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Electromagnetic energy density manipulation and enhancement in a relativistic plasma: the role of relativistic nonlinearities (Part I & II)

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Electromagnetic energy density manipulation and enhancement in a relativistic plasma: the role of relativistic nonlinearities

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Outline

First lecture: Introductory remarks and calculations

- Relativistic kinematic nonlinearities
- Fluid and kinetic nonlinearities
- Role of nonlinearities: small scale generation
- Relativistic self-focussing: heuristic derivation
- Relativistic mirrors

Second lecture: Analytical 1D results and numerical simulations

- Electromagnetic and Langmuir waves with relativistic amplitudes
- Relativistic Solitons
- Energy density manipulation: soliton reflection by a relativistic mirror
- Conclusions

For a recent review of relativistic effects on light propagation in plasmas see:
Optics in the relativistic regime by G.A. Mourou, T. Tajima and S.V. Bulanov, *Rev. Mod. Phys.*, **78**, 309 (2006).

Relativistic single particle kinematics

The motion of a test particle in a large amplitude electromagnetic wave provides a good example of the nonlinearities that are intrinsic to relativistic kinematics. Let us use standard 3D notation[1] and write

$$\frac{d(\gamma \mathbf{v})}{dt} = \frac{e}{m}(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}) \tag{1}$$

with $\gamma \equiv (1 - v^2/c^2)^{-1/2}$. Let us consider a monochromatic wave with $\mathbf{E} = \mathbf{E}_0 \exp{(i\mathbf{k}\cdot\mathbf{x} - i\omega t)}$, $\mathbf{B} = \mathbf{B}_0 \exp{(i\mathbf{k}\cdot\mathbf{x} - i\omega t)}$, $|\mathbf{E}_0| = |\mathbf{B}_0|$, $\mathbf{k}\cdot\mathbf{E}_0 = \mathbf{k}\cdot\mathbf{B}_0 = 0$, $\mathbf{k}\times\mathbf{E}_0 = \omega\mathbf{B}_0$.

[1] Covariant notation in terms of four-vectors is theoretically more satisfactory but may hide features that the standard 3D notation leaves more evident. In these lectures c.g.s. units are used.

The charged particle motion depends nonlinearly[2] on the electromagnetic fields because of

- the dependence of the Lorentz factor γ on v,
- the magnetic term in the force,
- and the coordinate dependence of the wave phase.

In the reference frame where the particle is initially at rest, in the small amplitude limit we can assume $v/c \ll 1$ in which case $\gamma \sim 1$, the magnetic force term can be neglected and the particle motion is driven by the electric field only.
As a consequence $\mathbf{k} \cdot \mathbf{x} = const$.
In this limit Eq.(1) is linear and describes a particle oscillating with frequency ω along the direction of the electric field.

[2] i.e. the superposition principle does not apply.

It is easy to verify that the small amplitude limit is restricted by the condition

$$a \equiv \frac{eE_0}{mc\omega} \ll 1, \quad \text{with } a \text{ the dimensionless vector potential.} \tag{2}$$

Present day laser technology allows us to generate ultraintense laser pulses with intensities approaching 10^{23} W/cm^2 with $\omega \sim 2 \times 10^{15}\, sec^{-1}$.
For such intensities Eq.(2) is not valid and Eq.(1) has to be solved as is.
In the case of a propagating plane wave Eq.(1) is not difficult to solve because there are three integrals of motion in convolution, i.e., the Hamiltonian from which Eq.(1) has been derived is integrable. The three integrals of motion correspond to three components of the canonical four-momenta that are conserved because of Noether's theorem[3]. In the case of a 1D propagating wave, with $\mathbf{k}$ say along x, the Hamiltonian is invariant under translations in the transverse plane (y-z) and under combined x and t translations that leave $x - ct$ invariant.

[3] applied to the corresponding Lagragian http://en.wikipedia.org/wiki/Noether's_theorem.

Exercise: using the above result calculate the energy of a particle initially at rest under the action of the electromagnetic fields of a 1D, linearly polarized wave packet. Then calculate the particle energy after the wave packet has passed the particle.

Exercise: draw by numerical integration the particle trajectory and show that in the case of a monochromatic wave, in the reference frame where the particle mean velocity along x is zero, the particle trajectory has the shape of the number 8.

Exercise: calculate the particle trajectory in the case of a monochromatic plane wave using a perturbative expansion in terms of $(eE_0)/(mc\omega)$ up to quadratic terms.

Exercise: generalize these results to a circularly polarized plane wave packet.

Exercise: analyse the electron motion in the case of a standing wave and show that in this case the motion is no longer integrable. What are the consequences?

This example introduces us to two important features:

• relativistic effects bring new nonlinearities in the particle response to a time-varying electromagnetic field,

• in a 1D configuration the effect of these nonlinearities can be investigated analytically,

• • In addition, the case of circular polarized electromagnetic fields (see exercise) is easier to investigate because $|\mathbf{E}|^2$ and $|\mathbf{B}|^2$ do not oscillate with frequency 2ω

These features will also remain valid in the case of a plasma, not of a single test particle, that we will discuss in these two lectures.

Actually even in the case of a single particle, in the presence of ultraintense electromagnetic fields not much more intense than those we already obtain in the laboratory, we need to abandon the test particle point of view and must include the effect of the radiation reaction.

