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I. INTRODUCTION

For a better understanding of linear and weakly nonlinear behavior of fluctuations in plasmas, the Hamiltonian formulation in the dissipationless limit is quite informative and to specify the action-angle variables for linear waves is a fundamental issue. However, in contrast to finite-dimensional Hamiltonian systems and quantum mechanics, linearized equations for plasmas are generally infinite-dimensional and non-Hermitian. In particular, the treatment of the continuous spectrum is still nontrivial and highly involved. The continuous spectrum is known to occur in either fluid or kinetic description of plasmas due to inhomogeneities (gradient and shear) of background fields [1–5] and often play a central role in the linear stability analysis. Recently, we have proposed a general technique for the action-angle representation by making use of the spectral theory [6, 7], which is applicable even in the presence of the continuous spectrum. This lecture note provides a short review of some requisite Hamiltonian theory and introduces our method from a renewed point of view.

The simplest example of the action-angle variables would be a single Harmonic oscillator, whose Hamiltonian is given by

$$H = \omega \frac{q^2 + p^2}{2} = \omega\mu \tag{1}$$

in terms of the canonical coordinate q and momentum p, where ω corresponds to the frequency of the oscillation. In the rightmost expression, an action variable $\mu = (q^2 + p^2)/2$ has already appeared. By introducing an angle variable as $\theta(t) = \omega t + \theta_0$ with an appropriate initial value θ_0, the solution may be written as $q(t) = \sqrt{2\mu}\mathrm{Re}[e^{-i\theta(t)}]$ and $p(t) = \sqrt{2\mu}\mathrm{Im}[e^{-i\theta(t)}]$. Thus, the angle variable indicates the phase angle of oscillation, whereas the action variable measures the amplitude in such a way that the action variable multiplied by the frequency amounts to the modal energy.

For the case of linear waves in plasmas, a care is needed to evaluate the action variable, since the canonical variables (q, p) are not obvious from the fluid and kinetic equations. These systems are known to be noncanonical and a certain kinematical (or topological) constraint on dynamics, so-called the dynamical accessibility, must be taken into account [8]. For example, the continuity equation imposes the mass conservation law on the hydrodynamic motion. If a linear perturbation violates the mass conservation law, such a perturbed state is not dynamically accessible from the reference state. Then, we cannot expect the perturbation to behave like the canonical variables (q, p). On the other hand, if an initial perturbation is assumed to satisfy all the kinematical conservation laws, the perturbed state would be dynamically accessible and the subsequent wave would have the same energy expression as (1). Note that, however, the wave generally involves multiple eigenmodes as well as continuum modes. For simplicity, let us consider the situation that all modes are neutrally stable. In this note, we will see that the wave energy is formally represented by

$$\tilde{H} = \int_{\mathbb{R}} \omega\mu(\omega)d\omega, \tag{2}$$

where $\mu(\omega)$ represents the frequency spectrum of the wave action. For the case of discrete spectra at $\{\omega_n \in \mathbb{R}, n = 1, 2, \dots\}$, we simply observe the corresponding delta functions $\mu(\omega) = \mu_n\delta(\omega - \omega_n)$ and obtain $\tilde{H} = \sum_n \omega_n\mu_n$, where μ_n, $n = 1, 2, \dots$, correspond to action variables for eigenmodes. Therefore, nonzero distribution of $\mu(\omega)$ on a continuous spectrum is understood as uncountable set of action variables for a continuum mode. The derivation of $\mu(\omega)$ has been performed in the cases of electrostatic oscillation [9] and parallel shear flow [10]. Our method can reproduce them and apply to more complicated problems [6, 7].

In general, the wave energy can take negative value, in which case the corresponding wave tends to release the energy stored in the background fields so that an instability occurs [11]. The sign of the modal energy for each eigenmode is essential for understanding the loss of stability, namely, the bifurcation of equilibrium state. According to Krein's theorem [12], linear instability (local bifurcation) is possible when pairs of eigenvalues of positive and negative energy modes collide. Although we have obtained some evidence suggesting that the same idea is true for the case of continuum modes [6], the bifurcation theory in the presence of continuum modes is under development and our method for evaluating $\mu(\omega)$ would hopefully form the foundation of it.

The wave action is traditionally well-studied in the eikonal approach [13–15], where the wave action *density*, say $\hat{\mu}(\boldsymbol{x}, t)$, satisfies a conservation law, $\partial_t\hat{\mu} + \nabla\cdot(\boldsymbol{v}_g\hat{\mu}) = 0$ ($\boldsymbol{v}_g$: group velocity). Our spectral approach agrees with this theory through the relation $\int \mu(\omega)d\omega = \int \hat{\mu}(\boldsymbol{x}, t)d^3x$ and can deal with general waves for which the eikonal approximation is not always suitable.

In Sec. II, we will review the noncanonical Hamiltonian theory that is common to various models of plasmas and especially highlight the linear theory. For example, incompressible fluid, magnetohydrodynamics and the Vlasov-Poissson system will be introduced. Spectral decomposition of non-Hermitian operators in the linearized equations is highly associated with the action-angle representation of waves. In Sec. III, we will show that, the Fourier-Laplace transform is, in practice, useful for calculating the action-angle variables in the presence of continuous spectrum [6]. This formulation will be demonstrated in Sec. IV by using the simplest and classical model, called the Case-Van Kampen equation [1, 2].

