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About nonlinear elliptic problems in plasma physics

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ABOUT NONLINEAR ELLIPTIC PROBLEMS IN PLASMA PHYSICS

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ABSTRACT. We consider a set of nonlinear elliptic partial differential equations involved in some usual models in plasma physics and introduced to some modern techniques to deal with these.

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1. Introduction

Nonlinear elliptic partial differential equations appear naturally in many models in Plasma Physics. We will concentrate our attention in nonlinear second order elliptic PDEs, specially involved in equilibrium questions. In particular we deal with existence and multiplicity of solutions for problems related the so called *nonlinear Poisson equation*, namely

$$-\Delta u = f(x, u), \tag{1.1}$$

where $x \in \Omega \subseteq \mathbb{R}^n$, Ω a domain (i. e. an open and connected set), $\Delta := \operatorname{div} \nabla$ is the usual Laplace operator in $\mathbb{R}^n$ and

$$u : \Omega \longrightarrow \mathbb{R}$$

and

$$f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$$

are functions (f is the so called *nonlinearity*).

2. Function spaces

We will give a partial list of the more usual spaces of functions involved in the study of equations like (1.1) (for details see for instance [28] among many others).

From now on Ω will be a *bounded domain* in $\mathbb{R}^n$ with smooth boundary.

- $C^0(\overline{\Omega})$: continuous functions on $\overline{\Omega}$ with the norm

$$\| v \|_{C^0(\overline{\Omega})} := \sup_{\overline{\Omega}} |v|. \tag{2.1}$$

- $C^k(\overline{\Omega})$ where $k \in \mathbb{N}, k \geq 0$: k times differentiable functions on Ω with the norm

$$\| v \|_{C^k(\overline{\Omega})} := \sum_{0 \leq |\gamma| \leq k} \| D^\gamma u \|_{C^0(\overline{\Omega})} . \tag{2.2}$$

- $C_0^\infty(\Omega)$: $C^\infty(\Omega)$ functions with compact support in Ω.
- $C^{0,\alpha}(\overline{\Omega})$ where $0 < \alpha \leq 1$: Hölder [1] continuous functions on Ω, with the norm

$$\| v \|_{C^{0,\alpha}(\overline{\Omega})} := \| v \|_{C^0(\overline{\Omega})} + \sup_{\substack{x,y \in \Omega, \\ x \neq y}} \frac{|v(x) - v(y)|}{|x - y|^\alpha} \tag{2.3}$$

[1] Roughly a function is Hölder-α for $0 < \alpha \leq 1$, if $|f(s_1) - f(s_2)| \leq |s_1 - s_2|^\alpha$ for all $s_1, s_2 \in \Omega$.

- $C^{k,\alpha}(\overline{\Omega})$ where $k \geq 0$ and $0 < \alpha \leq 1$: $C^k(\overline{\Omega})$ functions with Hölder-α derivatives of order k, endowed with the norm

$$\| v \|_{C^{k,\alpha}(\overline{\Omega})} := \| v \|_{C^k(\overline{\Omega})} + \sup_{\substack{x,y\in\Omega,\\ x\neq y,\\ |\beta|=k}} \frac{|D^\beta v(x) - D^\beta v(y)|}{|x-y|^\alpha} \tag{2.4}$$

- $L^p(\Omega)$ where $1 \leq p < \infty$: (*Lebesgue spaces*) p-integrable functions on Ω, furnished with the norm

$$\| v \|_{L^p(\Omega)} := \left(\int_\Omega |v|^p \right)^{\frac{1}{p}}. \tag{2.5}$$

- $L^\infty(\Omega)$: essentially bounded functions on Ω, endowed with the norm [2]

$$\| v \|_{L^\infty(\Omega)} := \operatorname{ess\,sup}_\Omega |v|. \tag{2.6}$$

- $H^{k,p}(\Omega)$ where $k \geq 1$ and $1 \leq p \leq \infty$: (*Sobolev spaces*) measurable functions with weak derivatives (i. e. in the sense of distributions) up to order k in $L^p(\Omega)$ endowed with the norm

$$\| u \|_{H^{k,p}(\Omega)} := \sum_{0\leq|\gamma|\leq k} \| D^\gamma u \|_{L^p(\Omega)}\ . \tag{2.7}$$

[2] $\operatorname{ess\,sup}|v| := \inf\{a \geq 0 : \mu(\{|f| > a\}) = 0\}$, where μ in our cases is the Lebesgue measure.

- $H_0^1(\Omega)$: the closure of $C_0^\infty(\Omega)$ in $H^{1,2}(\Omega)$.

  For functions in $H_0^1(\Omega)$ the Poincaré inequality holds, namely:

$$\int_\Omega |u|^2 dx \leq c \int_\Omega |\nabla u|^2 dx, \tag{2.8}$$

where $c = c(\Omega)$ is a constant (possibly depending on Ω but independent of u). As a consequence of the Poincaré inequality it follows that

$$\|u\| := \left(\int_\Omega |\nabla u|^2 dx \right)^{\frac{1}{2}} \tag{2.9}$$

is a norm equivalent to the standard one $\|\cdot\|_{H_0^1}$.

3. Some embeddings

Let $k \geq 1$ be. For $1 \leq p < \infty$ we will denote

$$p_k^* = \begin{cases} \frac{pn}{n-kp} = \left(\frac{1}{p} - \frac{k}{n}\right)^{-1}, & \text{if } n > kp; \\ +\infty, & \text{if } n = kp. \end{cases}$$

Theorem 3.1. *(Sobolev embedding theorem) Let Ω be a bounded domain in $\mathbb{R}^n$ with Lipschitz boundary $\partial\Omega$ and let $k \geq 1$, $1 \leq p \leq \infty$. Then,*

$$\begin{array}{lcl} H^{k,p}(\Omega) & \longrightarrow & L^q(\Omega) \ \textit{more integrability} \\ kp < n & \textit{cont. embed.} & 1 \leq q \leq \frac{pn}{n-kp} > p \\ kp < n & \textit{comp. embed.} & 1 \leq q < \frac{pn}{n-kp} > p \\ kp = n & \textit{comp. embed.} & 1 \leq q < \infty \end{array}$$

$$H^{k,p}(\Omega) \longrightarrow C^{0,\alpha}(\Omega) \ \textit{more regularity}$$

$$\underbrace{kp > n}_{0<k-\frac{n}{p}} \quad \textit{cont. embed.} \quad \alpha \in \begin{cases} \{k - \frac{n}{p}\}, & \textit{if } k - \frac{n}{p} < 1; \\ [0,1), & \textit{if } k - \frac{n}{p} = 1; \\ \{1\}, & \textit{if } k - \frac{n}{p} > 1. \end{cases}$$

One particularly useful case of *Theorem 3.1* is the following:

Theorem 3.2. *(Sobolev embedding for $H^{1,2}(\Omega)$) Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and let*

$$2^* = \begin{cases} \frac{2n}{n-2}, & \text{if } n > 2; \\ +\infty, & \text{if } n = 2. \end{cases}$$

Then,

$$\begin{array}{ccc} H^{1,2}(\Omega) & \longrightarrow & L^q(\Omega) \\ 2 < n & \text{cont. embed.} & 1 \leq q \leq 2^* \\ 2 < n & \text{comp. embed.} & 1 \leq q < 2^* \\ 2 = n & \text{comp. embed.} & 1 \leq q < \infty \end{array}$$

$$\begin{array}{ccc} H^{1,2}(\Omega) & \longrightarrow & C^{0,\alpha}(\Omega) \\ 2 > n & \text{cont. embed.} & \alpha \in 1 - \frac{n}{2}. \end{array}$$

In particular this holds for $H_0^{1,2}(\Omega)$.