In the non relativistic limit its well known expression, usually derived from the Larmor formula for the radiated power, is

$$\mathbf{F} = \tau_o d(m\mathbf{a})/dt,$$

where $\tau_o = (2e^2)/(3mc^3)$ is the time it takes the radiation to cross a distance of the order of the so called classical electron radius.

This equation, and its relativistic generalization, has theoretical difficulties arising from the third derivative (the time derivative of the acceleration).

In the relativistic limit the Landau form is generally used

$$\begin{aligned}
\frac{d\mathbf{p}}{dt} = \mathbf{F} \quad &= \quad e\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right) + e\tau_0\gamma\left[\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)\mathbf{E} + \frac{\mathbf{v}}{c} \times \left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)\mathbf{B}\right] \\
&+ \quad \tau_0\frac{e^2}{mc}\left[\mathbf{E}\times\mathbf{B} + \left(\frac{\mathbf{v}}{c}\times\mathbf{B}\right)\times\mathbf{B} + \left(\frac{\mathbf{v}}{c}\cdot\mathbf{E}\right)\mathbf{E}\right] \\
&- \quad \tau_0\frac{e^2}{mc}\gamma^2\left[\left(\mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{B}\right)^2 - \left(\frac{\mathbf{v}}{c}\cdot\mathbf{E}\right)^2\right]\frac{\mathbf{v}}{c}. \qquad (3)
\end{aligned}$$

Note that Eq.(3) introduces additional nonlinearities in the equation for the particle motion.

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In relativistic covariant notation Eq.(3) reads

$$mc\frac{du^\mu}{d\tau} = eF^{\mu\nu}u_\nu + e\tau_0\left(\partial_\alpha F^{\mu\nu}u_\nu u^\alpha + \frac{e}{mc}F^{\mu\nu}F_{\nu\alpha}u^\alpha + \frac{e}{mc}(F^{\nu\beta}u_\beta F_{\nu\alpha}u^\alpha)u^\mu\right) \quad (4)$$

where $F^{\mu\nu}$ is the electromagnetic field tensor, τ is the proper time and $u^\mu = (\gamma, \gamma\mathbf{v}/c)$ is the particle four-velocity.

Is it interesting to note also this equation can be solved exactly in the 1D case[4].

Exercise. Using Eq(4), prove that

$$\frac{d(u_\mu u^\mu)}{d\tau} = 2u_\mu\frac{du^\mu}{d\tau} = 0$$

as required by the definition of the particle four-velocity u^μ.

[4] A. di Piazza , *Lett Math. Phys.*, **83**, 305 2008.

Fluid and kinetic nonlinearities

In these two lectures we are not interested exclusively in the single particle motion but we aim at addressing the behaviour of a large system of charged particles such as a plasma in which the electromagnetic fields are not externally imposed but are generated in large part by the charge and current densities that are present in the plasma.

This leads to the main difference e.g. between a particle bunch in an accelerator (where the particles are considered at least in the simplest description as test particles) and a plasma and is at the foundation of what is called the selfconsistent (i.e., **nonlinear**) plasma response:

$$\nabla \cdot \mathbf{E} = 4\pi \sum_j Z_j \int d^3 v f_j, \quad c\nabla \times \mathbf{B} = \partial \mathbf{E}/(\partial t) + 4\pi \sum_j Z_j \int d^3 v \mathbf{v} f_j,$$

where the time evolution of the distribution functions $f_j(x, v, t)$ depends on the electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ through Vlasov equation (if collisions are inessential).

The reduction in the detail of description that is brought about by the transition from a kinetic to a fluid description of the plasma brings about new nonlinearities. In the continuity equation a quadratic nonlinearity appears

$$\partial n/\partial t + \nabla \cdot (n\mathbf{u}) \equiv [\partial/\partial t + \mathbf{u}\cdot\nabla]\, n + n\nabla\cdot\mathbf{u} = 0.$$

In the momentum equation (Euler equation) a quadratic and a cubic nonlinearity appear in the inertia term.

$$nm\,[\partial/\partial t + \mathbf{u}\cdot\nabla]\,\mathbf{u} \equiv m\,[\partial(n\mathbf{u})/\partial t + \nabla\cdot n\mathbf{u}\mathbf{u}] = -\nabla p + \text{ external forces}$$

where $n(x,t) = \int d^3v f(x,v,t)$ is the particle density, $p(x,t) = ...$ is the pressure[5] and $\mathbf{u}(x,t) = \int d^3v \mathbf{v} f(x,v,t)/\int d^3v f(x,v,t)$ is the fluid velocity (not to be confused with the single particle four velocity u_μ used before).

[5] I will not discuss here the additional problem that arises in this reduction from the conceptual *a priori* impossibility of closing the fluid equations in the absence of thermodynamic equilibrium: $p = ?$.

