The Abdus Salam International Centre for Theoretical Physics

2050-10

Targeted Training Activity: Predictability of Weather and Climate: Theory and Applications to Intraseasonal Variability

27 July - 7 August, 2009

Predictability Theory

3. Linear Stability Analysis

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Stability

The decay or growth of errors influences the accuracy of predictions.

A solution is *stable* if any other sufficiently close solution remains arbitrarily close (a).

Otherwise, the solution is *unstable* and the nearby solution diverges (b).

If the solution is *stable*, it is *periodic* because when an approximate repetition of a previous state occurs, future states must remain arbitrarily close to the previous history (c).

If the solution is *nonperiodic*, it is necessarily *unstable* (d).

The deciding factor in predictability is stability versus instability.

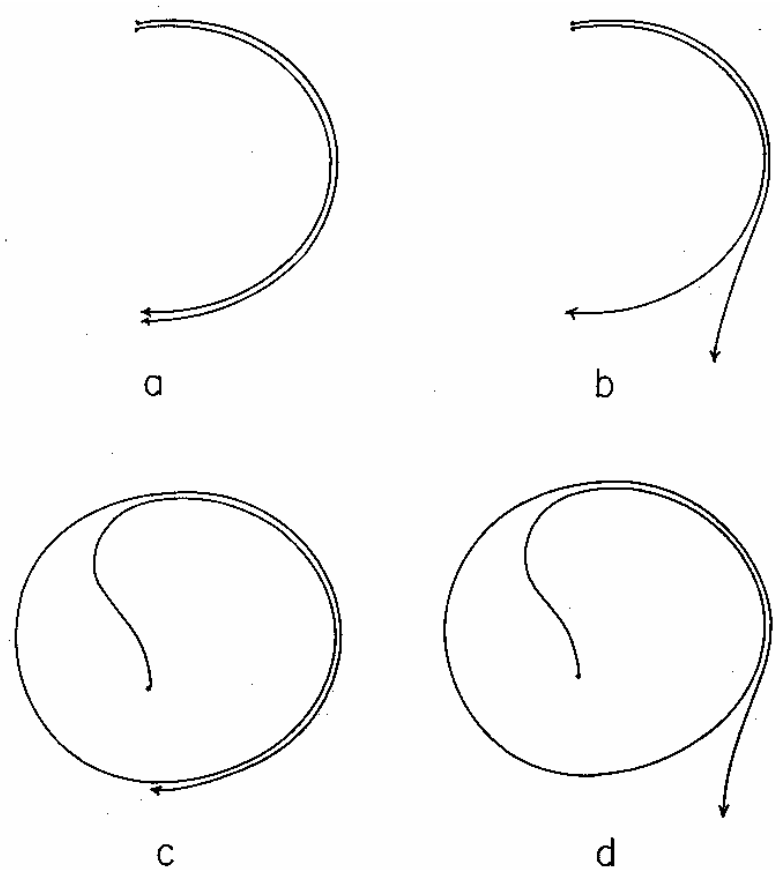


FIGURE 3. Schematic trajectories in phase space; (a) neighboring stable trajectories; (b) neighboring unstable trajectories; (c) stability implying periodicity (after the transient flow has died out); (d) nonperiodicity implying instability.

Lorenz, E. N., 1963: The predictability of hydrodynamic flows. *Trans. New York Acad. Sci.*, Ser II, **25**, 409-423.

Simple Predictability Experiments: Quadratic map

"Identical twin" experiment

Find the basic solutions with certain initial condition. Introduce a small error in the initial condition and find the perturbed solutions. Compare the two solutions and study the evolution of the error.

Example 1

$$X_{n+1} = X_n^2 - c, \qquad c = 0.5$$

X_n = basic ("true") solution

X_0' = "observed" value

X_n' = perturbed ("predicted") solution

$X_0' - X_0$ = "observed error"

$X_n' - X_n$ = error at time n

n	X_n	X_n'	$X_n' - X_n$
0	0.40000	0.40100	0.0010000
1	-0.34000	-0.33920	0.0008010
2	-0.38440	-0.38494	-0.0005440
3	-0.35224	-0.35182	0.0004186
4	-0.37593	-0.37622	-0.0002947
5	-0.35868	-0.35846	0.0002216
6	-0.37135	-0.37151	-0.0001590
7	-0.36210	-0.36198	0.0001181
8	-0.36888	-0.36897	-0.0000855
9	-0.36392	-0.36386	0.0000631
10	-0.36756	-0.36761	-0.0000459
11	-0.36490	-0.36487	0.0000338
12	-0.36685	-0.36687	-0.0000246
13	-0.36542	-0.36540	0.0000181
14	-0.36647	-0.36648	-0.0000132
15	-0.36570	-0.36569	0.0000097
16	-0.36626	-0.36627	-0.0000071
17	-0.36585	-0.36585	0.0000052
18	-0.36615	-0.36616	-0.0000038
19	-0.36593	-0.36593	0.0000028
20	-0.36609	-0.36610	-0.0000020
.			
.			

Simple Predictability Experiments

Example 1 (contd.)

$$X_{n+1} = X_n^2 - c\,, \qquad c = 0.5$$

The error steadily decreases and decays to zero. The steady state solution is stable.

Lorenz, E. N., 1985: The growth of errors in prediction. *Turbulence and predictability in geophysical fluid dynamics and climate dynamics*, M. Ghil and R. Benzi, Eds., LXXXVIII Corso Soc. Itiliana di Fisica, Bologna, Italy, 243-265.

n	X_n	X_n'	$X_n' - X_n$
.			
.			
81	-0.36603	-0.36603	0.0000000
82	-0.36603	-0.36603	0.0000000
83	-0.36603	-0.36603	0.0000000
84	-0.36603	-0.36603	0.0000000
85	-0.36603	-0.36603	0.0000000
86	-0.36603	-0.36603	0.0000000
87	-0.36603	-0.36603	0.0000000
88	-0.36603	-0.36603	0.0000000
89	-0.36603	-0.36603	0.0000000
90	-0.36603	-0.36603	0.0000000
91	-0.36603	-0.36603	0.0000000
92	-0.36603	-0.36603	0.0000000
93	-0.36603	-0.36603	0.0000000
94	-0.36603	-0.36603	0.0000000
95	-0.36603	-0.36603	0.0000000
96	-0.36603	-0.36603	0.0000000
97	-0.36603	-0.36603	0.0000000
98	-0.36603	-0.36603	0.0000000
99	-0.36603	-0.36603	0.0000000
100	-0.36603	-0.36603	0.0000000

Simple Predictability Experiments

Example 2

$$X_{n+1} = X_n^2 - c\,, \qquad c = 1.2$$

X_n = basic ("true") solution

X_0' = "observed"

X_n' = perturbed ("predicted") solution

$X_0' - X_0$ = "observed error"

$X_n' - X_n$ = error at time n

The error amplifies during the first few time steps and then undergoes damped oscillations.

n	X_n	X_n'	$X_n' - X_n$
0	0.50000	0.50100	0.0010000
1	-0.95000	-0.94900	0.0010010
2	-0.29750	-0.29940	-0.0019009
3	-1.11149	-1.11036	0.0011346
4	0.03542	0.03290	-0.0025210
5	-1.19875	-1.19892	-0.0001722
6	0.23699	0.23740	0.0004129
7	-1.14384	-1.14364	0.0001959
8	0.10836	0.10791	-0.0004481
9	-1.18826	-1.18836	-0.0000969
10	0.21196	0.21219	0.0002303
11	-1.15507	-1.15498	0.0000977
12	0.13420	0.13397	-0.0002257
13	-1.18199	-1.18205	-0.0000605
14	0.19710	0.19725	0.0001431
15	-1.16115	-1.16109	0.0000564
16	0.14827	0.14814	-0.0001310
17	-1.17802	-1.17806	-0.0000388
18	0.18772	0.18781	0.0000915
19	-1.16476	-1.16473	0.0000344
20	0.15667	0.15659	-0.0000800
.			
.			