4. The solution operator (Green op.)

Let the Dirichlet problem

$$-\Delta u = h \text{ on } \Omega \qquad u = 0 \text{ on } \partial\Omega, \tag{4.1}$$

where h is a given function on Ω. If $h \in L^2(\Omega)$, a weak solution of (4.1) is a function $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega vh, \tag{4.2}$$

for all $v \in C_0^\infty(\Omega)$.

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain.*

Sobolev: *If $h \in L^p(\Omega)$, $1 < p < +\infty$, then (4.1) has a unique solution $u \in H_0^1(\Omega) \cap H^{2,p}(\Omega)$ such that*

$$\|u\|_{H^{2,p}(\Omega)} \leq C\|h\|_{L^p(\Omega)},$$

where C is a real constant independent of h and u.

Schauder: *If Ω is $C^{2,\alpha}$ and $h \in C^{0,\alpha}(\overline{\Omega})$, then (4.1) has a unique solution $u \in C^{2,\alpha}(\overline{\Omega})$ such that*

$$\|u\|_{C^{2,\alpha}(\overline{\Omega})} \leq C\|h\|_{C^{0,\alpha}(\overline{\Omega})},$$

where C is a real constant independent of h and u.

So we can define the following *solution operator* K of (4.1) (i. e. $(-\Delta)^{-1}$):

$$\underbrace{L^2(\Omega) \xrightarrow{K} H^1_0(\Omega) \xhookrightarrow[\text{compact}]{\text{Sobolev}} L^2(\Omega)}_{K \text{ is compact}}$$

Analogously and by the theorem of Ascoli

$$\underbrace{C^{0,\alpha}(\Omega) \xrightarrow{K} C^{2,\alpha}_0(\Omega) \hookrightarrow C^{2,\alpha}(\Omega)}_{K \text{ is compact}}$$

where $C^{2,\alpha}_0(\Omega) := \{v \in C^{2,\alpha}(\Omega) : v|_{\partial\Omega} = 0\}$.

See [6, p. 7-11] for a presentation of the linear Poisson equation in Sobolev and Hölder spaces (See also [14, p.176]).

5. The Dirichlet nonlinear problem

Let the Dirichlet problem associated to (1.1) be, namely

$$-\Delta u = f(\cdot, u) \text{ on } \Omega \qquad (5.1)$$
$$u = 0 \text{ on } \partial\Omega.$$

By a weak solution of (5.1) we will understand a function $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v dx = \int_\Omega f(\cdot, u) v dx, \qquad (5.2)$$

for all $v \in C_0^\infty(\Omega)$.

To illustrate the analysis we will assume the following conditions for the nonlinearity $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$:

$(f.0)$: $f \in C^0(\overline{\Omega} \times \mathbb{R}) \bigcap C^1(\Omega \times \mathbb{R})$ and is positive on $\Omega \times \mathbb{R}_{>0}$.

$(f.1)$: $|f(x,t)| \leq a(x)|t|^p$, where a is positive on $\overline{\Omega}$ and $1 < p \leq 2^* + 1$. The equality holds for t large.

Applying $(f.1)$ it is possible to prove that the operator

$$\begin{array}{ccc} L^{2^*}(\Omega) & \longrightarrow & L^{\frac{2n}{n+2}}(\Omega) \\ u(\cdot) & \longmapsto & f(\cdot, u(\cdot)) \end{array} \tag{5.3}$$

is continuous. Moreover,

$$F(x,t) := \int_0^t f(x,s)ds$$

satisfies

$$|F(x,t)| \leq a(x)\frac{1}{p+1}|t|^{p+1} \leq \sup_{\overline{\Omega}} a \, \frac{1}{p+1} \, |t|^{p+1}, \tag{5.4}$$

where $2 < p + 1 \leq 2^*$. So, if $u \in L^{2^*}(\Omega)$

$$|F(\cdot, u(\cdot))| \leq \sup_{\overline{\Omega}} a \, \frac{1}{p+1} \, |u(\cdot)|^{p+1}, \tag{5.5}$$

and as consequence

$$\begin{array}{ccccc} H_0^1(\Omega) & \overset{\text{Sobolev}}{\hookrightarrow} & L^{2^*}(\Omega) & \longrightarrow & L^1(\Omega) \\ u & \longmapsto & u & \longmapsto & F(\cdot, u(\cdot)). \end{array}$$

Thus,

(5.6)

$$\begin{array}{rcl} J: \ H_0^1(\Omega) & \longrightarrow & \mathbb{R} \\ u & \longmapsto & \dfrac{1}{2}\displaystyle\int_\Omega |\nabla u|^2 dx - \int_\Omega F(\cdot, u) dx \end{array}$$

is well defined and it is not difficult to prove that $\forall v \in C_0^\infty(\Omega)$ holds (see the definition of differentiability below)

$$\begin{aligned} J'(u)[v] &:= \frac{d}{dt} J(u+tv)\Big|_{t=0} \\ &= \int_\Omega \langle \nabla u, \nabla v \rangle dx - \int_\Omega f(\cdot, u) v dx. \end{aligned}$$

Hence, the critical points of J in $H_0^1(\Omega)$ are weak solutions of (5.1).

Note 5.1. In order to fix the ideas it would be useful to take in mind the example

$$\left\{ \begin{array}{l} f(x,t) := a(x)\,|t|^p \text{ on } \overline{\Omega} \times \mathbb{R} \\ a(x) > 0 \text{ on } \overline{\Omega} \text{ and } 1 < p \le 2^* + 1. \end{array} \right. \tag{5.7}$$

Differentiability: [11, p. 16] Let $\mathbf{X}$ be a Banach space; a function $J : \mathbf{O} \to \mathbb{R}$, $\mathbf{O} \subseteq \mathbf{X}$ an open subset; $x \in \mathbf{X}$ and $\mathbf{p}$ a continuous linear functional on $\mathbf{X}$.