II. HAMILTONIAN EQUATION AND ITS LINEARIZATION

Various nonlinear equations governing ideal plasmas are regarded as Hamiltonian systems. The common Hamiltonian structure is however known to be noncanonical and the linearized system also inherits the noncanonical property.

We denote an abstract dynamical variable by u, which may be a set of time-dependent scalar and vector fields. Let the space of u be a Hilbert space L^2 with an inner product $\langle , \rangle$. A Hamiltonian equation can be written as

$$\partial_t u = \mathcal{J}_u \frac{\delta H}{\delta u}, \tag{3}$$

where $\mathcal{J}_u$ is a linear operator depending on u and $\delta H/\delta u$ denotes the functional derivative of the Hamiltonian function $H : L^2 \to \mathbb{R}$. The Poisson bracket of $F, G : L^2 \to \mathbb{R}$ is defined by

$$\{F, G\} = \left\langle \frac{\delta F}{\delta u}, \mathcal{J}_u \frac{\delta G}{\delta u} \right\rangle. \tag{4}$$

Therefore, in order for (3) to be indeed Hamiltonian, the operator $\mathcal{J}_u$ is required to be anti-symmetric and have a special property associated with the Jacobi identity [16].

The system (3) is essentially different from the standard canonical one in the following two points. First, the Poisson operator $\mathcal{J}_u$ itself depends on u, which brings about another nonlinearity in the Hamiltonian equation. Second, $\mathcal{J}_u$ is usually singular ($\mathcal{J}_u^{-1}$ does not exist), implying that the system is subject to some kinematical constraints (or conservation laws). These noncanonical properties are pointed out by Arnold (see Ref. 17, 18) and then found in various dynamical systems [19]. We will exhibit a few examples below.

Now, let us discuss linear perturbations $\tilde{u}(t)$ on a given *steady* solution u; $\mathcal{J}_u \delta H/\delta u = 0$. The linearized Hamiltonian equation for $\tilde{u}(t)$ will be denoted by

$$\frac{\partial \tilde{u}}{\partial t} = \mathcal{K}\tilde{u}, \tag{5}$$

where the generator is defined by

$$\mathcal{K}\tilde{u} := \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \mathcal{J}_{u+\epsilon\tilde{u}} \frac{\delta H}{\delta u}(u + \epsilon\tilde{u}). \tag{6}$$

It is very important to note that, by introducing the adjoint operator $\mathcal{K}^*$ of $\mathcal{K}$, a relation

$$\mathcal{K}\mathcal{J}_u = -\mathcal{J}_u \mathcal{K}^* \tag{7}$$

must hold due to the Jacobi identity [16]. Let $\xi(t)$ be a solution of the adjoint equation given by

$$\frac{\partial \xi}{\partial t} = -\mathcal{K}^*\xi. \tag{8}$$

Then, the above relation (7) implies that $\tilde{u}(t) = \mathcal{J}_u\xi(t)$ for all t if $\tilde{u}(0) = \mathcal{J}_u\xi(0)$ initially. Since the linear map $\mathcal{J}_u$ is singular, such $\tilde{u}(t)$ is constrained into the range of $\mathcal{J}_u$. The class of linear perturbations belonging to the range of $\mathcal{J}_u$ is said to be dynamically accessible [8], for we can generate them by $\tilde{u} = \mathcal{J}_u \delta F/\delta u$ with some arbitrary Hamiltonian F. In physical terms, these perturbations will not violate any conservation laws and will preserve the topology of the basic solution u. In this paper, we restrict our attention to dynamically accessible (or iso-topological) perturbations by employing the assumption $\tilde{u}(0) = \mathcal{J}_u\xi(0)$. The resultant linear dynamics $\tilde{u}(t) = \mathcal{J}_u\xi(t)$ will conform to the canonical Hamiltonian formalism and admit the action-angle representation.

Example 1: Incompressible fluids

The Euler equation is known as a Hamiltonian system with a noncanonical Poisson bracket [17, 18]. Let $V \subset \mathbb{R}^3$ be a domain filled with a fluid. The velocity field $\boldsymbol{u}$ belongs to the space $\mathfrak{X}(V)$ of divergence-free vector fields in V which are tangent to the boundary wall ∂V. This space $\mathfrak{X}(V)$ is a Hilbert space with an inner product $\langle \boldsymbol{v}, \boldsymbol{u} \rangle = \int_V \boldsymbol{v} \cdot \boldsymbol{u} d^3x$ for $\boldsymbol{v}, \boldsymbol{u} \in \mathfrak{X}(V)$, The Hamiltonian function H and the Poisson operator $\mathcal{J}_u$ are given by

$$H(\boldsymbol{u}) = \frac{1}{2}\int_V |\boldsymbol{u}|^2 d^3x \tag{9}$$

$$\mathcal{J}_{\boldsymbol{u}} = \mathcal{P}[\circ \times (\nabla \times \boldsymbol{u})] \tag{10}$$

where $\mathcal{P}$ denotes the orthogonal projection to the space $\mathfrak{X}(V)$. The Hamiltonian equation (3) then coincides with the Euler equation.