Simple Predictability Experiments

Example 2 (contd.)

$X_{n+1} = X_n^2 - c$, $c = 1.2$

By about time step 90, the two solutions are identical up to five decimal places. The error has decayed. The periodic solution is stable.

n	X_n	X_n'	$X_n' - X_n$
.			
.			
81	-1.17083	-1.17083	0.0000000
82	0.17083	0.17083	0.0000001
83	-1.17082	-1.17082	0.0000000
84	0.17081	0.17081	-0.0000001
85	-1.17082	-1.17082	0.0000000
86	0.17083	0.17083	0.0000000
87	-1.17082	-1.17082	0.0000000
88	0.17081	0.17081	0.0000000
89	-1.17082	-1.17082	0.0000000
90	0.17083	0.17083	0.0000000
91	-1.17082	-1.17082	0.0000000
92	0.17082	0.17082	0.0000000
93	-1.17082	-1.17082	0.0000000
94	0.17082	0.17082	0.0000000
95	-1.17082	-1.17082	0.0000000
96	0.17082	0.17082	0.0000000
97	-1.17082	-1.17082	0.0000000
98	0.17082	0.17082	0.0000000
99	-1.17082	-1.17082	0.0000000
100	0.17082	0.17082	0.0000000

Simple Predictability Experiments

Example 3

$X_{n+1} = X_n^2 - c\,,\qquad c$ **= 1.8**

X_n = basic ("true") solution
$X_0{}'$ = "observed"
$X_n{}'$ = perturbed ("predicted") solution
$X_0{}' - X_0$ = "observed error"
$X_n{}' - X_n$ = error at time n

The error grows irregularly, gaining an order of magnitude in about five time steps and becomes comparable to X_n itself.

n	X_n	$X_n{}'$	$X_n{}' - X_n$
0	0.50000	0.50100	0.0010000
1	-1.55000	-1.54900	0.0010010
2	0.60250	0.59940	-0.0031021
3	-1.43699	-1.44072	-0.0037284
4	0.26495	0.27568	0.0107293
5	-1.72980	-1.72400	0.0058006
6	1.19221	1.17218	-0.0200341
7	-0.37863	-0.42600	-0.0473684
8	-1.65664	-1.61852	0.0381141
9	0.94445	0.81962	-0.1248300
10	-0.90802	-1.12823	-0.2202084
11	-0.97550	-0.52711	0.4483977
12	-0.84839	-1.52216	-0.6737677
13	-1.08023	0.51697	1.5971992
14	-0.63310	-1.53275	-0.8996509
15	-1.39919	0.54931	1.9485018
16	0.15773	-1.49826	-1.6559891
17	-1.77512	0.44477	2.2198931
18	1.35105	-1.60218	-2.9532298
19	0.02534	0.76697	0.7416291
20	-1.79936	-1.21175	0.5876038
.			
.			

Simple Predictability Experiments

Example 3 (contd.)

$$X_{n+1} = X_n^2 - c, \qquad c = 1.8$$

The error varies irregularly, but does not amplify forever because both X_n and X_n' are bounded.

When the error becomes comparable to X_n itself, the error has reached saturation. At this point, the prediction X_n' has become worthless.

The nonperiodic solution is unstable.

n	X_n	X_n'	$X_n' - X_n$
.			
.			
81	0.25407	1.09554	0.8414776
82	-1.73545	-0.59978	1.1356682
83	1.21179	-1.44026	-2.6520482
84	-0.33157	0.27435	0.6059283
85	-1.69006	-1.72473	-0.0346710
86	1.05630	1.17469	0.1183943
87	-0.68423	-0.42010	0.2641366
88	-1.33182	-1.62352	-0.2916942
89	-0.02624	0.83581	0.8620564
90	-1.79931	-1.10142	0.6978925
91	1.43752	-0.58688	-2.0243978
92	0.26647	-1.45558	-1.7220418
93	-1.72900	0.31870	2.0476958
94	1.18943	-1.69843	-2.8878565
95	-0.38527	1.08467	1.4699306
96	-1.65157	-0.62350	1.0280698
97	0.92769	-1.41125	-2.3389325
98	-0.93940	0.19162	1.1310159
99	-0.91753	-1.76328	-0.8457533
100	-0.95814	1.30917	2.2673060

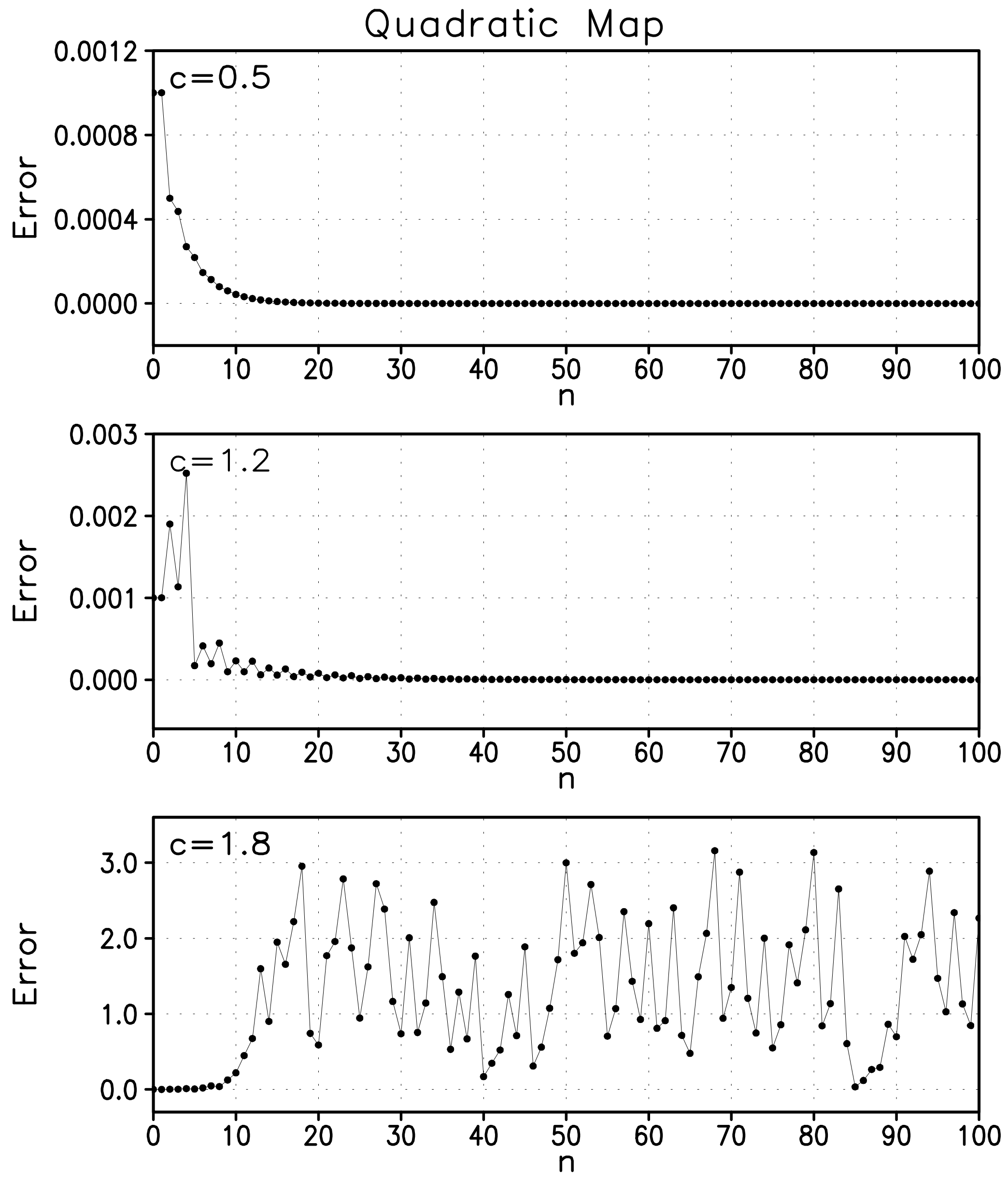
Error Growth in Quadratic Map

Absolute value of $(X_n' - X_n)$ is plotted as error.