In the relativistic case in the fluid equations become

$$\partial_\mu(n_o w^\mu) = 0, \quad \partial n/\partial t + \nabla \cdot n\mathbf{u} = 0 \tag{5}$$

where n_o is the proper density $n_o = n/\gamma$, and $w^\mu = (\gamma, \gamma\mathbf{u}/c)$ is the fluid velocity with $\gamma \equiv (1 - u^2/c^2)^{-1/2}$, and

$$\partial_\nu T^{\nu\mu} = F^{\mu\nu} j_\nu, \tag{6}$$

with $T^{\nu\mu} \equiv p_o\delta^{\mu\nu} + (\epsilon_o + p_o)w^\mu w^\nu$ the mechanical stress tensor with $\delta^{\mu\nu}$ the Minkowski metric tensor, ϵ_o is the internal + rest energy ($\epsilon_o \neq n_o mc^2$), p_o the pressure in the fluid rest frame and $j^\mu = \rho, \mathbf{j}/c$ the electric four current.
The spatial part of Eq.(6) can be rewritten in 3D notation as

$$\partial[(\epsilon_o + p_o)\gamma^2\mathbf{u}/c^2]/\partial t + \nabla[(\epsilon_o + p_o)\gamma^2\mathbf{uu}/c^2] + \nabla p_o = \gamma[\rho\mathbf{E} + (\mathbf{j}/c) \times \mathbf{B}]. \tag{7}$$

A fluid, a plasma, can be relativistic because its fluid velocity $\mathbf{u}$ approaches the velocity of light c or because its temperature is very high, $p_o \sim n_o m c^2$.

In these lectures we will be interested only Fwith the former case and will neglect all pressure contributions ($cold$ plasma approximation). Then ϵ_o reduces to $m n_o c^2$.
In addition we will consider a single charge fluid[6] where $j^\mu = n_o e w^\mu$.
In this case, using the continuity equation, we find the simpler equation

$$m[\partial/\partial t + \mathbf{u}\cdot\nabla](\gamma\mathbf{u}) = e[\mathbf{E} + (\mathbf{u}/c)\times\mathbf{B}], \qquad (8)$$

that resembles the single particle equation of motion (1) with $m\gamma\mathbf{u}$ the relativistic fluid momentum and $d/dt = \partial/\partial t + \mathbf{u}\cdot\nabla$.
Again relativistic kinematics introduces new nonlinearities in the fluid equations.

[6] i.e. we will write a fluid equation for each charge species in the plasma.

The so called convective derivative $d/dt = \partial/\partial t + \mathbf{u} \cdot \nabla$ actually corresponds to the transformation from the Eulerian variables x, t to Lagrangian variables x_o, t

$$x = x_o + \xi(x_o, t), \quad t = t,$$

where the position x_o of the fluid element at $t = 0$ plays the role of the fluid element label, $\xi(x_o, t)$ is its displacement at time t, and d/dt stands for $\partial/\partial t|_{x_o}$. It is interesting to note that the transformation from Eulerian to Lagrangian variables is an example of an important mathematical procedure whereby a linear equation can be associated to a nonlinear equation via a nonlinear transformation[7].

Exercise: Show that in a cold homogeneous plasma the nonrelativistic 1D equation for nonlinear Langmuir waves becomes linear in Lagrangian variables. Discuss whether a superposition principle applies. Show that relativistic effects keep this equation nonlinear also in Lagrangian variables.

[7] Other cases are e.g., the hodograph transformation in 1D hydrodynamics [see at the end of these notes] or the inverse scattering transform associated to the Korteweg-de Vries (KdV) equation.

Role of nonlinearities: small scale generation

A fundamental difference between linear and nonlinear processes is that linear processes preserve the initially imposed space and time scales[8] while nonlinear processes move energy through the space and time scale spectrum.
Example of the effect in Fourier space of a quadratic nonlinearity

$$k_1, k_2 \;\rightarrow\; k_1 - k_2,\, k_1 + k_2 \qquad \omega_1, \omega_2 \;\rightarrow\; \omega_1 - \omega_2,\, \omega_1 + \omega_2.$$

This is well known in hydrodynamics and in gas dynamics because of the $\mathbf{u} \cdot \nabla \mathbf{u}$ nonlinearity.

[8]Strictly speaking this is only valid for homogeneous conditions, if the system is not homogeneous small scales can be produced also by linear phenomena. A well known example is the phase mixing that occurs when solving Vlasov equation in phase space and that is ultimately responsible for the Landau damping of linear waves.

The energy transfer in time to increasingly small scales leads to the breaking of the waves, or to the formation of shocks when dissipative processes quench this transfer at some small spatial scale (of the order of the particle mean free path). This energy transfer is clearly at work also in collisionless systems where however this transfer of energy to increasingly small scales can be interrupted dynamically because of dispersion effects (in this case energy is reflected, not absorbed)

$$\text{if } k_1, \omega_1 \, \text{satisfy the dispersion relation}, \quad k_1 + k_1 = 2k_1, \, \omega_1 + \omega_1 = 2\omega_1 \, \text{do not.}$$

This may lead to a balance between nonlinear and dispersion effects that can allow for solitary waves (solitons for short) involving dispersive spatial scales (such as the electron skin depth c/ω_{pe} or the electron Debye length $\lambda_{De} = v_{the}/\omega_{pe}$ with ω_{pe} the Langmuir frequency and v_{the} the electron thermal velocity) instead of dissipative scales. A similar balance can also occur between nonlinear and diffraction effects (self-focussing).

The nonlinear transport of energy to small scales may mean the onset of turbulence which is a fundamental phenomenon of nature, but need not be very useful for applications (aside e.g., for faster mixing processes and transport).

But the nonlinear transport of energy to small scales can also mean concentration of energy (in space) or of power (in time) and, if controllable, it can lead to a completely new variety of applications.

In the rest of these lectures I will illustrate a few examples of how relativistic nonlinearities in a plasma can serve this purpose.