Gâteaux: We said that $\mathbf{p}$ is the Gâteaux derivative of J at x, if for any $\nu \in \mathbf{X}$

$$\lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \frac{1}{h} \left[J(x + h\nu) - J(x) - \langle \mathbf{p}, h\nu \rangle \right] = 0. \tag{5.8}$$

Frêchet: We said that $\mathbf{p}$ is the Frêchet derivative of J at x, if

$$\lim_{\substack{h \to 0 \\ h \in \mathbf{X}}} \frac{1}{\|h\|_{\mathbf{X}}} \left[J(x + h) - J(x) - \langle \mathbf{p}, h \rangle \right] = 0. \tag{5.9}$$

A such $\mathbf{p}$ is usually denoted by $J'(x)$.

Clearly Frêchet differentiability implies Gâteaux differentiability. The converse is false.

With these definitions, the functional introduced in (5.6) is Frêchet differentiable. Even more, "J is C^1 as a functional on $H_0^1(\Omega)$ with values in $\mathbb{R}$".

6. The maximum principle

Let Ω be a bounded domain in $\mathbb{R}^n$ and let L be a linear second order *uniformly elliptic operator* on Ω of the form

(6.1)
$$Lu := -\sum_{i,j=1}^{n} a^{ij}(x)u_{x_i x_j} + \sum_{i=1}^{n} b^i(x)u_{x_i} + c(x)u,$$

where the coefficients are continuous functions. *Uniformly elliptic* means $a^{ij} = a^{ji}$ and there exists a positive constant θ such that $a^{ij}\xi_i\xi_j \geq \theta|\xi|^2$ for a.e. x and $\forall \xi \in \mathbb{R}^n$.

Theorem 6.1. *(Weak Maximum Principle) Assume $u \in C^2(\Omega) \bigcap C^0(\overline{\Omega})$ and $c \equiv 0$.*

i: *If*

(6.2) $$Lu \leq 0 \text{ on } \Omega \quad \textit{(subsolution)}$$

then $\max_{\overline{\Omega}} u = \max_{\partial\Omega} u$.

ii: *If*

(6.3) $$Lu \geq 0 \text{ on } \Omega \quad \textit{(supersolution)}$$

then $\min_{\overline{\Omega}} u = \min_{\partial\Omega} u$.

Theorem 6.2. *(Strong Maximum Principle) Assume* $u \in C^2(\Omega) \bigcap C^0(\overline{\Omega})$ *and* $c \geq 0$ *in* Ω.

i: *If*

$$Lu \leq 0 \text{ on } \Omega \tag{6.4}$$

and $\max_{\overline{\Omega}} u$ *is attained in* Ω, *then* u *is constant in* $\overline{\Omega}$.

ii: *If*

$$Lu \geq 0 \text{ on } \Omega \tag{6.5}$$

and $\min_{\overline{\Omega}} u$ *is attained in* Ω, *then* u *is constant in* $\overline{\Omega}$.

For the proofs of these principles and several extensions see for instance [28, chap. 6].

Example 6.3. Let the Dirichlet problem

$$\begin{aligned} -\Delta u - \lambda u &= f \text{ on } \Omega \\ u &= 0 \text{ on } \partial\Omega, \end{aligned} \tag{6.6}$$

where $\lambda \leq 0$. If f is nonnegative and u sufficiently regular verifies (6.6), then u is positive in Ω or $u \equiv 0$ in Ω. In other words, the solution operator of $-\Delta - \lambda Id$ is positive if $\lambda \leq 0$ (really it holds for $\lambda < \lambda_1$, the principal eigenvalue of $-\Delta$ with Dirichlet conditions on the boundary).

Example 6.4. Consider the problem (5.1) with f nonnegative. So, if u is a sufficiently regular solution of (5.1), then by the strong maximum principle u is positive on Ω or $u \equiv 0$ on Ω.

Notice that if we look for positive solutions, without lose of generality, we can modify the nonlinearity in order to avoid the case $u \equiv 0$. In synthesis we will assume also

$(f.2)$: f is positive for $t \leq 0$ and $f(x,t) = a(x)|t|^p$ for t large.

7. About sub-super solutions

Some words about the *Method of Sub and Super-solutions* (see [19], [2, p. 648], [6, p. 46], [35]).

Theorem 7.1. *Let $\Omega \subset \mathbb{R}^n$ a smooth bounded domain and $f \in C^1(\overline{\Omega} \times \mathbb{R})$. Assume that there exist $\underline{u}, \overline{u} \in C^2(\Omega) \bigcap C^0(\overline{\Omega})$ such that*

$$-\Delta \underline{u} \leq f(\cdot, \underline{u}) \text{ on } \Omega \qquad (7.1)$$
$$\underline{u} \leq 0 \text{ on } \partial\Omega$$

and

$$-\Delta \overline{u} \geq f(\cdot, \overline{u}) \text{ on } \Omega \qquad (7.2)$$
$$\overline{u} \geq 0 \text{ on } \partial\Omega.$$

If $\underline{u} \leq \overline{u}$, then there exists u with $\underline{u} \leq u \leq \overline{u}$ verifying of (5.1).

A function $\underline{u}$ (respectively $\overline{u}$) that verifies (7.1) (respectively (7.2)) is called a *sub-solution* (respectively *super-solution*) of (5.1).

Example 7.2. Let $0 < q < 1$ be. In order to apply Theorem 7.1 to

$$-\Delta u = u^q \text{ on } \Omega$$
$$u = 0 \text{ on } \partial\Omega, \tag{7.3}$$

it is sufficient to note that

- $-\Delta(\epsilon u_1) = \lambda_1 \epsilon u_1 \leq \underbrace{\lambda_1 \epsilon^{1-q} \|u_1\|_\infty^{1-q}}_{\leq 1} (\epsilon u_1)^q,$
  where $\lambda_1$is the principal eigenvalue of $-\Delta$ with Dirichlet conditions on the boundary and u_1 its corresponding normalized eigenfunction.
- Let e be the normalized solution of (4.1) with $h \equiv 1$. By the maximum principle $e > 0$ in Ω.
  $-\Delta(te) = t = t^{1-q} e^{-q} (te)^q \geq \underbrace{t^{1-q} \|e\|_\infty^{-q}}_{\geq 1} (te)^q$
- With ϵ eventually smaller, there results

$$\underline{u} := \epsilon u_1 \leq te =: \overline{u}.$$

Then, by Theorem 7.1 there exists a positive solution of (7.3) with $\underline{u} \leq u \leq \overline{u}$.

8. Some variational results

Among others, the following are important references for the arguments in this section and more [39, 14, 43, 28, 3, 6, 11].

By the previous discussions in order to look for positive solutions of (5.1) we are interested in the critical points of (5.6), i. e. functions in $H_0^1(\Omega)$ where $J'(u)[v] = 0$ for all $v \in C_0^\infty(\Omega)$.