Steady State:
The error decays to zero and the system is stable.

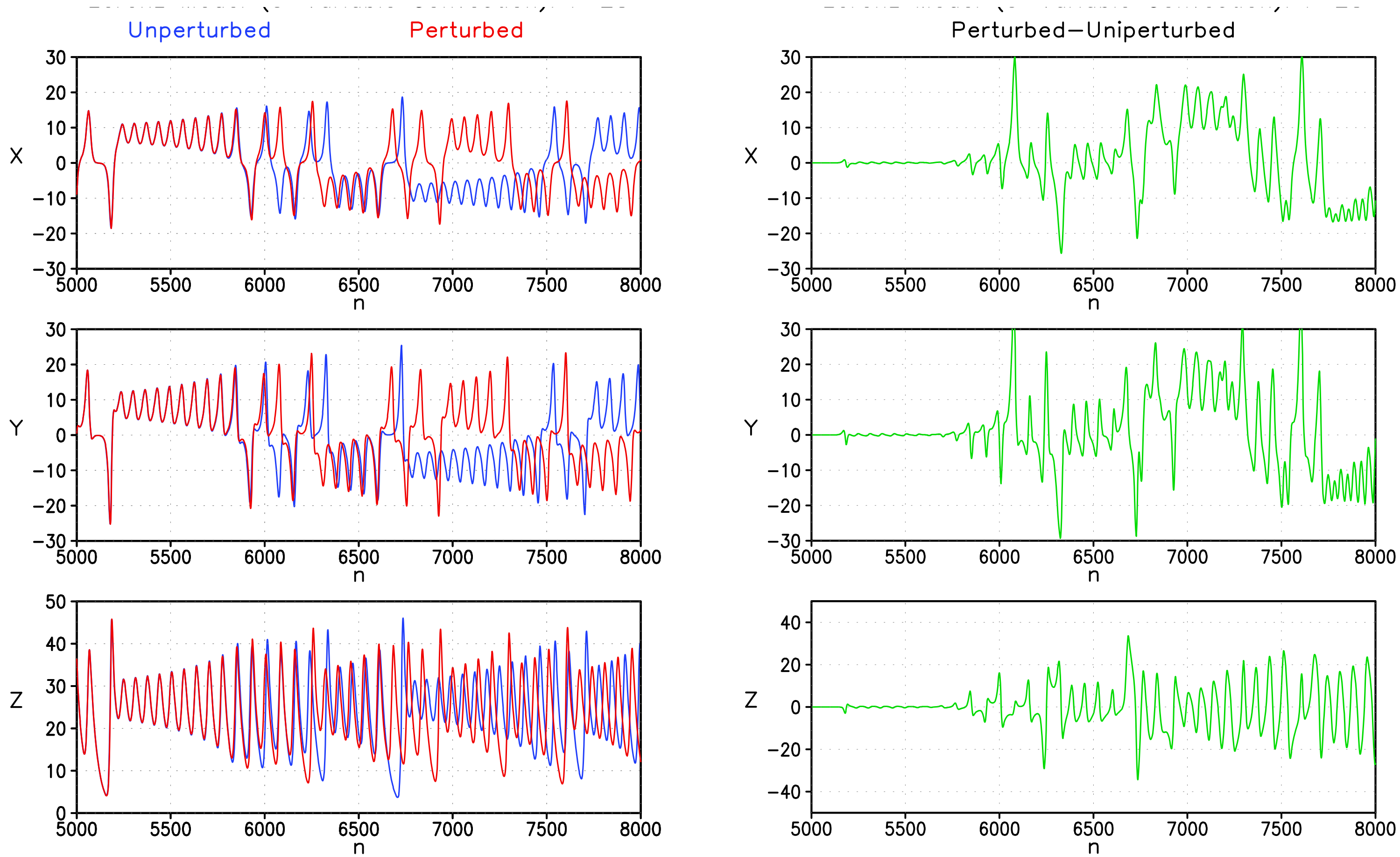
Periodic:
The error decays to zero and the system is stable.

Nonperiodic:
The error grows and becomes as large as the difference between two randomly selected states of the system. The system is unstable.



Lorenz model: "Identical twin" experiment

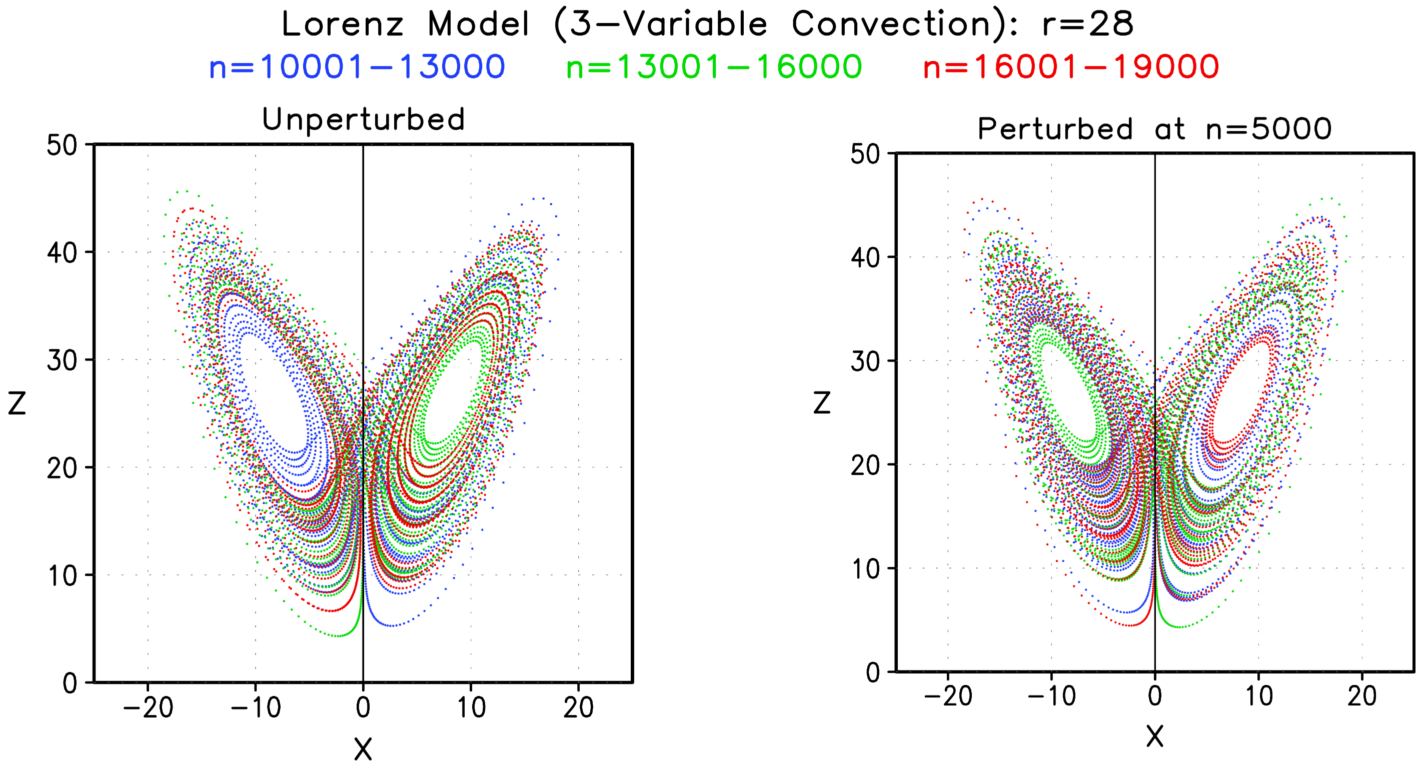
A new integration with a small perturbation added to the original solution at time 5000 is carried out. The two solutions stay close for a while and then diverge. The difference between the two solutions become as large as the variables themselves by step 6000. The nonperiodic solution at $r = 28$ is unstable.