Relativistic self-focussing: heuristic derivation

In vacuum or in a linear medium a wave packet with a Gaussian transverse amplitude spreads because of diffraction.

Exercise: Consider a wave packet propagating in vacuum of the form[9] at $t = 0$

$$E_o \mathbf{e_z} \exp\left(-[x^2/L^2 + y^2/D^2] + i k_{x0} x\right) \quad \text{with} \quad L \gg D \quad \text{and} \quad k_0 D > 1,$$

where $\mathbf{e_z}$ is the unit vector along z, L is the length and D the width of the wave packet. Calculate the time evolution of this wave packet, e.g., using a Fourier expansion in plane waves and approximating the wave dispersion relation in the form

$$\omega(k) = c[k_x^2 + k_y^2]^{1/2} \sim c k_x [1 + k_y^2/(2k_x^2)] \sim c k_x [1 + k_y^2/(2k_{x0}^2)].$$

[9] Prove that the magnetic field in this wave packet must have a component along x. This would also be the case for the electric field if $r^2 \equiv y^2 + z^2$ would be substituted for y^2.

Show that its effective transverse width increases with time as

$$[D^2 + 4(ct/k_{x0})^2]^{1/2} = [D^2 + 4(X/k_{x0})^2]^{1/2}$$

where $X(t) \equiv ct$ is the propagation distance along x at time t [see at the end of these notes]. Note that for $X = k_{x0}D^2/2$ the width of the wave packet has doubled: *Rayleigh length.*

Nonlinear effects con either compensate for or increase the diffraction spreading. Relativistic effects in a plasma can compensate for the spreading[10] and allow for focussed laser pulse to propagate without spreading over distances much longer than the Rayleigh length. Let us take for granted for the moment that the "nonlinear" dispersion relation of a large amplitude (circularly polarized) transverse electromagnetic wave in a plasma be given by:

$$\omega^2 = k^2c^2 + \omega_{pe}^2/\gamma,$$

with $\omega_{pe} \equiv [4\pi ne^2/m]^{1/2}$ the Langmuir frequency and $\gamma = (1+a^2)^{1/2}$ the Lorentz factor of an electron in an e.m. wave of dimensionless amplitude $a = eE_0/mc\omega$. In the next lecture a more formal derivation of this result will be given.

[10] G.A. Askar'yan, *Sov. Phys. JETP*, **15**, 8, (1962).

First let us recall that the phase velocity (i.e., the velocity of constant phase planes) of transverse waves in a plasma is superluminous (larger than c) and that it is an increasing function of ω_{pe}.
For a pulse inhomogeneous in the transverse direction, in the local approximation we have $\gamma(a) = \gamma(a(r))$ (see exercise above, r is the transverse coordinate). At the centre of the pulse where the wave amplitude a is larger, electrons appear to be heavier and as a consequence the effective Langmuir frequency squared ω_{pe}^2/γ becomes smaller and thus the phase velocity less superluminous.
As sketched in Fig.(1) this bends the phase planes and focuses the pulse as in a linear optical fiber with index of refraction decreasing with r.
For the sake of simplicity in this heuristic derivation we have taken the electron density to remain uniform. This indeed is not the case because ponderomotive effects (radiation pressure) contribute to the pulse focussing by reducing the electron density where the pulse intensity is larger[11] (pulse self-channelling).

[11] *Exercise*: modify this reasoning and show that on the contrary for a pulse propagating in a neutral gas ionization of the gas induced by the laser pulse leads to spreading and thus adds to its diffraction.