$$\frac{\partial \boldsymbol{u}}{\partial t} = \mathcal{P}[\boldsymbol{u} \times (\nabla \times \boldsymbol{u})], \tag{11}$$

$$= -\mathcal{P}[(\boldsymbol{u} \cdot \nabla)\boldsymbol{u}]. \tag{12}$$

The linearized equation is straightforward,

$$\frac{\partial \tilde{\boldsymbol{u}}}{\partial t} = \mathcal{P}[\boldsymbol{u} \times (\nabla \times \tilde{\boldsymbol{u}}) + \tilde{\boldsymbol{u}} \times (\nabla \times \boldsymbol{u})], \tag{13}$$

and the adjoint equation for $\boldsymbol{\xi} \in \mathfrak{X}(V)$ is given by

$$\frac{\partial \boldsymbol{\xi}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{\xi}) + \mathcal{P}[\boldsymbol{\xi} \times (\nabla \times \boldsymbol{u})]. \tag{14}$$

If $\boldsymbol{\xi}(t)$ solves (14), the dynamically accessible perturbation $\boldsymbol{u}(t) = \mathcal{J}_{\boldsymbol{u}}\boldsymbol{\xi}(t)$, namely,

$$\tilde{\boldsymbol{u}} = \mathcal{P}[\boldsymbol{\xi} \times (\nabla \times \boldsymbol{u})], \tag{15}$$

solves the linearized Euler equation.

It is interesting to note that the equation (14) corresponds to the definition of the Lagrangian displacement field [20, 21],

$$\frac{\partial \boldsymbol{\xi}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{\xi} - (\boldsymbol{\xi} \cdot \nabla)\boldsymbol{u} = \tilde{\boldsymbol{u}}. \tag{16}$$

Namely, the vector field $\boldsymbol{\xi}$ physically represents the infinitesimal displacement of the fluid particle orbits.

The constraint (15) implies that the vorticity field $\boldsymbol{w} = \nabla \times \boldsymbol{u}$ must be *frozen* to the displacement $\boldsymbol{\xi}$ and preserve its topology (the Kelvin's circulation theorem). For instance, the vorticity perturbation $\tilde{\boldsymbol{w}} = \nabla \times \tilde{\boldsymbol{u}}$ cannot occur in the absence of the background vorticity $\boldsymbol{w} = 0$. This topological aspect of ideal fluid was observed by Arnold, who call (15) isovortical perturbation [17, 18].

Example 2: Magnetohydrodynamics

The above Hamiltonian structure for the Euler equation can be extended to compressible fluids and, furthermore, to magnetohydrodynamics (MHD) [22]. The number of dynamical variables are increased and written by $u = (\rho\boldsymbol{u}, \boldsymbol{B}, \rho, s)$ for the MHD case, where $\boldsymbol{B}$ is the magnetic field, ρ is the mass density and s is the specific entropy.

The Hamiltonian function becomes

$$H(u) = \int_V \left(\frac{\rho}{2}|\boldsymbol{u}|^2 + \frac{1}{2}|\boldsymbol{B}|^2 + \rho U(\rho, s)\right) d^3x,$$

where a given function $U(\rho, s)$ of ρ and s denotes the internal energy per unit mass.

The Poisson bracket is determined by the Poisson operator,

$$\mathcal{J}_u = \begin{pmatrix} \rho \circ \times (\nabla \times \boldsymbol{u}) - \rho \nabla (\boldsymbol{u} \cdot \circ) - \boldsymbol{u} \nabla \cdot (\rho \circ) & -\boldsymbol{B} \times (\nabla \times \circ) & -\rho \nabla \circ & \circ \nabla s \\ \nabla \times (\circ \times \boldsymbol{B}) & 0 & 0 & 0 \\ -\nabla \cdot (\rho \circ) & 0 & 0 & 0 \\ - \circ \cdot \nabla s & 0 & 0 & 0 \end{pmatrix}, \tag{17}$$

which indeed recovers the ideal MHD equations [22].

By denoting the adjoint variables by $\xi = (\boldsymbol{\xi}, \boldsymbol{\eta}, \alpha, \beta)$, the dynamically accessible perturbations are

$$\tilde{\boldsymbol{u}} = \rho \boldsymbol{\xi} \times (\nabla \times \boldsymbol{u}) - \rho \nabla (\boldsymbol{u} \cdot \boldsymbol{\xi}) - \boldsymbol{B} \times (\nabla \times \boldsymbol{\eta}) - \rho \nabla \alpha + \beta \nabla s, \tag{18}$$

$$\tilde{\boldsymbol{B}} = \nabla \times (\boldsymbol{\xi} \times \boldsymbol{B}), \tag{19}$$

$$\tilde{\rho} = - \nabla \cdot (\rho \boldsymbol{\xi}), \tag{20}$$

$$\tilde{s} = - \boldsymbol{\xi} \cdot \nabla s. \tag{21}$$

It is known that these constrained perturbations satisfy the conservation laws of magnetic flux, mass, entropy and so on. The vector field $\boldsymbol{\xi}$ still corresponds to the Lagrangian displacement field. The similarity between this dynamically accessible perturbation and the famous Frieman-Rotenberg theory [21] is discussed in Ref. 6.