Projection of “identical twin” trajectories

The evolutions of the unperturbed and perturbed trajectories are shown as projections on two-dimensional space.

The projections of unperturbed and perturbed trajectories are shown in different colors for different segments of time and the divergence of trajectories is clearly evident.

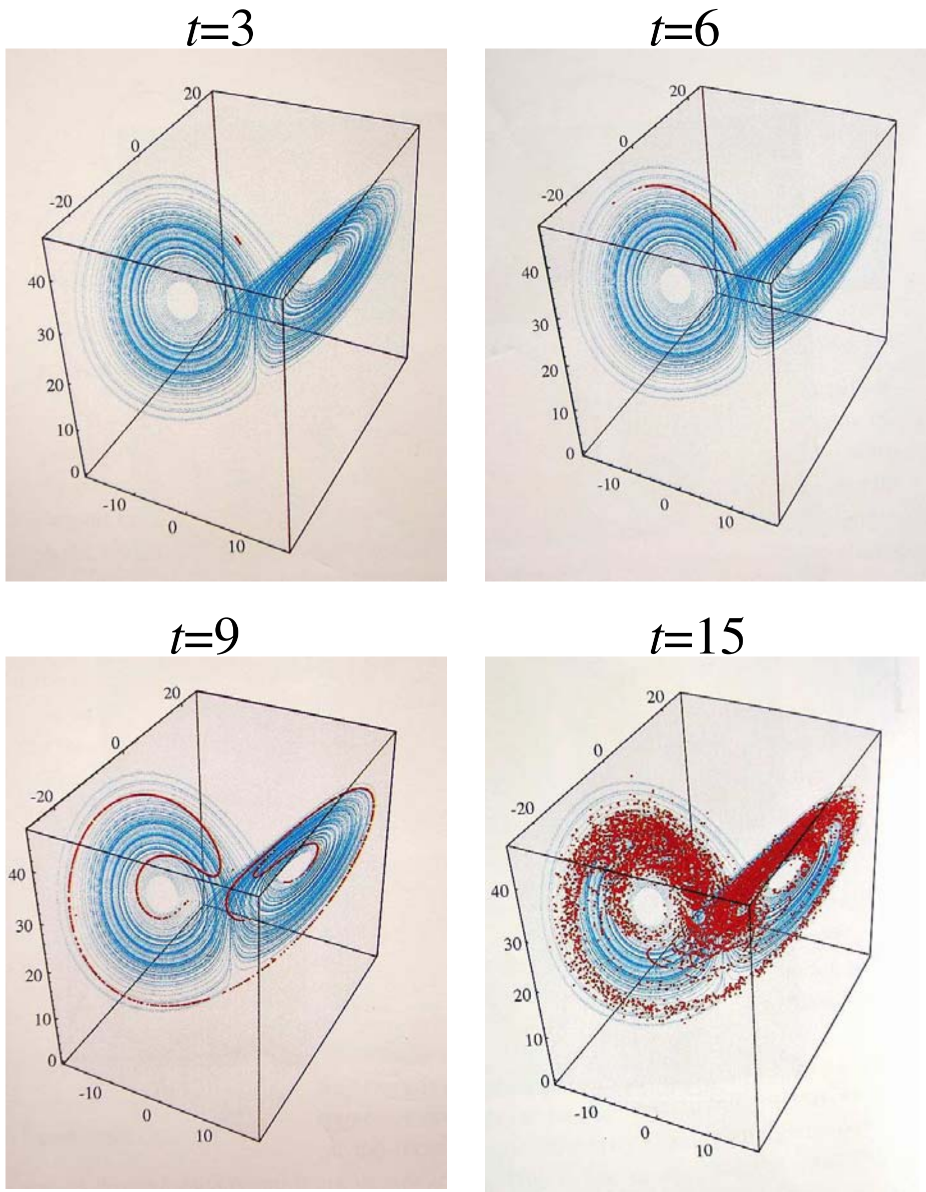


Lorenz model
Predictability experiment

An ensemble of 10000 nearby points at an initial $t = 0$ around a basic state is allowed to evolve.
Blue points are from unperturbed integration.
Red points show the evolution of the perturbed initial states.

"As each point moves according to Lorenz equations, the blob is stretched into a thin filament… Ultimately, the points spread over … showing that the final state could be almost anywhere, even though the initial conditions were almost identical."

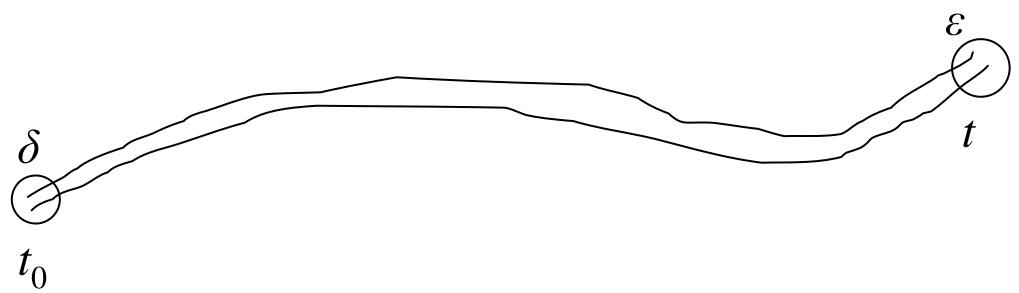
From Strogatz, S. H., 1994: *Nonlinear dynamics and chaos*, Westview Press



Stability and Instability

An orbit is called ***stable*** at a point $X(t_0)$ if any other orbit passing sufficiently close to $X(t_0)$ at time t_0 remains close to $X(t)$ as $t \to \infty$.

An orbit $X(t)$ is ***stable*** at $t = t_0$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $|Y(t_0) - X(t_0)| < \delta$ and $t > t_0$, $|Y(t) - X(t)| < \varepsilon$.



This is called ***Lyapunov stable***: "start near stay near"

Otherwise, $X(t)$ is un***stable***.

If $X(t)$ is stable at $t = t_0$, it is stable for all $t > t_0$ (and also at $t < t_0$ if the system is defined by differential equations).