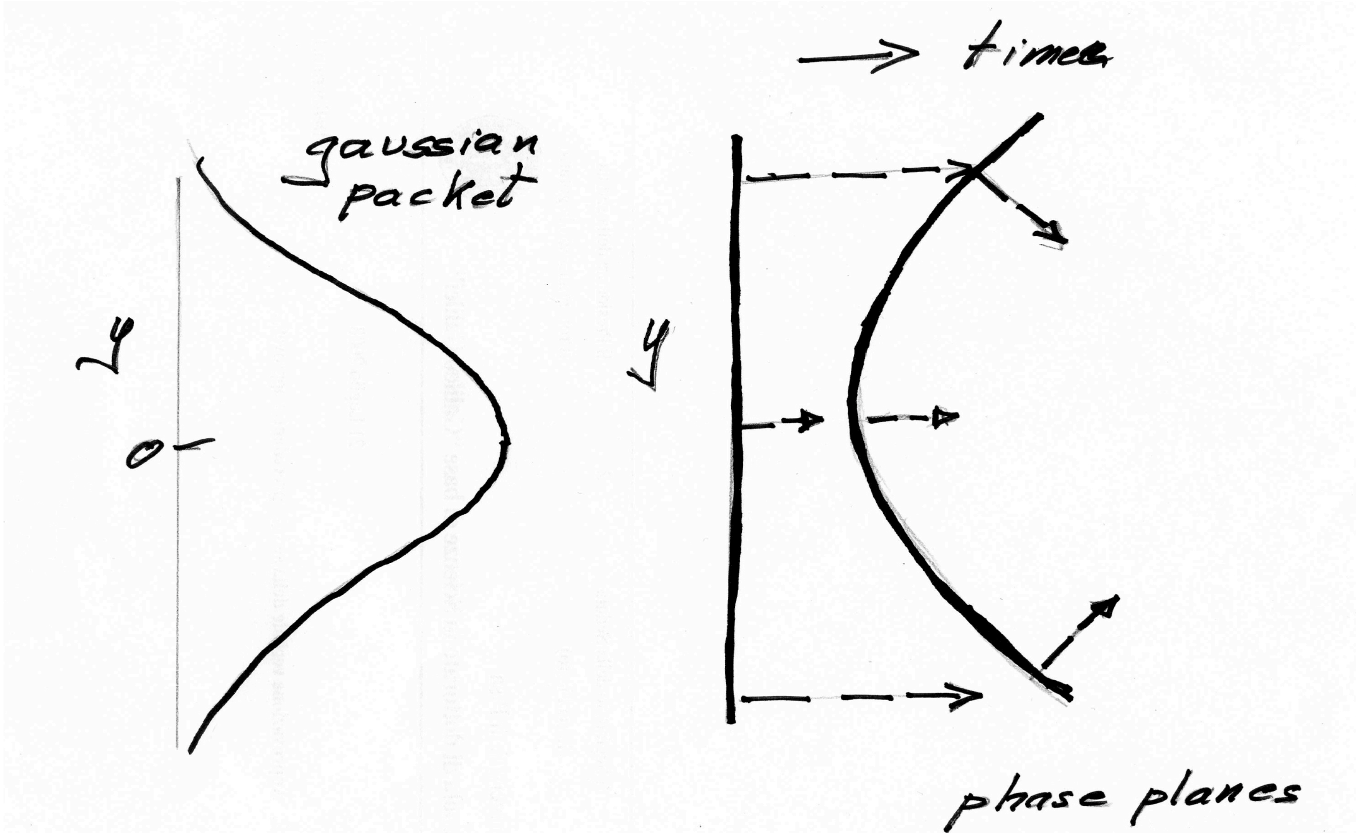


Figure 1: Bending of the phase planes

A simplified equilibrium type argument can be given using an infinitely long pulse and adopting a perturbative approach by expanding $\gamma(a)$ in powers of a^2 up to first order (cubic nonlinearity).
Start from the nonlinear wave equation

$$\nabla^2 a - \frac{\omega^2}{c^2}\left[1 - \frac{\omega_{pe}^2}{\gamma(a)\omega^2}\right] = 0$$

with $1/\gamma(a) \sim 1 - a^2/2$ and a of the (imposed) form

$$a = a_0 \exp -(r^2/D^2)\ [\mathbf{e_y}\cos(kx - \omega t) + \mathbf{e_z}\sin(kx - \omega t)]$$

Expanding the wave equation around $r = 0$ we obtain

$$\omega^2 = k^2c^2 + \omega_{pe}^2, \quad \text{and} \quad D^2/4 = (c^2/\omega_{pe}^2)\,[1/(2a^2)]$$

where c/ω_{pe} is the collisionless electron skin depth[12].

[12]Note that this inverse dependence of the size of the structure on the amplitude of the fields (the strongest, the smallest) is a common feature of nonlinear phenomena.

The latter condition can be expressed in terms of a critical pulse power P_{cr} (integrated over the pulse cross section) for self-focussing to take place: a more accurate treatment gives

$$P_{cr} = (mc^5/e^2)\,(\omega^2/\omega_{pe}^2) \sim 17\,GW\;(\omega^2/\omega_{pe}^2).$$

Well above this threshold the pulse can divide into self focussed filaments.
PIC numerical simulations (see Appendix) can be exploited to illustrate this phenomenon.

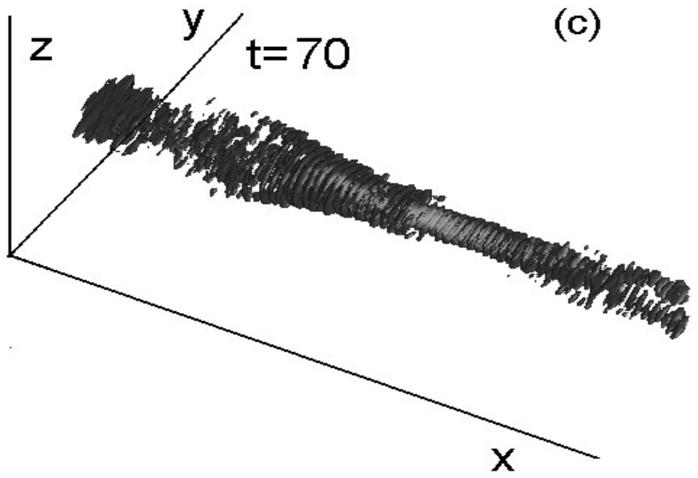


Fig. 1c

Relativistic Mirrors

Relativistic effects in a high energy plasma provide also a matching condition that makes it possible to exchange very effectively ordered kinetic energy and momentum between the e.m. fields and the plasma (provided the plasma is and remains non-transparent).

Radiation

$$\frac{\text{energy} \quad \text{density} \quad \text{flux}}{\text{momentum} \quad \text{density} \quad \text{flux}} \propto c,$$

Non relativistic matter

$$\frac{\text{kinetic} \quad \text{energy} \quad \text{density} \quad \text{flux}}{\text{momentum} \quad \text{density} \quad \text{flux}} \propto v.$$

where v is the matter velocity (assumed $\ll c$).

Suppose radiation momentum is absorbed (or reflected) by matter. Then **the ratio between the (ordered kinetic) energy**[13] **and the momentum gained by matter scales as** v/c. On the contrary it can be shown that the efficiency of laser energy conversion into ordered kinetic energy of matter tends to unity when matter moves at relativistic velocities.