Example 3: Vlasov-Poisson system

Consider a collisionless plasma consisting of electrons and ions with charges $q_{e,i}$ and masses $m_{e,i}$, whereas the ions are assumed to be immobile and form a uniform background with a charge density $q_i n_i = \text{const}$. Let $\boldsymbol{x}, \boldsymbol{v} \in \mathbb{R}^3$ denote the position and velocity of electrons and $f(\boldsymbol{x}, \boldsymbol{v}, t)$ be the distribution function. The Vlasov-Poisson equations for electrons are

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f}{\partial \boldsymbol{x}} + \frac{q_e}{m_e} \boldsymbol{E} \cdot \frac{\partial f}{\partial \boldsymbol{v}} = 0, \tag{22}$$

$$\mathrm{div} \boldsymbol{E} = \frac{1}{\epsilon_0} \left(q_e \int f d^3 v + q_i n_i \right). \tag{23}$$

This is also written as a noncanonical Hamiltonian system [23, 24],

$$\frac{\partial f}{\partial t} = \left[\frac{\delta H}{\delta f}, f \right], \tag{24}$$

where the Hamiltonian function is given by

$$H(f) = \iint m_e \frac{|\boldsymbol{v}|^2}{2} f d^3 x d^3 v + \epsilon_0 \int \frac{|\boldsymbol{E}|^2}{2} d^3 x, \tag{25}$$

and the Poisson operator ($\mathcal{J}_u$) now corresponds to

$$\mathcal{J}_f = [\circ, f] := \frac{1}{m_e} \left(\frac{\partial \circ}{\partial \boldsymbol{x}} \cdot \frac{\partial f}{\partial \boldsymbol{v}} - \frac{\partial \circ}{\partial \boldsymbol{v}} \cdot \frac{\partial f}{\partial \boldsymbol{x}} \right). \tag{26}$$

The linearization about an equilibrium state f gives

$$\frac{\partial \tilde{f}}{\partial t} = \left[\frac{\delta H}{\delta f}, \tilde{f}\right] + \left[\frac{\delta^2 H}{\delta f^2}\tilde{f}, f\right]. \tag{27}$$

The dynamically accessible perturbation is $\tilde{f} = [\zeta, f]$, where the adjoint variable $\zeta(\boldsymbol{x}, \boldsymbol{v}, t)$ is governed by the adjoint equation [9]

$$\frac{\partial \zeta}{\partial t} = \left[\frac{\delta H}{\delta f}, \zeta\right] + \frac{\delta^2 H}{\delta f^2}[\zeta, f]. \tag{28}$$

This linear problem will be considered later in Sec. IV.

III. SPECTRAL APPROACH TO ACTION-ANGLE VARIABLES FOR LINEAR WAVES

A. Action integral for linear waves

For dynamically accessible perturbations $\tilde{u}(t) = \mathcal{J}_u(t)\xi(t)$, we can naturally define the perturbation energy $\tilde{H}$ in terms of $\xi(t)$ as

$$\tilde{H} := \frac{1}{2}\left\langle \frac{\partial \xi}{\partial t}, \tilde{u}\right\rangle = -\frac{1}{2}\left\langle \xi, \mathcal{K}\mathcal{J}_u\xi\right\rangle. \tag{29}$$

It is easy to confirm that this is a constant of motion for the linearized system (5) [or (8)]. From the property (7), $\tilde{H}$ is a symmetric quadratic functional of ξ. If an eigenmode is substituted into $\tilde{H}$, the time derivative $\partial/\partial t$ turns into an eigenvalue $-i\omega$ and we immediately obtain the modal energy in the form of (1). However, this naive procedure does not apply to continuum mode that is composed of infinite number of singular (or improper) eigenmodes. If a singular eigenmode was substituted, the energy $\tilde{H}$ would diverge.

In order to handle multiple eigenmodes and continuum modes as well, we make the following general observations. Suppose that we have a closed family of solutions $\xi(t, \theta_0)$ labeled by a parameter $0 \le \theta_0 < 2\pi$ that satisfies $\xi(t, \theta_0) = \xi(t, \theta_0 + 2\pi)$. Since the evolution of ξ is deterministic, the dependence on θ_0 must originate from a closed family of initial values. Then, we claim that the Poincaré's invariant (or the action integral associated with the family of solutions) is given by the ensemble average,

$$S := \frac{1}{4\pi}\int_0^{2\pi}\left\langle \frac{\partial \xi}{\partial \theta_0}, \tilde{u}\right\rangle d\theta_0 = -\frac{1}{4\pi}\int_0^{2\pi}\left\langle \xi, \frac{\partial \tilde{u}}{\partial \theta_0}\right\rangle d\theta_0, \tag{30}$$

which is actually invariant $\partial S/\partial t = 0$.

Suppose that the family of solutions is attributed to only a single oscillatory eigenmode such as $\xi(t, \theta_0) = 2\mathrm{Re}[\hat{\xi}e^{-i\omega t - i\theta_0}]$, where $\hat{\xi}$ is a complex eigenfunction for an eigenvalue $\omega \in \mathbb{R}$. Then, the Poincaré's invariant is reduced to

$$S = \frac{1}{4\pi}\int_0^{2\pi}\left\langle \hat{\xi}e^{-i\omega t - i\theta_0} + \bar{\hat{\xi}}e^{i\omega t + i\theta_0}, i\mathcal{J}_u\left(\hat{\xi}e^{-i\omega t - i\theta_0} - \bar{\hat{\xi}}e^{i\omega t + i\theta_0}\right)\right\rangle d\theta_0 \tag{31}$$

$$= \left\langle \bar{\hat{\xi}}, i\mathcal{J}_u\hat{\xi}\right\rangle, \tag{32}$$

where $\overline{\hat{\xi}}$ denotes the complex conjugate of $\hat{\xi}$. Since $\partial\xi/\partial t = \omega\partial\xi/\partial\theta_0$ holds for such eigenmode, this is related to the perturbation energy by

$$\omega S = \frac{1}{4\pi}\int_0^{2\pi}\left\langle \frac{\partial\xi}{\partial t}, \tilde{u}\right\rangle dt = \tilde{H}. \tag{33}$$

Therefore, S is understood as the action variable for the eigenmode. The angle variable is obviously $\theta(t) = \omega t + \theta_0$.