Thus we will consider some arguments that imply the existence of stationary points of real functions defined on a Banach space.

If the function J is bounded from above or below it is reasonable to try to show that it attains a maximum or minimum. For convex functions, a classical result in this direction is

Theorem 8.1. [39, p. 277] *Let J be a lower-semi-continuous convex function, bounded from below, defined on a reflexive Banach space, such that $J(x) \to +\infty$ as $x \to \infty$, i.e. J is coercive. Then J attains its minimum.*

8.2. *Semi-continuity*: [11, p. 11] Let $\mathbf{X}$ be a Banach space. A function $J : \mathbf{X} \to \mathbb{R} \cup \{+\infty\}$ is lower semi-continuous (lsc) if for each point $\bar{x} \in \mathbf{X}$, we have

$$\liminf_{x \to \bar{x}} J(x) \geq J(\bar{x}). \tag{8.1}$$

This means that the set (usually called the epigraph)

$$\operatorname{epi} J := \{(x, a) \in \mathbf{X} \times \mathbb{R} : a \geq J(x)\} \tag{8.2}$$

is a closed subset of $\mathbf{X} \times \mathbb{R}$.

Results about the existence of stationary points make use of some kind of *compactness*. In the Theorem 8.1 the compactness is hidden. The proof uses the fact that a closed bounded convex set in a reflexive Banach space is compact in the weak topology.

If J is not convex, it does not need achieve its infimum (see [39]).

An usual compactness condition for special sequences employed to prove the existence of stationary points is the Palais-Smale condition **(PS)** for a C^1 function J (see for instance [39],[11, p. 269-270]).

8.3. *Palais-Smale condition*: Let $\mathbf{X}$ a Banach space and $J : \mathbf{X} \to \mathbb{R}$ a C^1 function. We say that J satisfies the Palais-Smale condition **(PS)** on $\mathbf{X}$, if any sequence $\{x_n\}_n$ in $\mathbf{X}$, with

$$|J(x_n)| \leq \text{ constant} \tag{8.3a}$$

$$J'(x_n) \to 0 \text{ in } \mathbf{X}^* \tag{8.3b}$$

has a strongly convergent subsequence $\{x_{n_k}\}_k$.

It is clear that in a such case, if $x = \lim_k x_{n_k}$, then $J'(x) = 0$, i.e. x results a critical point of J.
A sequence in $\mathbf{X}$ verifying the conditions (8.3) is called a Palais-Smale sequence (for short (**PS**)-sequence).

The following result holds:

Theorem 8.4. *Let J be a real C^1 function on a Banach space satisfying* (**PS**) *and bounded from below. Then J achieves a minimum at some point.*

8.5. *Existence of almost-minimizing sequences*: A central question in the proof of *Theorem 8.4* is to show the existence of (**PS**)-sequences. A pioneering result in this direction is (see [27, p. 452, Thm. 8]):

Let $\mathbf{X}$ be a Banach space. If $J : \mathbf{X} \to \mathbb{R}$ a C^1 function bounded from below, then there exists a sequence $\{y_n\}_n$ such that

$$J(y_n) \to \inf J \tag{8.4}$$

$$J'(y_n) \to 0 \text{ in } \mathbf{X}^*, . \tag{8.5}$$

where $\mathbf{X}^*$ is the topological dual of $\mathbf{X}$.

8.6. *Ekeland variational principle*:[27, Thm 1 bis] Let $\mathbf{V}$ be a complete metric space, and $F : \mathbf{V} \to \{+\infty\} \cup \mathbb{R}$ a l.s.c. function, $\not\equiv +\infty$, bounded from below. For any $\epsilon > 0$, there is some point $v \in \mathbf{V}$ with:

$$F(v) \leq \inf_{\mathbf{V}} F + \epsilon, \tag{8.6}$$

$$\forall w \in \mathbf{V}, F(w) \geq F(v) - \epsilon d(v, w). \tag{8.7}$$

Analogously, it is possible to prove that:

Theorem 8.7. *Let J be a real C^1 function on a Banach space satisfying* (**PS**) *and bounded from below. If $C \subset \mathbf{X}$ is bounded closed convex set with* $\operatorname{int} C \neq \phi$ *(for instance a ball) and*

$$\inf_{\operatorname{int} C} J < \inf_{\partial C} J,$$

then there exists $x_0 \in \operatorname{int} C$ such that

$$J(x_0) = \inf_{\operatorname{int} C} J.$$

And as consequence $J'(x_0) = 0$.

8.8. *Existence of almost-minimizing sequences*: To show the existence of (**PS**)-sequences, it is sufficient to apply [27, p. 445, Thm. 1 bis], to J restricted to the complete metric space C. Indeed, for any ϵ such that

$$0 < \epsilon < \inf_{\partial C} J - \inf_{\operatorname{int} C} J,$$

there exists $v_\epsilon \in C$ with

$$f(v_\epsilon) \leq \inf_{C} J + \epsilon < \inf_{\partial C} J \tag{8.8}$$

and for all $w \in C$

$$f(w) \geq f(v_\epsilon) - \epsilon \|v_\epsilon - w\|.$$

Notice, that (8.8) implies that $v_\epsilon \in \operatorname{int} C$.

Hence, the same procedure applied in [27, p. 452, Thm. 8], allow to infer the existence of a (**PS**)-sequence in $\operatorname{int} C$.

Saddle Points

In order to look for critical points of this type the central result is the pioneering *Mountain Pass Theorem* of Ambrosetti-Rabinowitz (see [6] and many references therein), namely:

8.9. *Mountain Pass Theorem*: Let $\mathbf{X}$ be a Banach space and $J : \mathbf{X} \to \mathbb{R}$ a $C^1(\mathbf{X}, \mathbb{R})$ and

(8.9a)

$$\exists \rho > 0 : m(\rho) := \inf\{J(x) : \| x \| = \rho\} > J(0)$$

(8.9b)

$$\exists z \in \mathbf{X} : \| z \| > \rho \text{ and } J(z) < m(\rho)$$

(8.9c)

J satisfies (**PS**).

Then there is a point $\bar{x} \in \mathbf{X}$ where

(8.10)

$$J(\bar{x}) = \inf_{\sigma \in \Sigma} \sup_{[0,1]} J \circ \sigma \geq m(\rho) \text{ and } J'(\bar{x}) = 0.$$

Example 8.10. *About* (**PS**) *and Mountain Pass - Brezis-Nirenberg example*: The only geometric conditions do not imply the existence of a Mountain Pass critical point, already in dimension 2. Indeed, let $J : \mathbb{R}^2 \to \mathbb{R}$

$$J(x, y) = x^2 + (1 - x)^3 y^2. \quad (8.11)$$

J verifies (8.9a) and (8.9b), but not (8.10), thus (**PS**) is not verified (see [6, p. 118 and 121-122 —]).

