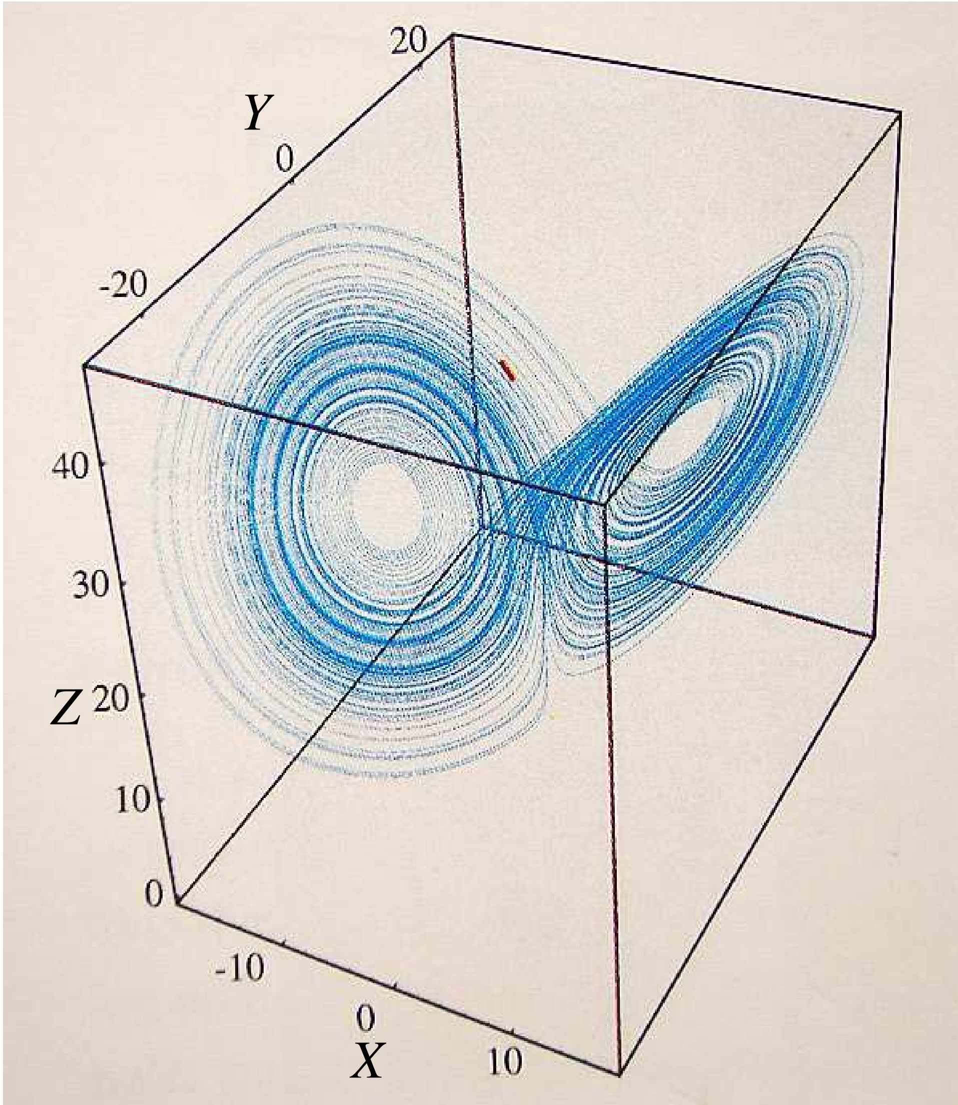
Projection on two-dimension

The trajectory of the solutions are projected on Y-Z, X-Z and X-Y planes. The trajectory moves around the steady state C, crosses a plane and moves around C' and returns to the neighborhood of C. This process continues at irregular intervals. The trajectory never passes through C, C' or O as all three steady states are unstable.