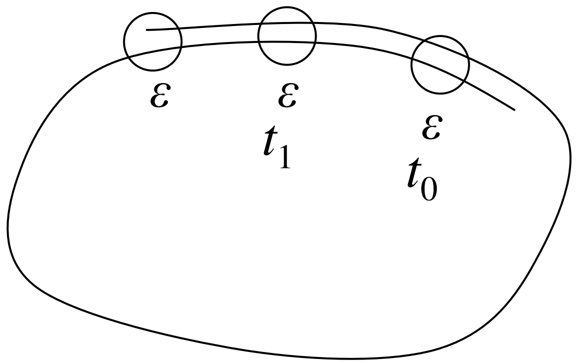
If $X(t)$ is Lyapunov stable and if $|Y(t) - X(t)| \to 0$ as $t \to \infty$ (i.e., attracting), then $X(t)$ is ***asymptotically stable***.

Periodicity

Since each point lies on a unique orbit, any orbit passing through a point through which it has previously passed must continue to repeat its past behavior and must be *periodic*.

An orbit $X(t)$ is *quasi-periodic* if for some arbitrary large time interval τ, $X(t+ \tau)$ ultimately remains arbitrarily close to $X(t)$.

$X(t)$ is *quasi-periodic* at if, for any $\varepsilon > 0$ and τ_0, there exists a $\tau > \tau_0$ such that $|X(t+ \tau) - X(t)| < \varepsilon$ if $t > t_0$.



Periodic orbits are special cases of quasi-periodic orbits.

An orbit with a stable limiting orbit is quasi-periodic (includes periodic orbits). These orbits are the periodic or quasi-periodic attractors.

Nonperiodic orbits

An orbit that is not quasi-periodic is called *nonperiodic*.

If $X(t)$ is nonperiodic, $X(t_1+\tau)$ may be arbitrarily close to $X(t_1)$ for some time t_1 and some arbitrarily large time interval τ, but if this is so, $X(t_1+\tau)$ cannot remain arbitrarily close to $X(t)$ as $t\rightarrow\infty$.

A nonperiodic orbit is *unstable*. It implies that two states differing by imperceptible amounts may eventually evolve into two considerably different states.

If there is any error in observing the present state, an acceptable prediction in the distant future may well be impossible.

Instability places a limit on the predictability of the system if the observations are less than perfect. The deciding factor in predictability is stability versus instability.

Attractors and Stability

Steady state attractors (fixed points) are stable.
Unstable steady states are not attractors.

Periodic attractors (limit cycles) are stable.
Unstable periodic solutions are not attractors.

Nonperiodic (chaotic) attractors consist of points that are only unstable.

Linear Stability Analysis

Study the stability of solutions with respect to small perturbations (or errors).
Consider a dynamical system

$$\frac{dX_i}{dt} = F_i(X_1, \cdots, X_M), \quad i = 1, \cdots, M$$

In compact notation,

$$\frac{d\mathbf{X}}{dt} = \mathbf{F}(\mathbf{X})$$

where $\mathbf{X} = (X_1, \cdots, X_M)$ and $\mathbf{F} = (F_1, \cdots, F_M)$

Consider two solutions $\mathbf{X}$ and $\mathbf{X}+\mathbf{x}$, where $\mathbf{x} = (x_1, \ldots, x_M)$ is small.

$$\frac{d}{dt}(\mathbf{X} + \mathbf{x}) = \mathbf{F}(\mathbf{X} + \mathbf{x})$$

Expand $\mathbf{F}(\mathbf{X}+\mathbf{x})$ in Taylor series around $\mathbf{X}$.

$$\mathbf{F}(\mathbf{X}+\mathbf{x}) = \mathbf{F}(\mathbf{X}) + \frac{\partial \mathbf{F}(\mathbf{X})}{\partial \mathbf{X}} \mathbf{x} + \cdots \text{higher order terms}$$

Neglecting higher order terms (because $\mathbf{x}$ is small), we obtain a linear equation for $\mathbf{x}$.

$$\frac{d\mathbf{x}}{dt} = \frac{\partial \mathbf{F}}{\partial \mathbf{X}} \ \mathbf{x}$$

or

$$\frac{dx_1}{dt} = \frac{\partial F_1}{\partial X_1} \ x_1 \ + \ \frac{\partial F_1}{\partial X_2} \ x_2 + \cdots + \frac{\partial F_1}{\partial X_M} \ x_M$$

$$\vdots$$

$$\frac{dx_M}{dt} = \frac{\partial F_M}{\partial X_1} \ x_1 \ + \ \frac{\partial F_M}{\partial X_2} \ x_2 + \cdots + \frac{\partial F_M}{\partial X_M} \ x_M$$

Write in compact matrix notation.

Error equation: $\dfrac{d\mathbf{x}}{dt} = \mathbf{H}\mathbf{x}$

where

$$\mathbf{H} = \begin{pmatrix} \dfrac{\partial F_1}{\partial X_1} & \cdots & \dfrac{\partial F_1}{\partial X_M} \\ \vdots & & \vdots \\ \dfrac{\partial F_M}{\partial X_1} & \cdots & \dfrac{\partial F_M}{\partial X_M} \end{pmatrix}$$

If $\mathbf{X}$ is a steady state, $\mathbf{H}$ is constant,

If $\mathbf{X}$ is periodic or chaotic, $\mathbf{H}$ is time-dependent.

Because $\mathbf{H}$ is linear, the error equation can be integrated from time t_0 to t_1 to obtain

$$x_i(t_1) = \sum a_{ij}(t_1,t_0)\, x_j(t_0), \qquad i = 1, \ldots, M$$

$$\mathbf{x}(t_1) = \mathbf{A}(t_1,t_0)\, \mathbf{x}(t_0)$$

$\mathbf{A}$ is a square matrix which depends on the behavior of $\mathbf{X}$ between t_0 and t_1.

$\mathbf{A}$ is multiplicative:

If $t_0 < t_1 < t_2$,

$$\mathbf{x}(t_2) = \mathbf{A}(t_2, t_0)\mathbf{x}(t_0)$$
$$\mathbf{x}(t_2) = \mathbf{A}(t_2, t_1)\mathbf{x}(t_1) = \mathbf{A}(t_2, t_1)\mathbf{A}(t_1, t_0)\mathbf{x}(t_0)$$

Equating the RHS of the two equations**,**

$$\mathbf{A}(t_2,t_0) = \mathbf{A}(t_2,t_1)\ \mathbf{A}(t_1,t_0)$$

Simple solution:

If $M = 1$ or
if $M > 1$ and $\mathbf{H}$ is constant,

$$\mathbf{x}(t_1) = \mathbf{x}(t_0)\exp\left(\int_{t_0}^{t_1} \mathbf{H}\,dt\right)$$

Linear Stability Analysis of Lorenz Model

Lorenz model

$$\frac{dX}{dt} = -\sigma X + \sigma Y$$
$$\frac{dY}{dt} = -XZ + rX - Y$$
$$\frac{dZ}{dt} = XY - bZ$$

Let the state of the system at time t be (X, Y, Z) and
let a state with a small perturbation be $(X+x, Y+y, Z+z)$ where x, y, z are small.

The linear perturbation equation is

$$\frac{d\mathbf{x}}{dt} = \mathbf{H}\mathbf{x}$$

$$\frac{\partial F_1}{\partial X} = -\sigma, \qquad \frac{\partial F_1}{\partial Y} = \sigma, \qquad \frac{\partial F_1}{\partial Z} = 0$$

$$\frac{\partial F_2}{\partial X} = -Z + r, \qquad \frac{\partial F_2}{\partial Y} = -1, \qquad \frac{\partial F_2}{\partial Z} = -X$$

$$\frac{\partial F_3}{\partial X} = -Y, \qquad \frac{\partial F_3}{\partial Y} = X, \qquad \frac{\partial F_3}{\partial Z} = -b$$

The linear perturbation equation (or stability equation) for the Lorenz model becomes

$$\begin{pmatrix} dx/dt \\ dy/dt \\ dz/dt \end{pmatrix} = \begin{pmatrix} -\sigma & \sigma & 0 \\ r-Z & -1 & -X \\ Y & X & -b \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

The stability equation is linear and should be integrated numerically when (X,Y,Z) is time-dependent.