Fortunately, in many applications, the associated functionals satisfy the Palais-Smail condition (or eventually some variant of this) in suitable spaces. For instance, for our example (5.1) with the conditions $(f.0)$ and $(f.1)$, the functional (5.6) verifies (**PS**). This is a consequence of essentially of two facts:

(1) A (**PS**)-sequence is bounded in $H_0^1(\Omega)$.
(2) The embedding of $H_0^1(\Omega)$ in $L^p(\Omega)$ for $1 < p < 2^*$ is compact (see the Sobolev embedding Theorem 3.2).

So, applying the Mountain Pass Theorem the following result holds (see [39, p. 281-282]).

Theorem 8.11. *Let Ω be a bounded domain in the $\mathbb{R}^n$ with smooth boundary and $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ verifying $(f.0)$, $(f.1)$ and $(f.2)$. Then there exists a positive solution of* (5.1).

9. Some operators in plasma physics

In the applications instead of $-\Delta$ frequently appear more general second order differential operators L linear or nonlinear and depending or not explicitly of the independent variable $x \in \Omega$. For instance:

- **Linear second order operators in non-divergence form**

(9.1)

$$Lu := -\sum_{i,j=1}^{n} a_{ij}(x)u_{x_i x_j} + \sum_{i=1}^{n} b_i(x)u_{x_i} + c(x)u,$$

with suitable hypothesis about the coefficients.

- **Linear second order operator in divergence form**

(9.2)

$$Lu := -\sum_{i,j=1}^{n} (a_{ij}u_{x_i})_{x_j} + \sum_{i=1}^{n} b_i(x)u_{x_i} + c(x)u,$$

with suitable hypothesis about the coefficients. Clearly $-\Delta$ belongs to this class of operators.

- **Grad-Shafranov equation in $\mathbb{R}^3_+$**

In the 80's Heyvaerts and coauthors introduced a two dimensional model of solar flares (the plane x, y is the surface of the sun) assuming that the magnetic field is force-free (i. e. $\mathbf{j} \times \mathbf{B} = 0$) and that its evolution is quasi-static. This brings to the study of a semi-linear elliptic equation in a half-plane depending of one parameter λ which describes the time evolution (see [30, 31, 32, 33, 34]; a recent review about these arguments is [41]).

In 89 Amari & Aly generalized this model in order to study the structure and quasi-static evolution of a two-dimensional x-invariant magnetostatic equilibria in the half-space $\{z > 0\}$, where the plasma pressure p and gravity are taken into account, but the field is shearless ($B_x = 0$) (see [1]). Their approach follows the Low's ideas in [36], namely.

Let $\mathbb{R}^3_+$ the Euclidean space and $\{\mathbf{e_x}, \mathbf{e_y}, \mathbf{e_z}\}$ the usual orthonormal frame.
The equation of magnetostatic equilibrium of the plasma is

$$0 = -\nabla p + \mathbf{j} \times \mathbf{B} + \rho \mathbf{g}, \tag{9.3}$$

where $p(\mathbf{x})$ is the gas pressure, $\mathbf{B}(\mathbf{x})$ is the magnetic field, $\rho(\mathbf{x})$ is the gas density,

$$\mathbf{g} = -g\mathbf{e_z} \tag{9.4}$$

is the uniform gravity acceleration in the negative $\mathbf{e}_z$ direction and

$$\mathbf{j} = \nabla \times \mathbf{B} = \text{ curl } \mathbf{B}, \tag{9.5}$$

is the current density. [3]
Equation (9.3) is a magnetohydrostatic balance among the pressure gradient, the Lorentz force ($\mathbf{j} \times \mathbf{B}$) and the gravitational force.

We assume the gas obeys the ideal gas law

$$\rho = \frac{p}{T}, \tag{9.6}$$

where $T(\mathbf{x})$ is the gas temperature.
Furthermore equation (9.5), we assume

$$\text{div } \mathbf{B} = 0. \tag{9.7}$$

The equations (9.7) and (9.5) are the Maxwell equations.

[3]Notice that except g, the other involved constants were normalized.

Consider now the case where the physical quantities depend only on the Cartesian coordinates y and z, so that they are x-invariant.

The magnetic field derives from a vector potential $\mathbf{A} = (A_x, A_y, A_z)$

$$\mathbf{B} = \operatorname{curl} \mathbf{A} = \begin{vmatrix} \mathbf{e_x} & \mathbf{e_y} & \mathbf{e_z} \\ \partial_x & \partial_y & \partial_z \\ A_x & A_y & A_z \end{vmatrix}$$
$$= (\underbrace{\partial_y A_z - \partial_z A_y}_{B})\mathbf{e_x} + \partial_z \underbrace{A_x}_{u} \mathbf{e_y} - \partial_y \underbrace{A_x}_{u} \mathbf{e_z}$$
$$= (B, \partial_z u, -\partial_y u),$$

where u is usually called a *flux function*. Clearly (9.7) is verified.

Since $B = B(y, z)$ and $u = u(y, z)$, the current

$$\mathbf{j} = \operatorname{curl} \mathbf{B} = \begin{vmatrix} \mathbf{e_x} & \mathbf{e_y} & \mathbf{e_z} \\ \partial_x & \partial_y & \partial_z \\ B & \partial_z u & -\partial_y u \end{vmatrix}$$
$$= -\Delta u \mathbf{e_x} + \partial_z B \mathbf{e_y} - \partial_y B \mathbf{e_z}$$
$$= (-\Delta u, \partial_z B, -\partial_y B)$$

where Δ is the Laplace operator, namely

$$\Delta u = \partial^2_{yy} u + \partial^2_{zz} u.$$

So,

$$\begin{aligned}\mathbf{j}\times\mathbf{B} = &\begin{vmatrix} \mathbf{e_x} & \mathbf{e_y} & \mathbf{e_z} \\ -\Delta u & \partial_z B & -\partial_y B \\ B & \partial_z u & -\partial_y u \end{vmatrix}\\ =&(-\partial_z B\partial_y u + \partial_y B\partial_z u)\mathbf{e_x}\\ &-(\Delta u\partial_y u + B\partial_y B)\mathbf{e_y}\\ &+(-\Delta u\partial_z u - B\partial_z B)\mathbf{e_z}.\end{aligned}$$

Thus, (9.6), (9.4) and since $p = p(y,z)$ and $T = T(y,z)$, (9.3) takes the form [4]

(9.8a) $$0 = \partial_z B\partial_y u - \partial_y B\partial_z u$$

(9.8b) $$0 = \partial_y p + \Delta u\partial_y u + B\partial_y B$$

(9.8c) $$0 = \partial_z p + \Delta u\partial_z u + B\partial_z B + g\frac{p}{T}.$$

The equation (9.8a) implies

(9.9) $$B(y,z) = \bar{B}(u(y,z)).$$

Then, (9.8b) and (9.8c) take the form

(9.10a) $$0 = \partial_y p + [\Delta u + \bar{B}\bar{B}']\partial_y u$$

(9.10b) $$0 = \partial_z p + [\Delta u + \bar{B}\bar{B}']\partial_z u + g\frac{p}{T}.$$

Now, making (9.10a)$\partial_z u-$(9.10b)$\partial_y u$, there results

(9.11) $$\partial_y p\partial_z u - \partial_z p\partial_y u = g\frac{p}{T}\partial_y u.$$

[4] See equations (7,8,9) in [36].