B. Fourier-Laplace analysis

Now, we consider general perturbations. Since the eigenmodes and continuum modes are respectively associated with the discrete and continuous spectra of the linear operator $\mathcal{K}$ (or $\mathcal{K}^*$), we will formally perform the spectral decomposition by invoking the Fourier-Laplace analysis.

While the perturbation $\tilde{u}$ (or ξ) must be always real, it is conventional in the spectral analysis to view it as a complex variable belonging to the complex Hilbert space with the inner product $\langle \overline{\circ}, \circ\rangle$. By simply multiplying the imaginary unit i, the evolution equations, (5) and (8), can look like non-Hermitian Schrödinger equations,

$$i\frac{\partial\tilde{u}}{\partial t} = \mathcal{L}\tilde{u}, \quad \tilde{u}(0) = \tilde{u}_0 = \mathcal{J}_u\xi_0, \tag{34}$$

$$i\frac{\partial\xi}{\partial t} = \mathcal{L}^*\xi, \quad \xi(0) = \xi_0 \tag{35}$$

where $\mathcal{L}^* = -i\mathcal{K}$ is the adjoint operator of $\mathcal{L} = i\mathcal{K}$ with respect to the inner product $\langle \overline{\circ}, \circ\rangle$.

The property (7) becomes $\mathcal{L}\mathcal{J}_u = \mathcal{J}_u\mathcal{L}^*$. The evolution equation is said to possess a pseudo-Hermitian structure [25, 26], since $\mathcal{L}$ is Hermitian with respect to an indefinite inner product given by $\langle \overline{\circ}, \mathcal{J}_j\circ\rangle$.

The solution of (34) [or (35)] is formally represented by the Dunford-Taylor integral [27],

$$\tilde{u}(t) = \frac{1}{2\pi i}\oint_{\Gamma(\sigma)} (\Omega - \mathcal{L})^{-1}\mathcal{J}_u\xi_0 e^{-i\Omega t} d\Omega = \mathcal{J}_u\xi(t), \tag{36}$$

$$\xi(t) = \frac{1}{2\pi i}\oint_{\Gamma(\sigma)} (\Omega - \mathcal{L}^*)^{-1}\xi_0 e^{-i\Omega t} d\Omega, \tag{37}$$

where the integral path $\Gamma(\sigma)$ in the complex plane $\Omega \in \mathbb{C}$ positively encircles the whole spectrum $\sigma \subset \mathbb{C}$ defined by

$$\sigma := \{\omega \in \mathbb{C} | (\Omega - \mathcal{L})^{-1}\mathcal{J}_u\xi_0 \text{ is not regular at } \Omega = \omega\}. \tag{38}$$

We can prove that the set σ is symmetric with respect to both real and imaginary axes; $\sigma = \overline{\sigma} = -\sigma = -\overline{\sigma}$, which is a common property of Hamiltonian systems [6]. Suppose that σ does not overlap the imaginary axis for simplicity and decompose it into $\sigma = \sigma_+ \cup \sigma_-$ such that σ_+ is inside the right half plane $\mathrm{Re}(\Omega) > 0$. Since $\xi(t)$ is in fact real-valued, (37) is rewritten as

$$\xi(t) = 2\mathrm{Re}\left[\frac{1}{2\pi i}\oint_{\Gamma(\sigma_+)} (\Omega - \mathcal{L}^*)^{-1}\xi_0 e^{-i\Omega t} d\Omega\right]. \tag{39}$$

Now, we can generate a family of solutions by replacing the initial condition ξ_0 by $\xi_0 e^{-i\theta_0}$,

$$\xi(t,\theta_0) = 2\mathrm{Re}\left[\frac{1}{2\pi i}\oint_{\Gamma(\sigma_+)}(\Omega-\mathcal{L}^*)^{-1}\xi_0 e^{-i\Omega t-i\theta_0}d\Omega\right]. \tag{40}$$

The corresponding Poincaré's invariant (30) is then expressed by

$$S = \mathrm{Re}\left\langle\overline{\frac{1}{2\pi i}\oint_{\Gamma'(\sigma_+)}(\Omega'-\mathcal{L}^*)^{-1}\xi_0 e^{-i\Omega' t}d\Omega'}, i\frac{1}{2\pi i}\oint_{\Gamma(\sigma_+)}(\Omega-\mathcal{L})^{-1}\tilde{u}_0 e^{-i\Omega t}d\Omega\right\rangle, \tag{41}$$

$$= \mathrm{Re}\frac{1}{(2\pi i)^2}\oint_{\Gamma'(\sigma_+)}\oint_{\Gamma(\sigma_+)}\left\langle\overline{\xi_0}, i(\Omega'-\mathcal{L})^{-1}(\Omega-\mathcal{L})^{-1}\tilde{u}_0\right\rangle e^{i(\Omega'-\Omega)t}d\Omega d\Omega', \tag{42}$$

$$= \frac{1}{2\pi i}\oint_{\Gamma(\sigma_+)}D(\Omega)d\Omega, \tag{43}$$

where a function $D:\mathbb{C}\to\mathbb{C}$ is defined by

$$D(\Omega) := \left\langle\overline{\xi_0}, i(\Omega-\mathcal{L})^{-1}\tilde{u}_0\right\rangle = \left\langle\overline{\xi_0}, i(\Omega-\mathcal{L})^{-1}\mathcal{J}_u\xi_0\right\rangle. \tag{44}$$