• This property will be exploited in some proposed experimental schemes that use relativistic plasma mirrors.

Under appropriate conditions, the reflection of ultraintense electromagnetic radiation from a thin plasma foil can be exploited in order to push the mirror to relativistic energies.

Viceversa the energy density of a pulse interacting with a counterpropagating relativistic mirror can be amplified as the pulse frequency gets upshifted and the pulse length shortened[14].

[13] The excess energy must be reflected or absorbed e.g., as heat.

[14] A basic point here is the coherent plasma response: photon frequency upshift by inverse Compton scattering from a bunch of relativistic electrons has already been obtained e.g., at SLAC where $10^{18}\,W/cm^2$ laser photons, backscattered by a $46.6\,GeV$ electron beam, interacted with the laser pulse and several electron-positron pairs were detected: D.L. Burke, et al., Phys. Rev. Lett. 79, 1626 (1997).

Direct approach

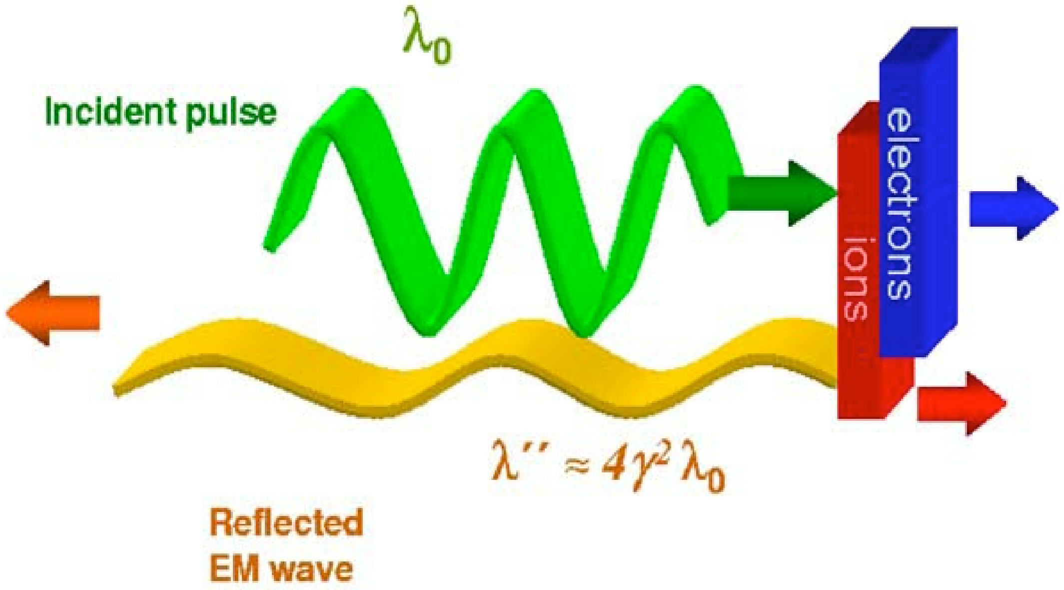
In this approach sheets of plasma are first accelerated to relativistic velocities through the interaction of an ultra intense laser pulse with a plasma, e.g,. through the process of controlled wavebreak of a plasma wake wave excited by the laser pulse. Under appropriate conditions, the electrons in these sheets can reflect coherently a counter-propagating laser pulse[15]. In the process the laser pulse gains energy and at the same time becomes shorter, which corresponds to a further increase in its energy density.

Reverse approach

Conversely the reflection of radiation from a plasma mirror can be exploited in order to push the mirror to relativistic energies at the expense of the laser energy. In the reflection process the laser pulse energy is transferred to the electrons of a thin plasma foil and then to the ions of the foil, through the formation of a

[15] S.V. Bulanov, *et al.*, Phys. Rev. Lett. 91, 085001 (2003).

charge separation electric field so intense as to keep electrons and ions locked together[16].



[16] T. Esirkepov, *et al.*, Phys. Rev. Lett. 92, 175003 (2004).

The mechanism of momentum transfer to charged particles by radiation pressure was first investigated long ago[17]. Particle acceleration by radiation pressure has been considered in high energy astrophysical environments[18].

The reflection properties of an e.m. field structure from a moving mirror can be derived by performing a Lorentz transformation to the reference frame where the mirror is at rest. In this frame the e.m. fields are Fourier transformed with respect to time and the appropriate (frequency dependent) reflection coefficient is used. For a perfect mirror the standard condition that the tangential component on the electric field vanishes in the mirror frame is used.
The form and amplitude of the reflected pulse in the laboratory frame are then obtained by adding the reflected Fourier components and by performing the inverse Lorentz transformation of the resulting e.m. fields.

[17] P. N. Lebedev, *Ann. Phys.*, (Leipzig) **6**, 433 (1901); A.S. Eddington, *MNRAS* **85**, 408 (1925).

[18] P. Goldreich, *Phys. Scripta* **17**, 225 (1978); T. Piran, *ApJ* **257**, L23 (1982); V. S. Berezinskii, *et al.*, *Astrophysics of Cosmic Rays*, (Elsevier, Amsterdam, 1990).