When (X,Y,Z) is time-independent, the stability equation can be solved by assuming some form of the solution for (x,y,z).

Stability of steady states

Denote the steady states by (X_0, Y_0, Z_0).
Lorenz model has three steady states O, C and C'. $\mathbf{H}$ is constant.

Solve the linear stability equation by assuming

$$x = x_0 \exp(\lambda t)$$
$$y = y_0 \exp(\lambda t)$$
$$z = z_0 \exp(\lambda t)$$

Solve the characteristic equation

$$\begin{vmatrix} -\sigma - \lambda & \sigma & 0 \\ r - Z_0 & -1-\lambda & -X_0 \\ Y_0 & X_0 & -b-\lambda \end{vmatrix} = 0$$

$$(\lambda + \sigma)[(\lambda + b)(\lambda + 1) + X_0^2] + \sigma[(Z_0 - r)(\lambda + b) + X_0 Y_0] = 0$$

Equivalently, perform an eigenanalysis of $\mathbf{H}$ for each steady state.

Solve

$$\mathbf{Hv}=\lambda\mathbf{v}$$

where $\mathbf{v}$ is an eigenvector with a corresponding eigenvalue λ.

For distinct eigenvalues $(\lambda_1,\lambda_2,\lambda_3)$ with eigenvectors $(\mathbf{v_1},\mathbf{v_2},\mathbf{v_3})$, the general solution is

$$\mathbf{x}(t) = c_1\exp(\lambda_1 t)\mathbf{v_1} + c_2\exp(\lambda_2 t)\mathbf{v_2} + c_3\exp(\lambda_3 t)\mathbf{v_3}$$

where c_1, c_2 and c_3 depend on the initial perturbation (x_0,y_0,z_0).

The stability of the steady state depends on λ.

Stability of O

$X_0 = 0,\ \ Y_0 = 0,\ \ Z_0 = 0$

Solve $\quad (\lambda+b)[\lambda^2 + (\sigma+1)\lambda + \sigma(1-r)] = 0$

When $r > 0$, the characteristic equation has three real roots.

$$\lambda = -b$$

$$\lambda = -\frac{1}{2}(\sigma+1) \pm \frac{1}{2}[(\sigma+1) - 4\sigma(1-r)]^{1/2}$$

When $0 < r < 1$, all three roots are negative. This means that the perturbation decays at an exponential rate. The steady state O is stable in this case.

When $r > 1$, one root is positive indicating that the perturbation grows at an exponential rate. The steady state O is now unstable.

Stability of C and C'

When $r > 1$, there are two more steady states

$$C: \qquad X_0 = Y_0 = \sqrt{b(r-1)}\,, \qquad Z_0 = r-1$$

$$C': \qquad X_0 = Y_0 = -\sqrt{b(r-1)}\,, \qquad Z_0 = r-1$$

For stability analysis, solve the characteristic equation:

$$\lambda^3 + (\sigma + b + 1)\lambda^2 + (r + \sigma)b\lambda + 2\sigma b(r-1) = 0$$

When $r > 1$, the equation has one real root and two complex conjugate roots.
The complex roots become pure imaginary if

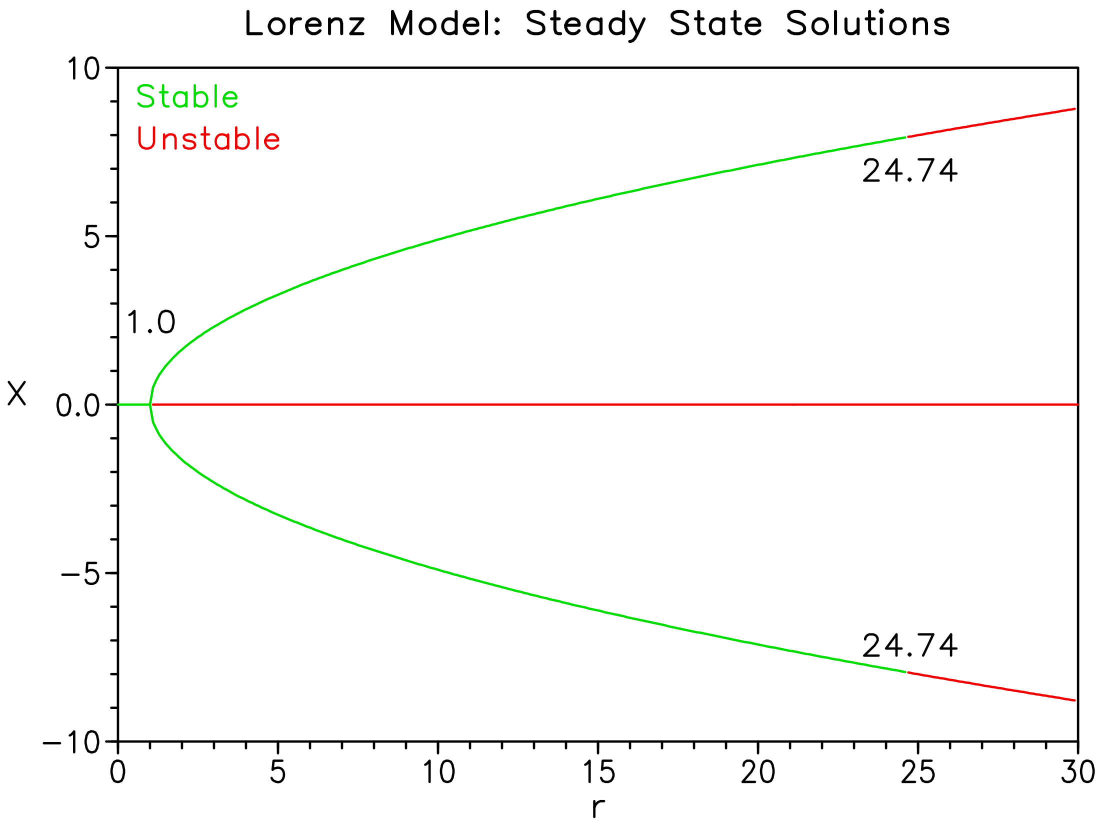
$$r = r_c = \frac{\sigma(\sigma + b + 3)}{(\sigma - b - 1)}$$

i.e., if the complex root is $\lambda = \lambda_r + i\lambda_i$, $\lambda_r < 0$ for $1 < r < r_c$ and $\lambda_r > 0$ for $r > r_c$.
This is the critical value for the instability of C and C'.

If $\sigma > b+1$, the steady states C and C' will become unstable for sufficiently high Rayleigh numbers.
For $\sigma = 10$ and $b = 8/3$, the instability occurs at the critical value of $r_c = 24.74$.