Here, it should be remarked that the Poincaré's invariant S is defined for general perturbations that are not necessarily periodic in time, since σ_+ may include complex eigenvalues and continuous spectrum. Our definition of S however corresponds to the wave action (or the action variables) for neutrally stable modes, where the integration over the phase angle $0 < \theta_0 \le 2\pi$ is transformed to an integral path in $\mathbb{C}$ enclosing the spectrum σ_+. Note that if the integral path were to enclose the whole spectrum σ, it would always result in

$$\frac{1}{2\pi i}\oint_{\Gamma(\sigma)}D(\Omega)d\Omega = 0, \tag{45}$$

and mislead us into obtaining 'zero wave action'. We suggest that the correct wave action can be obtained by counting only the contribution from σ_+, i.e. the right half of the spectrum.

C. Spectral decomposition of wave action

We analytically deform the integral path $\Gamma(\sigma_+)$ such that it consists of many closed paths that individually enclose each isolated singularity of $D(\Omega)$.

Let us introduce a notation $\mathsf{U}(\Omega) = (\Omega-\mathcal{L})^{-1}\tilde{u}_0$, which is essentially equivalent to the Laplace transform of $\tilde{u}(t)$. If there are semi-simple eigenvalues ω_n, $n = 1, 2, 3, \ldots$, the $\mathsf{U}(\Omega)$ must have poles in the Ω plane,

$$\mathsf{U}(\Omega) = \frac{\hat{\tilde{u}}_n}{\Omega-\omega_n} + \ldots. \tag{46}$$

where $\hat{\tilde{u}}_n$ is the projection of $\tilde{u}_0$ onto the eigenspace for ω_n. An integral path $\Gamma(\omega_n)$ surrounding ω_n gives the action variable for the eigenmode,

$$\mu_n = \frac{1}{2\pi i}\oint_{\Gamma(\omega_n)}D(\Omega)d\Omega = \left\langle\overline{\xi_0}, i\hat{\tilde{u}}_n\right\rangle = \left\langle\overline{\xi_0}, i\mathcal{J}_u\hat{\xi}_n\right\rangle. \tag{47}$$

Strictly speaking, this is not the conventional action variable when ω_n is complex, for which the eigenmode is not oscillatory, but exponentially growing or damping. Nevertheless, μ_n is naturally derived from the Poincaré's invariant (or the action integral), and hence we refer to it as an 'action variable' in a generalized sense. A little attention needs to be paid to the fact that μ_n is complex when ω_n is complex. Due to the symmetry $\overline{\sigma_+} = \sigma_+$ of spectra, an eigenvalue $\overline{\omega_n}$ also exists in σ_+, and let $\hat{\tilde{u}}_{\overline{n}} = \mathcal{J}_u \hat{\xi}_{\overline{n}}$ be the corresponding projection and $\mu_{\overline{n}}$ be the 'action variable'. Using the orthogonality of the projection [27], we obtain

$$\mu_n = \left\langle \overline{\hat{\xi}_{\overline{n}}}, i\mathcal{J}_u \hat{\xi}_n \right\rangle = \overline{\left\langle \overline{\hat{\xi}_n}, i\mathcal{J}_u \hat{\xi}_{\overline{n}} \right\rangle} = \overline{\mu_{\overline{n}}}, \tag{48}$$

and hence the sum $\mu_n + \mu_{\overline{n}}$ of action variables for growing and damping modes is always real. When ω_n is a real eigenvalue, there is no distinction between ω_n and $\overline{\omega_n}$, and $\mu_n = \left\langle \overline{\hat{\xi}_n}, i\mathcal{J}_u \hat{\xi}_n \right\rangle$ agrees with the previous result (32).

As for the continuous spectrum $\sigma_c \subset \mathbb{R}$ on the real axis, the path of integration is deformed into the two paths that run parallel to σ_c at the slightly upper and lower sides;

$$\frac{1}{2\pi i} \oint_{\Gamma(\sigma_c)} \mathsf{U}(\Omega) e^{-i\Omega t} d\Omega \;=\; \lim_{\varepsilon \to 0} \frac{i}{2\pi} \int_{\sigma_c} \left[\mathsf{U}(\omega + i\varepsilon) - \mathsf{U}(\omega - i\varepsilon) \right] e^{-i\omega t} d\omega.$$

Hence, it is reasonable to define a singular eigenfunction for $\omega \in \sigma_c$ by

$$\hat{\tilde{u}}(\omega) := \frac{i}{2\pi} \left[\mathsf{U}(\omega + i0) - \mathsf{U}(\omega - i0) \right]. \tag{49}$$

This definition of $\hat{\tilde{u}}(\omega)$ agrees with the Fourier transform of $\tilde{u}(t)$ according to Sato's hyperfunction theory [28] (see also the Appendix of Ref. 29). Examples of singular eigenfunctions $\hat{\tilde{u}}(\omega)$ for various continuous spectra are found in literatures; see Van Kampen [1], Case [2, 3], Sedláček [4] and Tataronis [5].