In the case on an electromagnetic wave. the frequency of the reflected wave is given in the laboratory frame by [19]

$$\omega_r = \omega_0 \frac{1 + 2\beta_M \cos\theta_0 + \beta_M^2}{1 - \beta_M^2} \tag{9}$$

where $\beta_M c$ is the mirror velocity and θ_0 is the incidence angle, while the reflection angle is given by

$$\cos\theta_r = \frac{(1 + \beta_M^2)\cos\theta_0 + 2\beta_M^2}{1 + \beta_M^2 + 2\beta_M \cos\theta_0} \tag{10}$$

The wave amplitude transforms according to $E_0/\omega_0 = E_r/\omega_r$: the amplitude of the transverse component of the wave vector potential is the same in both frames.

[19] A. Einstein, *Ann. Phys.* (Leipzig) **17**, 891 (1905).
W. Pauli, in *Theory of Relativity*, Dover Publications, Inc., New York, (1981).

At normal incidence

$$\omega_r = \omega_0 \frac{(1 \pm \beta_M)^2}{1 - \beta_M^2} \tag{11}$$

i.e. for a co-propagating ($\beta_M > 0$) mirror (frequency downshift, the mirror gains energy from the wave)

$$\omega_r = \omega_0 \frac{1 + \beta_M}{1 - \beta_M} \sim 4/\gamma_M^2 \qquad \text{for} \quad \beta_M \to 1, \tag{12}$$

while for a counterpropagating mirror (frequency upshift, the mirror gives energy to the wave)

$$\omega_r = \omega_0 \frac{1 - \beta_M}{1 + \beta_M} \sim \gamma_M^2/4 \qquad \text{for} \quad \beta_M \to -1. \tag{13}$$

In the reflection the number of oscillations inside the e.m. wave is conserved.

How to construct a relativistic mirror

The mechanism envisaged is based on the results derived in S V. Bulanov, *et al.*, Phys. Rev. Lett. **91**, 085001 (2003), where it was shown (analytically and numerically), that when a laser pulse interacts with a *breaking Langmuir wake wave*, part of the pulse[20] is reflected in the form of a highly compressed and focused e. m. pulse with an up-shifted carrier frequency due to the Doppler effect.

The pulse enhancement of the pulse intensity and the pulse compression arise because the electron density modulations in the wake wave act as *parabolic relativistic mirrors*.

[20]The frequency dependent reflection coefficient $\rho(\omega') = -q/(q - i\omega')$ is used. Here ω' is the pulse frequency in the breaking wave frame, $q = 2\omega_{pe}(2\gamma_{ph})^{1/2}$ with γ_{ph} the Lorentz factor corresponding to the phase velocity of the wake wave that takes the place of γ_M in the previous formulae. This coefficient is obtained by solving the wave equation with a "foil current" linear in the vector potential

