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Summary College on Plasma Physics

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Introduction to Advanced Mathematical Methods in Nonlinear Plasma Theory

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I. Two Opposite Typical Dynamics

$$\frac{d}{dt} x = -a x$$

···· frictional damping.

$$x(t) = e^{-ta} x_0$$

Energy (Hamiltonian)

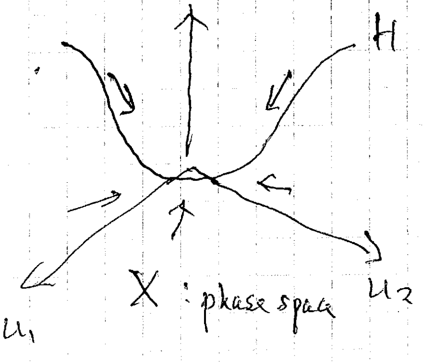
$$H = \frac{1}{2} a x^2 \Rightarrow \frac{1}{2}(x, ax) \quad x \in \mathbb{R}^n$$

$$\frac{d}{dt} x = -\partial_x H$$

For a convex $H(u)$

$$H(u+\delta) \geqq H(u) + (\delta, \partial_u H) \quad \forall \delta$$

Functional (observable)



$$\frac{d^2}{dt^2} x = -a x$$

···· oscillation

$$x(t) = \cos(\sqrt{a} t + \delta) x_0$$

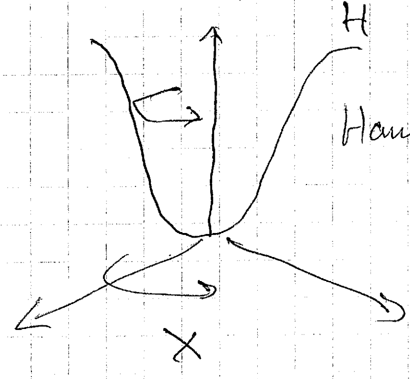
$$u = \begin{pmatrix} x \\ p \end{pmatrix}$$

$$\frac{d}{dt} u = \frac{d}{dt}\begin{pmatrix} x \\ p \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} ax \\ p \end{pmatrix}$$

$$H = \frac{1}{2} p^2 + \frac{1}{2} a x^2$$

$$\partial_u H = \begin{pmatrix} \partial_x H \\ \partial_p H \end{pmatrix} = \begin{pmatrix} ax \\ p \end{pmatrix}$$

$$\frac{d}{dt} u = \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{A} \partial_u H$$



Hamiltonian mechanics

$\boxed{A : \text{"Structure"}}$

" → non-trivial phenomena

II. Phase space $\mathbb{R}^n \to$ Hilbert space / ODE $\to$ PDE.

$$\partial_t u = \Delta u$$
$$(u|_{\partial\Omega} = 0)$$

$\cdots$ diffusion

$$H = \frac{1}{2}\int_\Omega |\nabla u|^2 dx = \frac{1}{2}\|\nabla u\|^2$$

$$\partial_u H = -\Delta u$$

$$\partial_t u = -\partial_u H(u)$$

$$\partial_t^2 u = \Delta u$$

$\cdots$ wave

$$U = \begin{pmatrix} u \\ p \end{pmatrix}$$

$$H = \frac{1}{2}\|p\|^2 + \frac{1}{2}\|\nabla u\|^2$$

$$\partial_U H = \begin{pmatrix} -\Delta u \\ p \end{pmatrix}$$

$$\partial_t U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \partial_U H$$
$$= \boxed{A}\, \partial_U H$$

A determines the direction to go

A : antisymmetric $(Au, u) \equiv 0$

$\to$ Energy conservation law.

III. Structured Phase Space

Dynamics (Kinematics) $\leftrightarrow$ "Structure" of X

(geometric / algebraic)

Kinematics : $u(t) = T(t)\, u_0$

group $\begin{cases} T(t)\cdot T(s) = T(t+s) \quad (= T(s+t)) \\ T(0) = I \quad (\text{id.}) \\ T(t)^{*} = T(t)^{-1} \quad (\text{inverse}) \end{cases}$

$\uparrow$ Abelian

$\frac{\partial}{\partial t} T(t) = \mathfrak{g}$: generator

structured phase space : $X = G = \{T(t) x\}$

Abelian group (diffeo.)

observables : $f(u) \in \mathrm{Fun}(X) \Rightarrow \mathrm{Fun}(G)$

Δ : $\Delta f(T(t), T(s)) = f(T(t)T(s))$: Func. Ring

generator ; $\mathfrak{g} f(0) = \frac{\partial}{\partial t} f(T(t))\Big|_{t=0}$: Lie Ring