The wave action for the continuous spectrum is then given as a function of ω;

$$\mu(\omega) = \frac{i}{2\pi} \left[D(\omega + i0) - D(\omega - i0) \right] = \left\langle \overline{\xi_0}, i\hat{\tilde{u}}(\omega) \right\rangle. \tag{50}$$

If the spectrum σ_+ is composed of such semi-simple discrete spectrum $\{\omega_n \in \mathbb{C} : n = 1, 2, \dots\}$ and a real continuous spectrum $\sigma_c \subset \mathbb{R}$, the solution is represented by

$$\tilde{u}(t) = \left[\sum_n \hat{\tilde{u}}_n e^{-i\omega_n t} + \int_{\sigma_c} \hat{\tilde{u}}(\omega) e^{-i\omega t} d\omega \right] + \text{c.c.}, \tag{51}$$

where the complex conjugate (c.c.) stems from the other spectrum σ_-. The Poincaré's invariant is decomposed into the action variables,

$$S = \sum_n \mu_n + \int_{\sigma_c} \mu(\omega) d\omega. \tag{52}$$

Similarly, the perturbation energy $\tilde{H}$ becomes

$$\tilde{H} = \sum_n \omega_n \mu_n + \int_{\sigma_c} \omega \mu(\omega) d\omega. \tag{53}$$

The action variable $\mu(\omega)$ for continuous spectrum has been already derived in a few problems [9, 10]. Our technique shown here is not only applicable to any noncanonical Hamiltonian system, but also suggesting an efficient way of derivation utilizing the Fourier-Laplace transform. In the next section, we revisit the problem tackled by Morrison & Pfirsch [9] as a demonstration of our method.

IV. EXAMPLE: VLASOV-POISSON EQUATION AND LANDAU DAMPING

A. The Case-Van Kampen equation

Our method developed in the previous sections is applicable to the Vlasov-Poisson system (see Example 3 of Sec. II). In order to demonstrate the action-angle representation as shortly as possible, let us restrict our consideration to spatially uniform steady states $f(\boldsymbol{v})$, and Fourier-transform $\tilde{f}$ in space

$$\tilde{f}(\boldsymbol{x},\boldsymbol{v},t) =\frac{1}{(2\pi)^{3/2}}\int \tilde{f}(\boldsymbol{k},\boldsymbol{v},t)e^{i\boldsymbol{k}\cdot\boldsymbol{x}}d^3k. \tag{54}$$

For fixed $\boldsymbol{k}$, the linearized equations, (27) and (28), are greatly simplified into a 1D problem for $\tilde{f}(v,t)$ along the $\boldsymbol{k}$ vector ($v := \boldsymbol{k}\cdot\boldsymbol{v}/k$). We further introduce a normalization $q_e = -1, m_e = \epsilon_0 = 1$ and finally obtain

$$i\frac{\partial \tilde{f}}{\partial t} =kv\tilde{f} + k\eta(v)\int_{\mathbb{R}}\tilde{f}dv, \quad \tilde{f}(v,0) = \tilde{f}_0(v) \tag{55}$$

$$i\frac{\partial \zeta}{\partial t} =kv\zeta + \int_{\mathbb{R}}k\eta(v)\zeta dv, \quad \zeta(v,0) = \zeta_0(v) \tag{56}$$

where $\eta(v)$ is a given function associated with the steady state $f(v)$,

$$\eta(v) = -\frac{1}{k^2}\frac{\partial f}{\partial v}. \tag{57}$$

This equation was studied by Van Kampen [1] and Case [2]. The dynamical accessibility condition $\tilde{f} = [\zeta, f]$ is now reduced to $\tilde{f} = -ik^3\eta\zeta$.

B. Laplace transform

Let $\mathsf{F}(v,\Omega)$ and $\mathsf{Z}(v,\Omega)$ be the solutions of

$$(\Omega - kv)\mathsf{F} =k\eta(v)\int_{\mathbb{R}}\mathsf{F}dv + \tilde{f}_0 \qquad (\tilde{f}_0 = -ik^3\eta\zeta_0), \tag{58}$$

$$(\Omega - kv)\mathsf{Z} = \int_{\mathbb{R}}k\eta(v)\mathsf{Z}dv + \zeta_0. \tag{59}$$

The relation $\mathsf{F}(v,\Omega) = -ik^3\eta(v)\mathsf{Z}(v,\Omega)$ follows immediately, and $\mathsf{Z}(v,\Omega)$ is explicitly solved as follows.

$$\mathsf{Z}(v,\Omega) =\frac{\zeta_0(v) - \Phi(\Omega)}{\Omega - kv}, \tag{60}$$

where we have put

$$\Phi(\Omega) = -\int_{\mathbb{R}} k\eta(v)\mathsf{Z}(v,\Omega)dv = -\frac{1}{\pi\Delta(\Omega)}\int_{\mathbb{R}} \frac{k\eta(v)\zeta_0(v)}{\Omega - kv}dv, \tag{61}$$

$$\pi\Delta(\Omega) = 1 - \int_{\mathbb{R}} \frac{k\eta(v)}{\Omega - kv}dv. \tag{62}$$

The function $D(\Omega)$ in (44) is now written as

$$D(\Omega) = \int_{\mathbb{R}} \overline{\zeta_0} i\mathsf{F}(v,\Omega)dv = \int_{\mathbb{R}} k^3\eta(v)\frac{|\zeta_0(v)|^2}{\Omega - kv}dv + k^2\pi\Delta(\Omega)\overline{\Phi(\overline{\Omega})}\Phi(\Omega). \tag{63}$$