Let us assume now that

$$p(y,z) = \bar{p}(u(y,z),z) \tag{9.12a}$$

and

$$T(y,z) = \bar{T}(u(y,z),z), \tag{9.12b}$$

physically this means that pressure and temperature vary with height z along any given magnetic field line $u(y,z) = constant$. Applying (9.12) to (9.11) the latter is transformed in

$$\partial_u \bar{p} \partial_y u \partial_z u - (\partial_u \bar{p} \partial_z u + \partial_z \bar{p}) \partial_y u = g \frac{\bar{p}}{\bar{T}} \partial_y u. \tag{9.13}$$

If we also consider $\partial_y u \neq 0$, (9.13) results equivalent to

$$\partial_z \bar{p} = -g \frac{\bar{p}}{\bar{T}} \tag{9.14}$$

and as consequence to

$$\partial_z \log \bar{p} = -g \frac{1}{\bar{T}}. \tag{9.15}$$

So

$$\bar{p}(u,z) = p_0(u) \exp\left[-\int_0^z g \frac{1}{\bar{T}(u,s)} ds \right], \tag{9.16}$$

where p_0 is an arbitrary function.

Thus, the equations (9.10) take the form

(9.17a)

$$0 = \partial_u \bar{p} \partial_y u + [\Delta u + \bar{B}\bar{B}'] \partial_y u$$

(9.17b)

$$0 = \partial_u \bar{p} \partial_z u + [\Delta u + \bar{B}\bar{B}'] \partial_z u + \underbrace{\partial_z \bar{p} + g\frac{\bar{p}}{\bar{T}}}_{=0}.$$

So if $\partial_y u \neq 0 \neq \partial_z u$, we reduce both equations to

$$0 = \partial_u \left\{ \bar{p} + \frac{1}{2}\bar{B}^2 \right\} + \Delta u \tag{9.18}$$

or equivalently, by (9.16), to the **Grad-Shafranov equation**

(9.19)

$$-\Delta u = \partial_u \left\{ p_0(u) \ \exp\left[-\int_0^z g\frac{1}{\bar{T}(u,s)} ds \right] + \frac{1}{2}\bar{B}^2 \right\}.$$

Thus, if we know p_0, $\bar{T}$ and $\bar{B}$ we have an equation of the type (1.1). The solutions of (9.19) define the associated magnetic fields.

The Heyvaerts and coauthors works correspond to (9.19) with $g = 0$ (no gravity) and $\partial_u p_0 = \lambda f$, where λ is a nonnegative real parameter (i.e., $\lambda \in \mathbb{R}_{\geq 0}$) and $\bar{B} = 0$. Or eventually $p_0 = 0$ and $\partial_u \bar{B}^2 = 2\lambda f$, again with $\lambda \in \mathbb{R}_{\geq 0}$.

More precisely,

$$\begin{aligned} -\Delta u &= \lambda f(u) \text{ on } \mathbb{R}^2_+ := \mathbb{R} \times (0, +\infty] \\ u &= h \text{ on } \mathbb{R} \times \{0\}, \end{aligned} \tag{9.20}$$

where $\lambda \in \mathbb{R}_{\geq 0}$.

The Amari and Aly work corresponds to (9.19) but in isothermal conditions, so that $\bar{T}$ is constant; the field is shearless, so that $\bar{B} = 0$ and $\partial_u p_0 = \lambda f$, where $\lambda \in \mathbb{R}_{\geq 0}$.

More precisely,

$$\begin{aligned} -\Delta u &= \lambda e^{-\beta z} f(u) \text{ on } \mathbb{R}^2_+ := \mathbb{R} \times (0, +\infty] \\ u &= h \text{ on } \mathbb{R} \times \{0\}, \end{aligned} \tag{9.21}$$

where $\lambda \in \mathbb{R}_{\geq 0}$ and $\beta \in \mathbb{R}_{>0}$.

In both cases the Authors consider the following set of hypothesis:

$f : \mathbb{R} \longrightarrow \mathbb{R}$ is a C^1 function satisfying

$(f-1)$ There exists $s_0 > 0$ such that $f(s) > 0 \; \forall \; s \in (0, s_0)$.
$(f-2)$ $f(s) = 0$ for $s \leq 0$ or $s \geq s_0$.
$(f-3)$ $f(s) \leq a s^{\sigma}$, where a is a positive constant and $\sigma > 3$.
$(f-4)$ There exists $l > 0$ such that $|f(s_1) - f(s_2)| \leq l|s_1 - s_2| \forall s_1, s_2 \in \mathbb{R}$.

and

the boundary condition $h : \mathbb{R} \longrightarrow \mathbb{R}$ is a $C^{2,\alpha}$ function satisfying

(h) Fixed $n(\sigma) \in \left(\frac{2}{\sigma - 1}, 1\right)$, there exists $\kappa > 0$ such that $h(y) \leq (P_\kappa(y))^{n(\sigma)} \; \forall y \in \mathbb{R}$, where $P_\kappa(y) := \frac{1}{\pi}\frac{\kappa}{\kappa^2 + y^2}$ is the Poisson kernel for the halfplane $\mathbb{R}^2_+$.

Low in [37] consider (9.20) with completely different hypothesis, namely

$$f(s) = -\,e^{-2s} \tag{9.22a}$$

$$h(y) = \log(1 + y^2). \tag{9.22b}$$

See some comments about the Low work in [33, p. 106].

For the problem (9.20), Heyvaerts and coAuthors proved, by techniques of sub-super solutions, the existence of at least two solutions with different behavior as $z \longrightarrow +\infty$. One exists for any positive value of λ and tends to s_0. The other one tends to 0 and exists if and only if λ belongs to a finite positive interval beside 0. They proved also that the possible behaviors as $z \longrightarrow +\infty$ are either 0 or s_0. Furthermore, that for λ large enough any solution approaches s_0.

Through numeric tests they conjectured the existence of a third solution for positive λ's near 0.

They associate the existence of turning points to the trigger of a catastrophic event: in their model a solar flare.