Stability of Steady States in Lorenz Model

$\sigma = 10$ and $b = 8/3$



Stability of Periodic Solutions and Transition to Chaos

Floquet theory

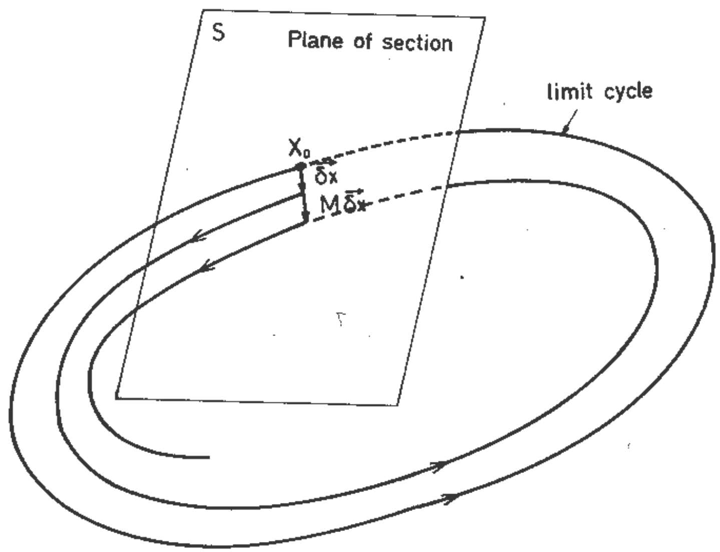
Consider a periodic solution
$\mathbf{X}(t+T) = \mathbf{X}(t)$, where T = period.

A periodic solution corresponds to a fixed point $\mathbf{X_0}$ on a Poincaré cross-section S. The stability of the periodic solution is the same as the stability of the fixed point on the Poincaré cross-section S.

Let $\delta\mathbf{X}$ be a small perturbation such that $\mathbf{X_0}+\delta\mathbf{X}$ is in S.

Linearizing the flow about the periodic orbit, the initial condition $\mathbf{X_0} + \delta\mathbf{X}$ is mapped to $\mathbf{X_0} + \mathbf{M}\,\delta\mathbf{X}$ at the end of the period T.

$$\delta\mathbf{X}(T) = \mathbf{M}\,\delta\mathbf{X}(0)$$



Bergé, Pomeau and Vidal, 1984: *Order within chaos*, John Wiley & Sons

$\mathbf{M}$ is a square matrix called the *Floquet matrix* and determines the stability of the periodic orbit.

$\mathbf{M}$ can be computed by numerically integrating

$$\frac{d(\delta\mathbf{X})}{dt} = \mathbf{H}(\delta\mathbf{X})$$

for exactly one period on the Poincaré cross-section.

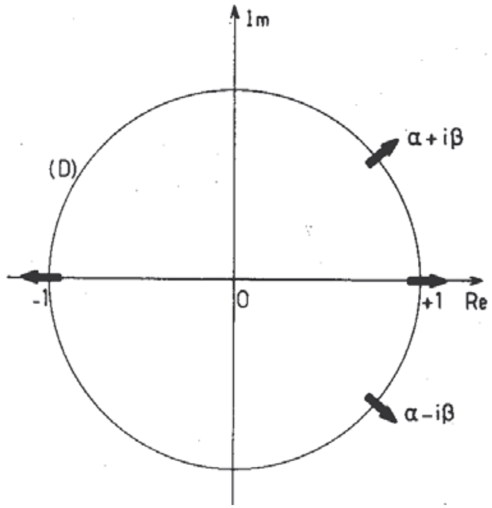
The stability of the periodic solution is determined by the eigenvalues λ_i of $\mathbf{M}$.
One of the eigenvalues will always be equal to one, corresponding to the direction of the flow.
If all other eigenvalues are located inside the unit circle complex plane, the periodic solution is *stable*.
i.e., the closed orbit is stable if $|\lambda_i| < 1$ all $i = 1, \ldots, M-1$.

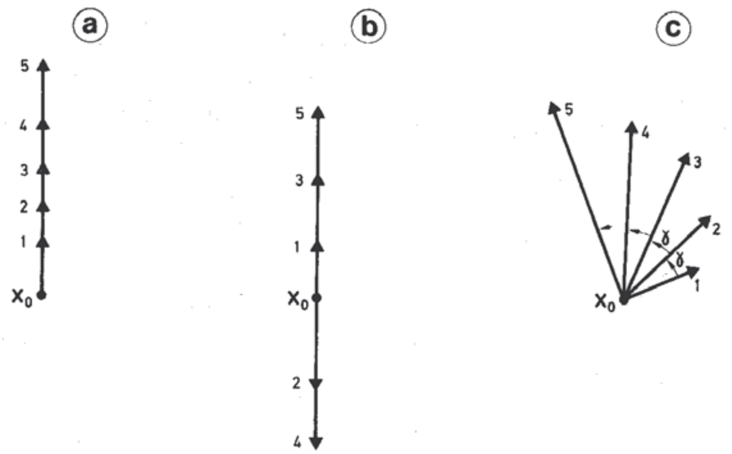
If at least one of the eigenvalues is outside the unit circle, the periodic solution is *unstable*.
i.e., the modulus of the eigenvalue is greater than one.

The λ_i are called the *Floquet multipliers*.

Instabilities of periodic solutions

There are three possibilities for an eigenvalue λ_i to cross the unit circle and cause instability.





(From Bergé, Pomeau and Vidal, 1984: *Order within chaos*, John Wiley & Sons)

(a) $\lambda_i > 1$: δX for each cycle is amplified in the same direction. This is *saddle-node* bifurcation.

(b) $\lambda_i < -1$: δX is amplified in the opposite direction alternately after each cycle. This is *subharmonic* or *period-doubling* bifurcation.

(c) $\lambda_i = \alpha + i\beta$ with $|\lambda_i| > 1$: δX rotates by an angle γ after each cycle, while their lengths increase. This is *Hopf* bifurcation.

Evolution of small errors in chaotic systems

Lorenz, E. N., 1965: A study of the predictability of a 28-variable atmospheric model. *Tellus*, 17, 321-333.

Linear tangent equation

As discussed earlier, the evolution of small perturbations in an M-dimensional dynamical system represented by

$$\frac{d\mathbf{X}}{dt} = \mathbf{F}(\mathbf{X})$$

is given by the *linear tangent equation*

$$\frac{d\mathbf{x}}{dt} = \mathbf{H}\mathbf{x}$$

where $\mathbf{X}$ and $\mathbf{X}$+$\mathbf{x}$ are basic and perturbed states of the system and

$$\mathbf{H} = \begin{pmatrix} \frac{\partial F_1}{\partial X_1} & \cdots & \frac{\partial F_1}{\partial X_M} \\ \vdots & & \vdots \\ \frac{\partial F_M}{\partial X_1} & \cdots & \frac{\partial F_M}{\partial X_M} \end{pmatrix}$$

When $\mathbf{X}$ is a state of a chaotic attractor, $\mathbf{H}$ is time-dependent.

Because $\mathbf{H}$ is linear, the tangent equation can be integrated from time t_1 to t_2 to obtain

$$\mathbf{x}(t_2) = \mathbf{A}(t_2,t_1)\ \mathbf{x}(t_1)$$

$$x_i(t_2) = \sum a_{ij}(t_2,t_1)\ x_j(t_1), \qquad i = 1, \ldots., M$$

$\mathbf{A}$ is an $M \times M$ square matrix which depends on the behavior of $\mathbf{X}$ between t_1 and t_2. The matrix $\mathbf{A}$ controls the growth of small errors during the interval t_1 to t_2, and is called the *error matrix*. It is also known as the *resolvent* or *propagator* of the tangent equation.

Note that

if $t_1 < t_2 < t_3$, $\quad \mathbf{A}(t_3,t_1) = \mathbf{A}(t_3,t_2)\ \mathbf{A}(t_2,t_1)$

$$\mathbf{A}(t_1,t_2) = \mathbf{A}(t_2,t_1)^{-1}$$

Growth of errors

An individual set of errors $\mathbf{x}$ can be treated as a point in the M-dimensional phase space. The amplitude of the error is defined as the distance of this point from the origin.