• **Discrete spectrum**

Some eigenvalues $\{\omega_n \in \mathbb{C}, n = 1, 2, \dots\}$ may exists due to the zeros of $\Delta(\Omega)$,

$$\pi\Delta(\omega_n) = 1 - \int_{\mathbb{R}} \frac{k\eta(v)}{\omega_n - kv}dv = 0. \tag{64}$$

By denoting the residue of $\Phi(\Omega)$ at $\Omega = \omega_n$ by

$$\hat{\phi}_n = \lim_{\Omega\to\omega_n}[(\Omega - \omega_n)\Phi(\Omega)] = -\frac{\int_{\mathbb{R}} \frac{k\eta(v)\zeta_0(v)}{\omega_n - kv}dv}{\int_{\mathbb{R}} \frac{k\eta(v)}{(\omega_n - kv)^2}dv}, \tag{65}$$

the corresponding eigenfunctions are represented by $\hat{\hat{f}}_n(v) = -ik^3\eta(v)\hat{\zeta}_n(v)$, where

$$\hat{\zeta}_n(v) = -\frac{\hat{\phi}_n}{\omega_n - kv}. \tag{66}$$

The action variable is immediately given by

$$\mu_n = \frac{1}{2\pi i}\oint_{\Gamma(\omega_n)} D(\Omega)d\Omega = \overline{\hat{\phi}_{\overline{n}}}\,\hat{\phi}_n \int_{\mathbb{R}} \frac{k^3\eta(v)}{(\omega_n - kv)^2}dv, \tag{67}$$

where $\hat{\phi}_{\overline{n}}$ is the residue of $\Phi(\Omega)$ at $\Omega = \overline{\omega_n}$.

• **Continuous spectrum**

There exists a continuous spectrum on the real axis of Ω,

$$\sigma_c = \{\omega \in \mathbb{R} \text{ s.t. } \eta(\omega/k) \neq 0\} \tag{68}$$

at which $\mathsf{Z}(v,\Omega)$ is not analytic with respect to Ω. Let us introduce the following shorthand notations,

$$\eta^{\natural}(\omega) = \int_{\mathbb{R}} \eta(v)\delta(\omega - kv)kdv = \eta(\omega/k), \tag{69}$$

$$\eta^{\dagger}(\omega) = -\frac{1}{\pi}\text{p.v.}\int_{\mathbb{R}} \frac{\eta(v)}{\omega - kv}kdv, \tag{70}$$

where the operation † corresponds to the Hilbert transform. Using the well-known formula

$$\frac{1}{\omega - kv \pm i0} = \text{p.v.}\frac{1}{\omega - kv} \mp \pi i \delta(\omega - kv), \tag{71}$$

we can evaluate $\Delta(\Omega)$ and $\Phi(\Omega)$ in the limit of $\Omega \to \omega \pm i0$ as

$$\Delta(\omega \pm i0) = \lambda(\omega) \pm i\eta^\natural(\omega), \tag{72}$$

$$\Phi(\omega \pm i0) = \zeta_0^\natural(\omega) - \alpha(\omega)\lambda(\omega) \pm i\alpha(\omega)\eta^\natural(\omega), \tag{73}$$

where $\lambda(\omega) = \frac{1}{\pi} + \eta^\dagger(\omega)$ and

$$\alpha(\omega) = -\frac{(\eta\zeta_0)^\dagger(\omega) - \lambda(\omega)\zeta_0^\natural(\omega)}{\lambda^2(\omega) + \eta^{\natural 2}(\omega)}. \tag{74}$$

Therefore, the singular eigenfunctions (called the Van Kampen modes) are obtained as

$$\hat{\zeta}(v,\omega) = \frac{i}{2\pi}\left[\mathsf{Z}(v,\omega+i0) - \mathsf{Z}(v,\omega-i0)\right] \tag{75}$$

$$= \alpha(\omega)\left[\frac{1}{\pi}\text{p.v.}\frac{\eta^\natural(\omega)}{\omega - kv} + \lambda(\omega)\delta(\omega - kv)\right], \tag{76}$$

$$\hat{\hat{f}}(v,\omega) = -ik^3\alpha(\omega)\eta^\natural(\omega)\left[\frac{1}{\pi}\text{p.v.}\frac{\eta(v)}{\omega - kv} + \lambda(\omega)\delta(\omega - kv)\right]. \tag{77}$$

Note that we have also derived the appropriate "amplitude" $\alpha(\omega)$ of the singular eigenmode, which depends on the initial value ζ_0. The wave action for the continuous spectrum $\omega \in \sigma_c$ turns out to be

$$\mu(\omega) = \frac{i}{2\pi}[D(\omega + i0) - D(\omega - i0)] = k^2\eta^\natural(\omega)[\lambda^2(\omega) + \eta^{\natural 2}(\omega)]|\alpha(\omega)|^2. \tag{78}$$

V. SUMMARY

In this lecture note, we have shown a technique for doing the action-angle representation of linear waves in plasmas, which is based on the noncanonical Hamiltonian structure of the linearized systems and exploits the spectral theory. The notion of action variable (or wave action) is naturally generalized to exponentially growing and damping eigenmodes as well as continuum modes by introducing a suitable action integral.

This method is demonstrated for the classical problem of electrostatic oscillations posed by Van Kampen [1] and Case [2], which is rather straightforward since the Laplace transform is analytically executable. When more complicated linear systems are considered, both elimination and transformation of variables are essential for reducing the systems to simpler ones [6]. We have investigated other continuous spectra in hydrodynamics and magnetohydrodynamics in this manner [6, 7].

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