In [15, 24] we looked for solutions of (9.20) and (9.21) under Heyvaerts type hypothesis by variational and expanding domains techniques. Notice that (9.21) is equivalent to

$$\begin{aligned} -\Delta v &= \lambda e^{-\beta z} f(v+\gamma) \text{ on } \mathbb{R}^2_+ := \mathbb{R}\times(0,+\infty] \\ v &= 0 \text{ on } \mathbb{R}\times\{0\}, \end{aligned} \tag{9.23}$$

where $v = u - \gamma$ and γ is the solution of

$$\begin{aligned} \Delta\gamma &= 0 \text{ on } \mathbb{R}^2_+ := \mathbb{R}\times(0,+\infty] \\ \gamma &= h \text{ on } \mathbb{R}\times\{0\}. \end{aligned} \tag{9.24}$$

Our approach consists in the study of the problems

$$\begin{aligned} -\Delta v &= \lambda e^{-\beta z} f(v+\gamma) \text{ on } D_R \\ v &= 0 \text{ on } \partial D_R, \end{aligned} \tag{9.25}$$

where D_R is a semidisk of radius R and center 0. The main result in [24] is the existence of a range of λ's for which (9.25) has at least three positive solutions if R is large enough. We do it by variational techniques, showing the existence of a *local minimum*, a *global minimum* and a *mountain pass solution* for the associated functional on a suitable Sobolev type space. The involved functionals are

$$\Phi_{\lambda,\gamma,R} : H^1_0(D_R) \longrightarrow \mathbb{R}$$

$$\Phi_{\lambda,\gamma,R}(u) = \frac{1}{2}\int_{D_R} |\nabla u|^2 - \lambda \int_{D_R} e^{-\beta y} F(u+\gamma),$$

where $F' = f$.

The existence of three solutions for large enough semidisks can be obtained by sub-super solutions techniques also (see [15]).

The variational approach adopted for $\beta > 0$ fails in the gravity free case, i.e. $\beta = 0$, because the Poincaré inequality is not satisfied in the whole positive half-plane $\mathbb{R}^2_+$. The role of the Poincaré inequality in $\mathbb{R}^2_+$ is play by the Hardy inequality

$$\int_{\mathbb{R}^2_+} u^2(y,z)e^{-\beta z}dxdy \leq \frac{1}{\beta^2}\int_{\mathbb{R}^2_+} |\nabla u|^2 dydz,$$

for all $u \in C_0^\infty(\mathbb{R}^2_+)$.

It is remarkable that when the radius $R \to +\infty$, the local minimum converges, the global minimum tends to $-\infty$ and we do not know what happens with the mountain pass solution, we only know that the mountain pass levels are bounded from below by a positive constant and are non increasing with R. Perhaps remains something in the limit?

More information about the Grad-Shafranov equation, can be obtained in [13, 38, 40] among many others papers and books.

In [16, 17] the Authors studied problems analogous to (9.23) with different linear part.

- **Generalized Grad-Shafranov equation**

Let $-L$ the generalized Grad-Shafranov operator, namely

$$-Lu := \nabla[(1 - M^2(u))\nabla u] + \left(\frac{M^2(u)}{2}\right)' |\nabla u|^2. \tag{9.26}$$

The generalized Grad-Shafranov equation is

$$-Lu + F'(u) = 0, \tag{9.27}$$

where F is a generic function.

If M is a constant $M_0 \neq 1$, then the latter takes the form (1.1), precisely

$$-\Delta u = -\frac{1}{1 - M_0^2} F'(u). \tag{9.28}$$

Motivations about the introduction of equation (9.27) can be obtained for instance in [29], [46].

One particular interesting case

Let $-L_\mu$ be the operator

$$-L_\mu u := \nabla[A(u)\nabla u] - \mu A'(u)|\nabla u|^2. \tag{9.29}$$

When

$$A(s) := 1 - M^2(s) \text{ and } \mu = \frac{1}{2}, \tag{9.30}$$

we obtain exactly (9.26).

Thus, if $\alpha \in \mathbb{R}$ and $u > 0$

$$-L_\mu(u^\alpha) = (1-\mu)A'(u^\alpha)|\nabla u^\alpha|^2 + A(u^\alpha)\Delta u^\alpha. \tag{9.31}$$

But,

$$\begin{aligned} \nabla u^\alpha &= \alpha u^{\alpha-1}\nabla u \\ \Delta u^\alpha &= \alpha[(\alpha-1)u^{\alpha-2}|\nabla u|^2 + u^{\alpha-1}\Delta u]. \end{aligned} \tag{9.32}$$

So,

$$\begin{aligned} -L_\mu(u^\alpha) &= (1-\mu)A'(u^\alpha)|\nabla u^\alpha|^2 + A(u^\alpha)\Delta u^\alpha \\ &= [(1-\mu)\alpha u^\alpha A'(u^\alpha) + (\alpha-1)A(u^\alpha)]\,\alpha u^\alpha \frac{|\nabla u|^2}{u^2} \\ &+ A(u^\alpha)\alpha u^\alpha \frac{\Delta u}{u}. \end{aligned} \tag{9.33}$$

[5]

[5]

$$\begin{aligned} -\frac{L(u^\alpha)}{u^\alpha} &= \frac{1}{\alpha-1}[(1-\mu)\alpha u^\alpha A'(u^\alpha) + (\alpha-1)A(u^\alpha)]\frac{\Delta u^\alpha}{u^\alpha} \\ &- (1-\mu)\frac{\alpha^2}{\alpha-1}u^\alpha A'(u^\alpha)\frac{\Delta u}{u}. \end{aligned}$$

Thus, if A verifies

$$(1-\mu)\alpha s A'(s) + (\alpha - 1)A(s) = 0, \tag{9.34}$$

then

$$-L_\mu(u^\alpha) = A(u^\alpha)\alpha u^\alpha \frac{\Delta u}{u}. \tag{9.35}$$

But the solutions of (9.34) are

$$A(s) = c|s|^{\frac{1}{1-\mu}\frac{1-\alpha}{\alpha}}, \tag{9.36}$$

with $c \in \mathbb{R}$ a constant. So (9.35) takes the form

$$-L_\mu(u^\alpha) = c\alpha u^{\frac{\mu}{1-\mu}(1-\alpha)}\Delta u. \tag{9.37}$$

Thus, if we look for positive solutions of (9.27) where

$$M^2(s) = 1 - c|s|^{2\frac{1-\alpha}{\alpha}}, \tag{9.38}$$

it is sufficient to obtain positive solutions of

$$-c\alpha v^{1-\alpha}\Delta v = F'(v^\alpha) \tag{9.39}$$

and take $u = v^\alpha$. Then, in this very particular case we return again to equations of the type (1.1) and their several boundary value problems.