Three-dimensional structure

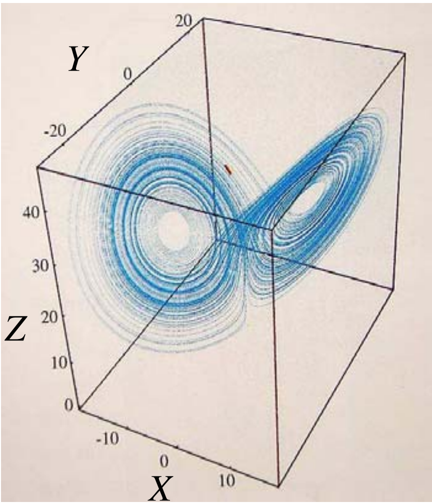
When the trajectory of the solutions is shown in 3-dimension, it is possible to imagine how a trajectory does not intersect itself. This is necessary for nonperiodic solutions. A trajectory may come arbitrarily close to a point that it has visited in the past but will soon diverge.



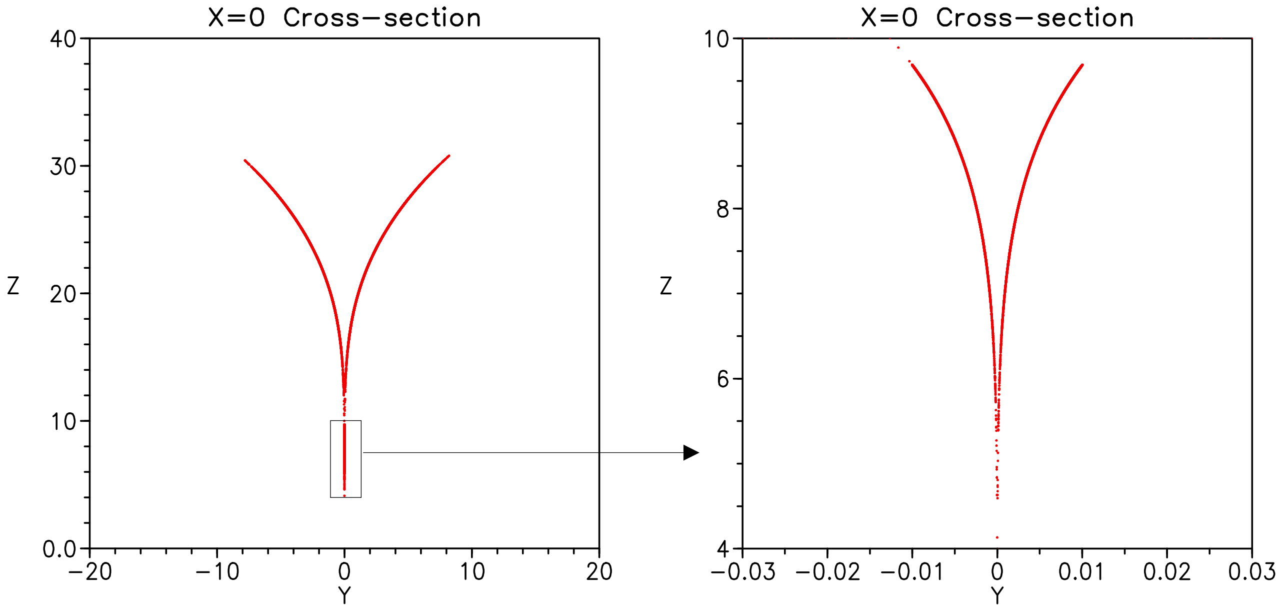
From Strogatz, S. H., 1994: *Nonlinear dynamics and chaos*, Westview Press.

The two surfaces merely appear to merge. However, there are actually an infinite number of surfaces. They are close to one or the other of two seemingly merging surfaces.

For example, the two $X=0$ cross-sections may appear to merge, but they are shown not to merge when magnified. The points (red) on the left branch correspond to when X crosses zero from a positive value (downcrossing) while those on the right branch correspond to when X crosses zero from a negative value (upcrossing).



Strogatz, S. H., 1994: *Nonlinear dynamics and chaos*, Westview Press

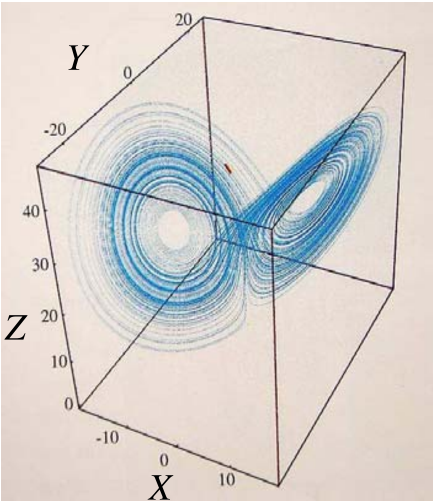


Left Branch: Downcrossing
Right Branch: Upcrossing

Left Branch: Downcrossing
Right Branch: Upcrossing

Further magnification of the surfaces show that the cross-sections have thickness. i.e., there are many (infinite) points in the Y-Z plane where X=0 intersects.