The squared-amplitude of the error at time t_1 is

$$\mathbf{x}(t_1)^T \mathbf{x}(t_1) = \sum_{i=1}^{M} x_i^2(t_1)$$

where the superscript T denotes the transpose of a matrix.

The squared-amplitude of the error at time t_2 is

$$\mathbf{x}(t_2)^T \mathbf{x}(t_2) = [\mathbf{A}(t_2,t_1)\mathbf{x}(t_1)]^T [\mathbf{A}(t_2,t_1)\mathbf{x}(t_1)]$$

$$\mathbf{x}(t_2)^T \mathbf{x}(t_2) = \mathbf{x}(t_1)^T \mathbf{A}(t_2,t_1)^T \mathbf{A}(t_2,t_1)\mathbf{x}(t_1)$$

The matrix $\mathbf{A}^T\mathbf{A}$ is symmetric positive-definite and possesses M real positive eigenvalues.

Growth of an initial sphere of errors

Consider an ensemble of random initial errors, each of amplitude ε at time t_1, occupying the surface of an M-dimensional sphere

$$\sum_{i=1}^{M} x_i^2(t_1) = \varepsilon^2$$

$$\mathbf{x}(t_1)^T \mathbf{x}(t_1) = \varepsilon^2$$

where the superscript T denotes the transpose of a matrix.

If each error in the ensemble evolves according to the propagator of the tangent equation, the sphere will be deformed into an ellipsoid:

$$\mathbf{x}(t_1) = \mathbf{A}(t_2,t_1)^{-1}\,\mathbf{x}(t_2)$$

$$[\mathbf{A}(t_2,t_1)^{-1}\,\mathbf{x}(t_2)]^T\,[\mathbf{A}(t_2,t_1)^{-1}\,\mathbf{x}(t_2)] = \varepsilon^2$$

$$\mathbf{x}(t_2)^T\,[\mathbf{A}(t_2,\mathrm{t}_1)^{-1}]^{\mathrm{T}}\,[\mathbf{A}(t_2,\mathrm{t}_1)^{-1}]\mathbf{x}(t_2) = \varepsilon^2$$

$$\mathbf{x}(t_2)^T[\mathbf{A}(t_2,\mathrm{t}_1)\mathbf{A}(t_2,\mathrm{t}_1)^T]^{-1}\mathbf{x}(t_2) = \varepsilon^2$$

Any matrix can be expressed in terms of two orthogonal matrices $\mathbf{U}$ and $\mathbf{V}$ by singular value decomposition (SVD):

$$\mathbf{A} = \mathbf{USV}^T$$

where $\mathbf{U}$ and $\mathbf{V}$ are $M \times M$ orthogonal matrices, $\mathbf{S}$ is a diagonal matrix containing the singular values of $\mathbf{A}$,

$$\mathbf{S} = \begin{pmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \sigma_M \end{pmatrix} \qquad \mathbf{S}^2 = \begin{pmatrix} \sigma_1^2 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \sigma_M^2 \end{pmatrix}$$

$$\mathbf{U}^T\mathbf{U} = \mathbf{I} \qquad \mathbf{U}\mathbf{U}^T = \mathbf{I} \qquad \mathbf{U}^T = \mathbf{U}^{-1}$$

$$\mathbf{V}^T\mathbf{V} = \mathbf{I} \qquad \mathbf{V}\mathbf{V}^T = \mathbf{I} \qquad \mathbf{V}^T = \mathbf{V}^{-1}$$

$$\mathbf{A}^T = \mathbf{VSU}^T$$

$$\mathbf{A}^T\mathbf{A} = \mathbf{VSU}^T\mathbf{USV}^T = \mathbf{VS}^2\mathbf{V}^T$$

$$\mathbf{A}^T\mathbf{AV} = \mathbf{VS}^2\mathbf{V}^T\mathbf{V} = \mathbf{VS}^2$$

$$\mathbf{AA}^T = \mathbf{USV}^T\mathbf{VSU}^T = \mathbf{US}^2\mathbf{U}^T$$

$$\mathbf{AA}^T\mathbf{U} = \mathbf{US}^2\mathbf{U}^T\mathbf{U} = \mathbf{US}^2$$

$$[\mathbf{AA}^T]^{-1} = (\mathbf{U}^T)^{-1}(\mathbf{S}^2)^{-1}\mathbf{U}^{-1} = \mathbf{U}(\mathbf{S}^2)^{-1}\mathbf{U}^T$$

The matrices $\mathbf{AA}^T$ and $\mathbf{A}^T\mathbf{A}$ are both symmetric positive-definite and possess the same M real positive eigenvalues. The matrix $\mathbf{S}^2$ is diagonal with diagonal elements consisting of these M real positive eigenvalues.

$$\mathbf{x}(t_2)^T[\mathbf{A}(t_2,\mathrm{t}_1)\mathbf{A}(t_2,\mathrm{t}_1)^T]^{-1}\mathbf{x}(t_2) = \varepsilon^2$$ becomes

$$\mathbf{x}(t_2)^T\,\mathbf{U}(\mathbf{S}^2)^{-1}\mathbf{U}^T\,\mathbf{x}(t_2) = \varepsilon^2$$

$$[\mathbf{U}^T\mathbf{x}(t_2)]^T\;(\mathbf{S}^2)^{-1}\;[\mathbf{U}^T\,\mathbf{x}(t_2)] = \varepsilon^2$$

Let $\mathbf{y}(t_2) = \mathbf{U}^T\,\mathbf{x}(t_2)$

$$\mathbf{y}(t_2)^T\;(\mathbf{S}^2)^{-1}\;\mathbf{y}(t_2) = \varepsilon^2$$

$$\frac{y_{21}^2}{\sigma_1^2} + \frac{y_{22}^2}{\sigma_2^2} + \cdots + \frac{y_{2M}^2}{\sigma_M^2} = \varepsilon^2$$

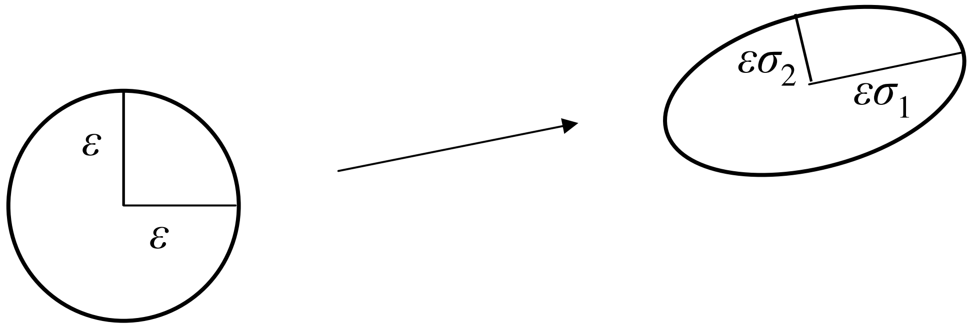
represents an ellipsoid. The sphere of initial errors has evolved into an ellipsoid.

$\sigma_1^2, \ldots, \sigma_M^2$ are the the eigenvalues of $\mathbf{AA}^T$

$\varepsilon\sigma_1, \ldots, \varepsilon\sigma_M$ are the lengths of the semiaxes of the ellipsoid.

$\sigma_1, \ldots, \sigma_M$ are the singular values of $\mathbf{A}$ and depend on t_1 and t_2.

Let $\sigma_1 > \sigma_2 > \ldots > \sigma_M$.



Whether or not any small errors grow between t_1 and t_2 depends on whether any semi-axis of the ellipsoid is greater than the radius ε of the sphere.
The error growth, therefore, depends on whether the singular value σ_1, or the eigenvalue σ_1^2 is greater than one.