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$\frac{\partial}{\partial t} u = A\, \partial_u H \qquad (A^{*} = -A)$

$\frac{\partial}{\partial t} f(u) = (\partial_u f, A \partial_u H) \equiv \{f, H\}$

: Poisson algebra

generator: $\mathfrak{g} = -\{H, \ \}$

IV. Nonlinear (singular) structures

General Hamiltonian mechanics

$$\frac{\partial}{\partial t} u = A_{(u)} \partial_u H_{(u)} \qquad (A_{(u)}^* = -A_{(u)} \ (\forall u))$$

$$\left(\begin{array}{ll} \text{ex. Canonical eq.} & A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ \text{Schrödinger eq} & A = -i \end{array} \right)$$

ex. Incompressible ideal flow

$$(E) \begin{cases} \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p \\ \nabla \cdot \mathbf{u} = 0 \\ (\mathbf{n} \cdot \mathbf{u} = 0) \end{cases}$$

$$(E') \begin{cases} \partial_t \mathbf{u} = -\Omega \times \mathbf{u} - \nabla \theta & (\Omega = \nabla \times \mathbf{u}, \ \theta = p + \frac{u^2}{2}) \\ \nabla \cdot \mathbf{u} = 0 \end{cases}$$

$$X = V(\Omega) = \{ \mathbf{u} \ ; \ \nabla \cdot \mathbf{u} = 0, \ \mathbf{n} \cdot \mathbf{u} = 0 \}$$

$$L^2(\Omega) = V(\Omega) \oplus \{ \nabla \theta \}$$

$\mathbb{P} : L^2(\Omega) \to V(\Omega)$: orthogonal projection

$\mathbb{P}(E') \Rightarrow$

$$(E'') \{ \ \partial_t \mathbf{u} = -\mathbb{P} \Omega \times \mathbf{u}$$

$H = \frac{1}{2} \| \mathbf{u} \|^2$ (norm of X)

$\partial_u H = \mathbf{u}$,

$A = -\mathbb{P} \Omega \times$: nonlinear

Show that
$(A\mathbf{u}, \mathbf{u}') = -(\mathbf{u}, A\mathbf{u}')$

$$(E''') \{ \ \partial_t \mathbf{u} = A \partial_{\mathbf{u}} H(\mathbf{u})$$

$A(u)\circ = -P(\nabla\times u)\times\circ$ has a "Kernel" (Central element) Casimir

(3D) $C_3 = \frac{1}{2}\int u\cdot\Omega\, dx$ (helicity)

$$\partial_u C_3 = \Omega \; (\equiv \nabla\times u)$$

$$A\partial_u C_3 = -P\Omega\times\Omega \equiv 0$$

or $\{f, C_3\} \equiv 0 \; (\forall f)$

(2D) $\Omega = \omega e_z \rightarrow u\cdot\Omega \equiv 0 \rightarrow C_3 = 0$

instead;

$$C_2 = \int \omega^n dx \quad \left(= \int f(\omega)\, dx\right)$$

$$\partial_u C_2 = n\nabla\omega^{n-1}\times e_z$$

$$A\partial_u C_2 = -P\,\omega e_z\times(n\nabla\omega^{n-1}\times e_z)$$

$$= -P(n\,\omega\nabla\omega^{n-1})$$

$$= -P((n-1)\nabla\omega^n) \equiv 0$$

enstrophy $(n=2)$

MHD

$$(\mathrm{MHD})\begin{cases}\dfrac{\partial}{\partial t}\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}=(\nabla\times\mathbf{B})\times\mathbf{B}-\nabla p\ , & \nabla\cdot\mathbf{u}=0\\[2ex] \dfrac{\partial}{\partial t}\mathbf{B}+(\mathbf{u}\times\mathbf{B})=0\end{cases}$$

$$(\mathrm{MHD}')\qquad \frac{d}{dt}\begin{pmatrix}\mathbf{u}\\ \mathbf{B}\end{pmatrix}=A\,\partial_{\binom{\mathbf{u}}{\mathbf{B}}}H$$

$$H=\frac{1}{2}\left(\|\mathbf{u}\|^2+\|\mathbf{B}\|^2\right).$$

$$A=\begin{pmatrix}-P\,\Omega\times\circ & P[(\nabla\times\circ)\times\mathbf{B}\\ \nabla\times(\circ\times\mathbf{B}) & 0\end{pmatrix}$$

Casimirs

$$\begin{cases}C_M=(A,\mathbf{B})\\ C_K=(v,\mathbf{B})\end{cases}$$

V. Casimir and Beltrami equilibria

if $\{f, C\} \equiv 0$ (or $\partial_u C \in \mathrm{Ker}(A)$)