Generalized Grad-Shafranov and scalar curvature

Equations of this type, appear also in the study of scalar curvature of semi-Riemannian manifolds. For instance the relation among the involved scalar curvatures in a warped product of Riemannian manifolds $(B_m \times F_k, g = g_B + f^2 g_F)$ can be written in the form:

$$f^2 S_g = 2k L_\mu f + f^2 S_{g_B} + S_{g_F} \tag{9.40}$$

where f is the so called warping function and

$$L_\mu f = -\nabla[A_0(f)\nabla f] + \mu A_0'(f)|\nabla f|^2 \tag{9.41}$$

with $\mu = \dfrac{3-k}{2}$ and $A_0(s) = s$.

The function A_0 verifies

$$(1-\mu)\alpha s A'(s) + (\alpha - 1)A(s) = 0, \tag{9.42}$$

if $\alpha = \dfrac{2}{k+1}$.

Thus,

$$L_\mu(u^{\frac{2}{k+1}}) = -\frac{2}{k+1} u^{\frac{4}{k+1}} \frac{\Delta_{g_B} u}{u} \tag{9.43}$$

and as consequence (9.40) takes the form

$$u S_g = -\frac{4k}{k+1}\Delta_{g_B} u + S_{g_B} u + S_{g_F} u^{1-\frac{4}{k+1}}. \tag{9.44}$$

This type of equation also appear in relativity (York-Lichnerowicz equation about the constraints for the Einstein equations), in filtration in porous media with absorption and in population dynamics among other areas. See [12, 23, 25, 26] and references therein.

The trick "change of variables of the type u^α in order to eliminate the terms with $|\nabla f|^2$", was applied already in the original paper of Yamabe [45] about the study of the behavior of the scalar curvature under conformal changes in the metric of a Riemannian manifold. This trick was also applied in [23] for the study of scalar curvature of warped products of Riemannian metrics (see also [25, 26] where this is also applied to Hessians). A P. J. Morrison generalization (see [44, footnote p. 453]) of this idea was applied after in plasma physics in [44] (see also [18]).

There is a set of existence results of Arcoya, Boccardo and collaborators for equations of the type

$$-\nabla[a(x,u)\nabla u] + \frac{1}{2}\partial_u a(x,u)|\nabla u|^2 = F'(u). \tag{9.45}$$

in dimension > 2 (see [8],[9],[10]), but again avoiding the singular situation (i. e. $a > 0$). In any case it is interesting to note the difficulties for the variational approach, since the associated functional

$$\frac{1}{2}\int_\Omega a(x,u)|\nabla u|^2 dx - \int_\Omega F(u)dx \tag{9.46}$$

is not Gâteaux differentiable in whole $H_0^1(\Omega)$, but only in $H_0^1(\Omega)\cap L^\infty(\Omega)$, even if a is smooth. They introduced a suitable notion of critical point and proved a variant of the mountain pass theorem applicable to functionals like (9.46).

We observe that there exist cases where (9.45) with Dirichlet boundary conditions has no non-negative solutions, for instance: If $\Omega \subset \mathbb{R}^n (n > 2)$ is a starshaped smooth bounded domain and let $m \geq 2^*$, then the problem (9.45) has no non trivial non negative solution in $H_0^1(\Omega) \cap L^\infty(\Omega)$ for

$$a(u) = (1+u)^{-2\gamma} \text{ and } F'(u) = u^{m-1}, \tag{9.47}$$

where $0 < \gamma \in \mathbb{R}$ suitable. This result was proved in [10, Theorem 5.1] applying a generalization of the so called Pohozaev identity (for the latter see for instance [6]).

- **Coupled systems of equations of the form**

$$\left\{ \begin{array}{l} -\Delta u = S(u,v) \\ -\Delta v = P(u,v), \end{array} \right. \tag{9.48}$$

on Ω. We will not deal with the study of these systems, but mention among others the following references and others therein
 (physical references): [46, p. 3663], [42, p. 11], [29].
 (mathematical references): [20, 21, 22].

- **Grad-Shafranov equation with discontinuous non linearity**

Equations of the type (1.1) where the nonlinearity is discontinuous appear frequently in plasma physics and fluid dynamics. These are also called free boundary problems. For instance:

$$-\Delta u = f(u-a) \text{ on } \Omega \tag{9.49}$$

where a is a positive real parameter and $f : \mathbb{R} \to \mathbb{R}$ is a non-decreasing bounded function in $C^1([0,+\infty))$, $\equiv 0$ in $(-\infty, 0)$ and > 0 in $[0,+\infty)$. Typical example of f is the Heaviside function.

We only give some references about these questions and others therein: [7, 4, 5, 3, 6].

Appendix A. Example in dimension 1

Example A.1. [6] Let $J : H_0^{1,2}(0,1) \to \mathbb{R}$ the functional

$$J(u) = \int_0^1 \left[\frac{1}{2}\dot{u}^2 + \frac{1}{2}u^2 - \frac{1}{4}u^4\right] dt. \tag{A.1}$$

One has that $J \in C^\infty(H_0^{1,2}(0,1), \mathbb{R})$ and there holds

$$J'(u)[v] = \int_0^1 \left[\dot{u}\dot{v} + uv - u^3 v\right] dt. \tag{A.2}$$

Then a critical point of J on $H_0^{1,2}(0,1)$ is an element $\zeta \in H_0^{1,2}(0,1)$ such that $\forall v \in H_0^{1,2}(0,1)$

$$\int_0^1 \left[\dot{\zeta}\dot{v} + \zeta v - \zeta^3 v\right] dt = 0 \tag{A.3}$$

and this means that ζ is a weak (and by regularity, see Theorem 1.16, classical) solution of the two point problem

$$-\ddot{u} = u - u^3, \tag{A.4}$$

$$u(0) = u(1) = 0. \tag{A.5}$$

Let us remark that the boundary conditions $\zeta(0) = \zeta(1) = 0$ are automatically satisfied because $\zeta \in H_0^{1,2}(0,1)$.

If we consider $J : H^{1,2}(0,1) \to \mathbb{R}$, then a critical point ζ of J satisfies $-\ddot{\zeta} = \zeta - \zeta^3$ together with the Neumann boundary conditions $\dot{\zeta}(0) = \dot{\zeta}(1) = 0$.

And if we take

(A.6)

$$J : \{u \in H^{1,2}(\mathbb{R}) : u(t+1) = u(t), \forall t \in \mathbb{R}\} \to \mathbb{R},$$

then a critical point ζ of J is a 1-periodic solution of $-\ddot{u} = u - u^3$.

Of course, the choice of Sobolev spaces like $H^{1,2}(0,1)$ is also related to the fact that (A.4) is a second order equation and the term $\int_0^1 \dot{u}^2 dt$ makes sense.

The preceding example is a model of the following variational technique: roughly, to look for solutions of boundary value problems consisting of a differential equation together with some boundary conditions, when these problem have a variational structure, namely they are the Euler-Lagrange equation of a functional J on a suitable space of functions E, chosen depending on the boundary conditions. The critical points of J on E (if they exist) give rise to solutions of these boundary value problems.

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