There are also gaps in between the intersections. The cross-sections clearly reveal *fractal* structure.



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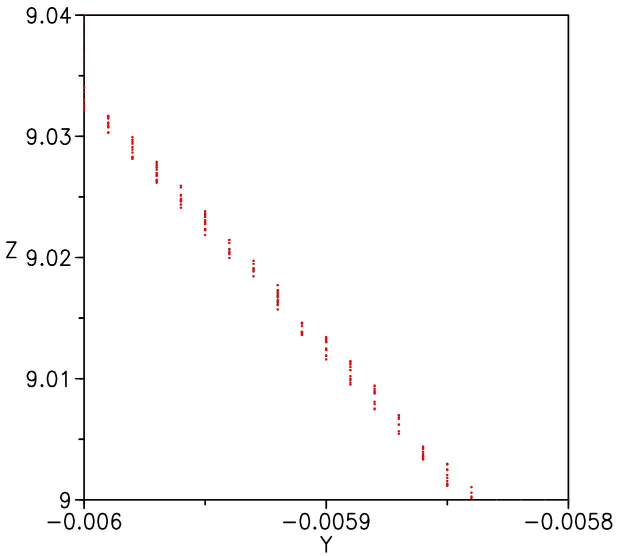
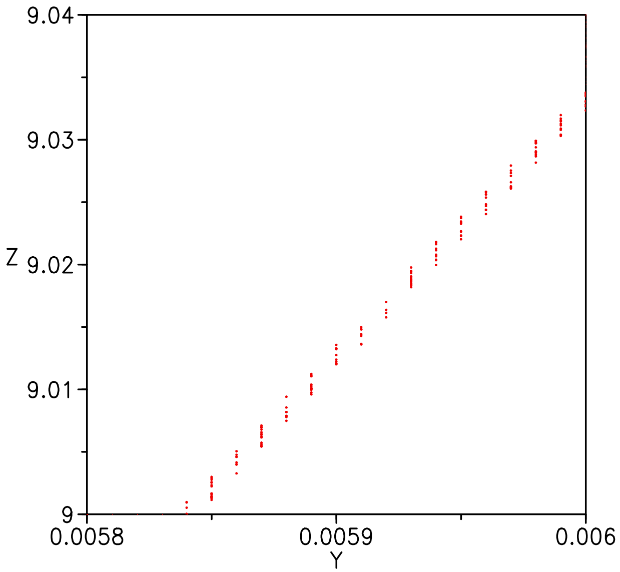


Left Branch: Downcrossing



Right Branch: Upcrossing

Attractor

Attractor is a set of phase space points to which all neighboring trajectories converge asymptotically.

Attractor is a closed invariant set. Any trajectory that starts in the attractor stays in the attractor.

The set of all initial conditions that reaches the attractor is the *basin of attraction*.

Attractor is minimal. There is no subset that satisfies the above conditions.

Attractor is relevant only for dissipative systems.

Types of attractors

Steady state attractor is the stable steady state solution of the system. It is just a point in the phase space.
It is also called *fixed point* or *equilibrium point*.

Periodic attractor is the stable periodic solution forming a closed loop or cycle in the phase space.
It is also called *limit cycle*.

Quasi-periodic attractor is the trajectory on a torus that almost (but not quite) repeats the same route periodically. Each trajectory winds around endlessly on the torus, never intersecting itself and yet never quite closing.

Chaotic attractor is the set of all nonperiodic solutions confined to a bounded region of phase space with volume zero. The cross-sections have fractal structure.
It is also called *strange attractor*.

Unstable steady states and unstable periodic orbits cannot be part of any attractor.

Steady state attractors in Lorenz model

$\underline{r < 1}$

$$X = 0, \quad Y = 0, \quad Z = 0$$

State of no convection (O)
There is only one steady state attractor.