$H \to \tilde{H} = H + \mu C$ does not change dynamics

Beltrami equilibrium : $\partial_u \tilde{H} = \partial_u (H + \mu C) = 0$

3D) $\tilde{H} = \frac{1}{2}\|u\|^2 + \frac{\mu}{2}(u, \nabla \times u)$

$\partial_u \tilde{H} = 0 \Rightarrow \underline{\nabla \times u = -\mu^{-1} u}$ Beltrami condition

2D) $\tilde{H} = \frac{1}{2}\|u\|^2 + \int f(\omega)\, dx$

$= \frac{1}{2}\|\nabla\phi\|^2 + \int f(-\Delta\phi)\, dx$

$(u = \nabla\phi \times e_z)$

* but ill-posed

Vortex eq. $\nabla \times (E) \to$

$\partial_t \omega + v \cdot \nabla \omega = 0$ $\left(\begin{array}{l} v = \nabla\phi \times e_z \\ \omega = \nabla \times v = -\Delta\phi \end{array}\right)$

$W = \frac{1}{2}\|\omega\|^2$ $\left(\to \int f(\omega)\, dx \quad (f'' > 0)\right)$

$A = (v \cdot \nabla) = \{\phi, \circ\}$ $\left(\to \{\phi, (f')^{-1} \circ\}\right)$

$\partial_t \omega = A \partial_\omega W$

$C_2' = \frac{1}{2}\|\nabla\phi\|^2 = \frac{1}{2}\left(\overset{\phi}{(-\Delta)^{-1}\omega}, \omega\right)$, $\partial_\omega C_2' = \phi$

$\tilde{W} = \|\omega\|^2 + \mu C_2'$

Well-posed $\Rightarrow$ $\partial_\omega \tilde{W} = \omega + \mu\phi = 0$ $\boxed{-\Delta\phi = \mu\phi}$

MHD

$$\tilde{H} = H + \mu C_M + \lambda C_K$$

$$= \frac{1}{2}\|\mathbb{V}\|^2 + \frac{1}{2}\|\mathbb{B}\|^2 + \mu(\mathbb{A}\cdot\mathbb{B}) + \lambda(\mathbb{V}+\mathbb{B})$$

$$\partial_u \tilde{H} = \begin{pmatrix} \mathbb{V} + \lambda\mathbb{B} \\ \mathbb{B} + \mu\mathbb{A} + \lambda\mathbb{V} \end{pmatrix} = 0$$

$$\Rightarrow \begin{cases} \nabla\times\mathbb{B} + \mu\mathbb{B} + \lambda\nabla\times\mathbb{V} = 0 \\ \mathbb{V} = -\lambda\mathbb{B} \end{cases}$$

Beltrami equilibria.

VI. Hoph Algebra (Quntum Groups)

Piosson-Lie Group.

$X \to G = \{T(t)X\}$ "diachronic" structure $\boxed{\Delta: X \to X \otimes X}$

Collective / interacting (scattering) dynamics

→ Quntum group (non-comutative / non-Co-Com.) Hoph Algebra)

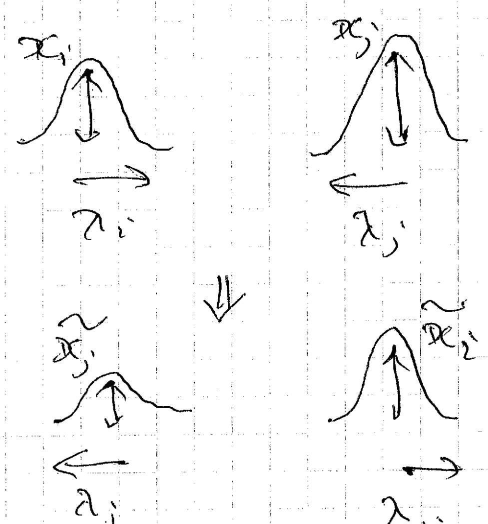
Binary interaction

$$R: X \otimes X \to X \otimes X$$

$$R: (x, y) \mapsto (\tilde{x}, \tilde{y}) = (B(y), A(x))$$

parameterize

$$R_{ij}(\lambda_i, \lambda_j): (x_i, x_j) \to (\tilde{x}_i, \tilde{x}_j)$$



(N-Soliton system)

- intertwiner

$$b_i = R_{i,i+1} : X_i \otimes X_{i+1} \to X_{i+1} \otimes X_i .$$

generally

$$b_{i+1} b_i \neq b_i b_{i+1}$$

but

$$b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1} \quad \text{: Yang-Baxter eq.}$$

two orbits of intertwiner must be identical

$$X_i \otimes X_{i+1} \otimes X_{i+2} \to X_{i+2} \otimes X_{i+1} \otimes X_i$$