$\underline{1 < r < 24.74}$

$$X = Y = \pm[b(r-1)]^{1/2}, \quad Z = r-1$$

Steady convection $(C,\ C')$
For all values of r in this range, there are two steady state attractors $(C,\ C')$. The basins of attraction for these two attractors are different. It is difficult to predict which initial state will reach which attractor (either C or C'). When there are two or more steady state attractors, they are also referred to as multiple equilibria.

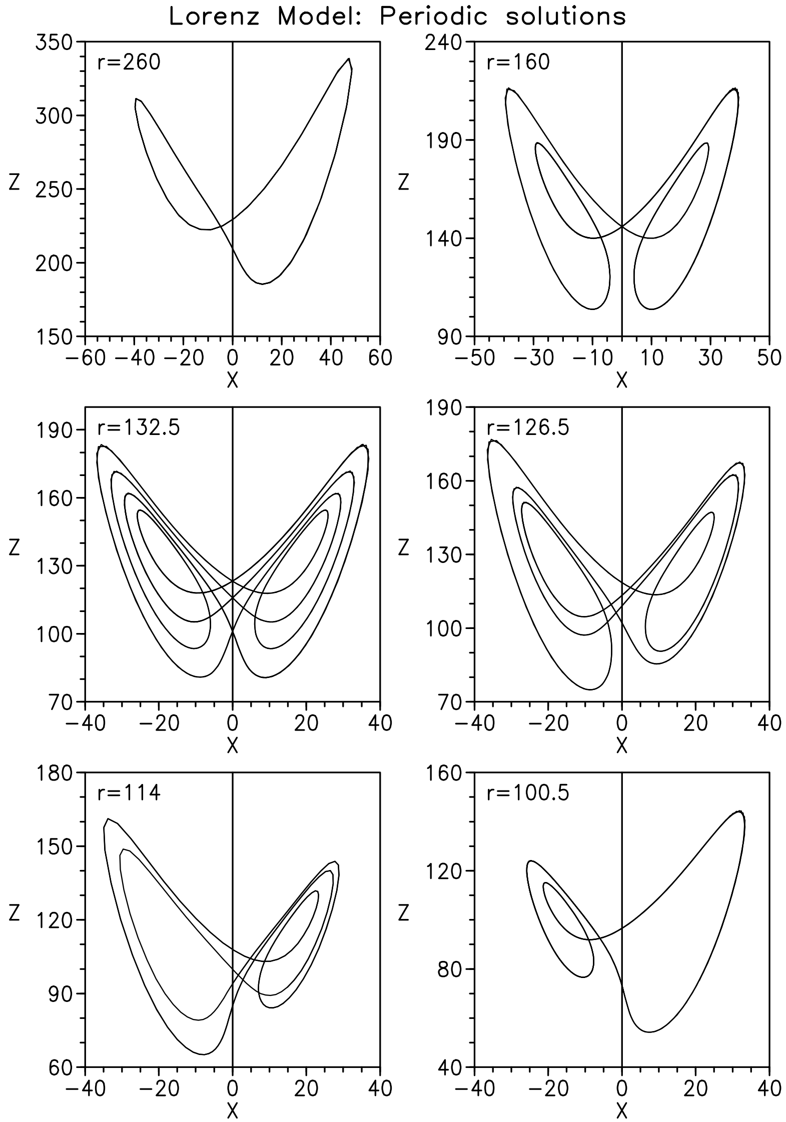
$\underline{r > 24.74}$

There are no steady state attractors.

Periodic attractors in Lorenz model

As *r* is varied, the behavior of the Lorenz model also varies from having chaotic attractors to periodic attractors at irregular intervals of *r*. Windows of periodic solutions and windows of nonperiodic solutions alternate as *r* varies.

$r = 260.0$	xy	Nonsymmetric
$r = 160.0$	x^2y^2	Symmetric
$r = 132.5$	$x^2y\,xy^2\,xy$	Symmetric
$r = 126.5$	$xy\,x^2y^2$	Nonsymmetric
$r = 114.0$	$x^2y\,xy$	Nonsymmetric
$r = 100.5$	xy^2	Nonsymmetric



Chaotic attractors in Lorenz model

Chaotic attractors exist at different values of r.

The attractors in different intervals of r have different characteristics.

The chaotic attractors at r=90 and r=180 are located in different parts of the phase space.