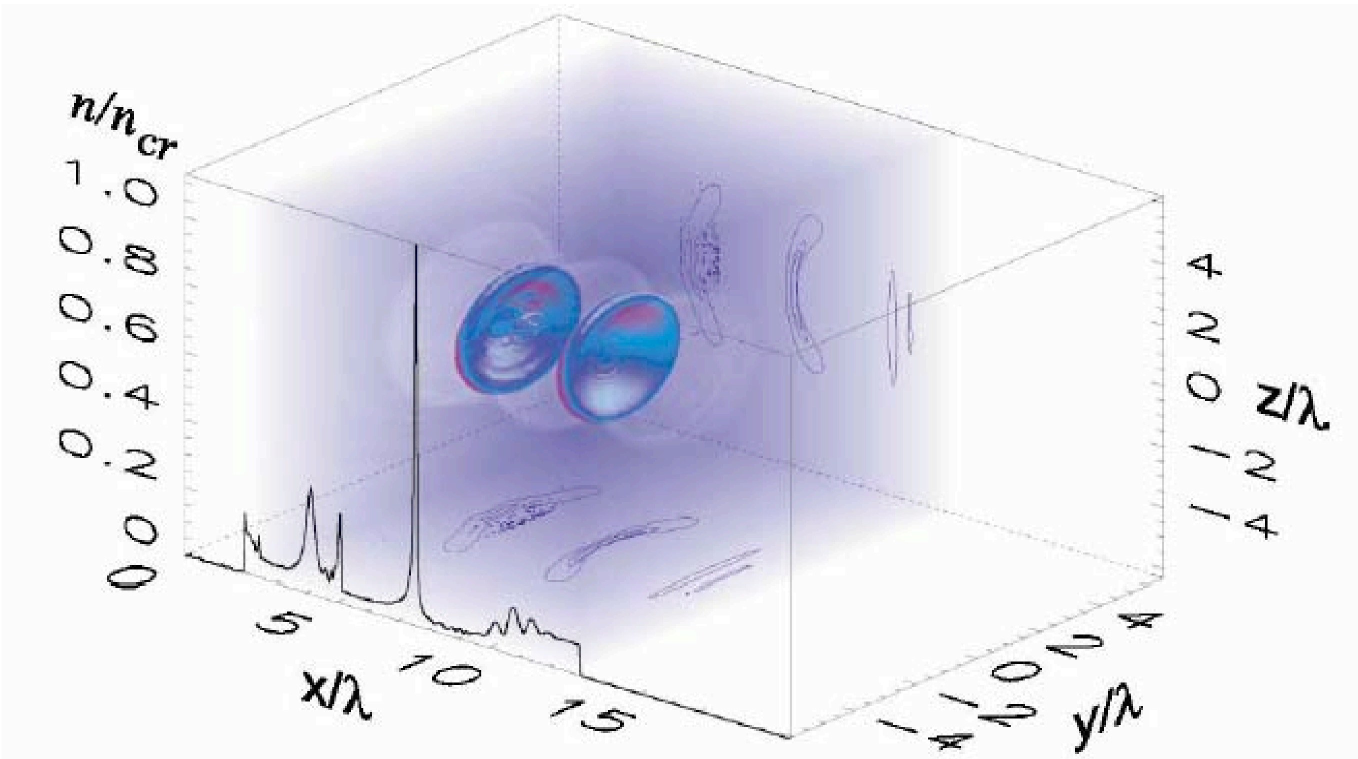


Figure 2: 3D relativistic Langmuir wake wave

Electromagnetic and Langmuir waves with relativistic amplitudes in underdense plasmas

It is well known that constant amplitude solutions of the equations for linear e.m. (transverse) and Langmuir (longitudinal) waves can be written in the form of waves propagating *with constant phase velocity* $v_{ph} \equiv \omega/k$:

$$u_t(x,t) = u_{t0}\cos[kx - (k^2c^2 + \omega_{pe}^2)^{1/2}t] \quad \text{and} \quad u_l(x,t) = u_{l0}\cos[kx - \omega_{pe}t].$$

For finite amplitude waves the frequency depends on the wave amplitude and the superposition principle does not apply.

In the theory of the interaction of high-intensity laser radiation with plasmas the paper A.I. Akhiezer R.V. Polovin, *Sov Phys. JETP* **30,** 915 (1956) has played a key role, searching for nonlinear solutions[21] that depend on $x - v_{ph}t$.

[21] Obviously, in these solutions v_{ph} depends on the wave amplitude. Furthermore these solutions do not form a complete basis since there is no superposition principle.

Here we will apply the relativistic cold plasma equations described before with some change of notation and using the relativistic version[22] of the identity $\mathbf{u} \cdot \nabla \mathbf{u} = \nabla u^2/2 - \mathbf{u} \times (\nabla \times \mathbf{u})$.
Maxwell's equations and cold collisionless plasma equations with ions at rest read

$$\Delta \mathbf{A} - \frac{1}{c^2}\partial_{tt}\mathbf{A} - \frac{1}{c}\nabla\partial_t\varphi - \frac{4\pi e n_e}{m_e c^2 \gamma}(\mathcal{P} + \frac{e}{c}\mathbf{A}) = 0, \tag{14}$$

$$n_e = n_i(x) + \frac{1}{4\pi e}\Delta\varphi, \tag{15}$$

$$\partial_t \mathcal{P} = \nabla(e\varphi - m_e c^2 \gamma) + \frac{1}{\gamma}(\mathcal{P} + \frac{e}{c}\mathbf{A}) \times \mathrm{rot}\mathcal{P}. \tag{16}$$

The continuity equation is implied by equations (14,15). These equations are written in the Coulomb gauge: $\mathrm{div}\ \mathbf{A} = 0\,.$ Here $\mathcal{P}$ is the canonical electron momentum, $\mathcal{P} = \mathbf{p} - e\mathbf{A}/c$, $\mathbf{p} = m_e \gamma \mathbf{p}$ and the relativistic Lorentz factor is $\gamma = \left[1 + (\mathcal{P} + e\mathbf{A}/c)^2/(m_e c^2)\right]^{1/2}$.

[22] *Exercise*: prove the identity $\mathbf{u} \cdot \nabla(\gamma \mathbf{u}) = (1/\gamma)[\gamma \mathbf{u} \cdot \nabla(\gamma \mathbf{u})] = \nabla\gamma\ - \mathbf{u} \times [\nabla \times (\gamma \mathbf{u})]$.

Let us assume that all the variables that characterize the fields and the plasma are independent of y and z (thus $A_x = 0$) and that $\mathcal{P}_y = \mathcal{P}_z = 0$.
Then we can rewrite equations (14)-(16) in components as

$$\partial_{xt}\varphi - 4\pi e n_e p_{||}/m_e c\gamma = 0 \ , \tag{17}$$

$$\partial_{xx}\mathbf{A}_\perp - \partial_{tt}\mathbf{A}_\perp - (4\pi e^2 n_e/m_e c^2\gamma)\mathbf{A}_\perp = 0 \ , \tag{18}$$

$$n_e = n_i(x) + \partial_{xx}\varphi/4\pi e, \tag{19}$$

$$\partial_t p_{||} + \partial_x(e\varphi - m_e c^2\gamma) = 0, \tag{20}$$

where $\gamma = [1 + (e\mathbf{A}_\perp/m_e c^2)^2 + (p_{||}/m_e c)^2]^{1/2}$.

Subscripts $||$ and $\perp$ denote the components of the vectors along and perpendicular to the x-axis while $e|\mathbf{A}_\perp|/(m_e c^2)$ corresponds to the normalized wave amplitude a used before.

Assume homogeneous ion density and wave propagating with constant velocity v_{ph}. Look for solutions that depend on the variable $X = x - v_{ph}t$. We obtain for the electron density

$$n_e = \frac{n_i m_e v_{ph} \gamma}{m_e c \beta_{ph} \gamma - p_{||}}, \tag{21}$$

and the two coupled equations for $\gamma = \gamma(p_{||}, |\mathbf{A}_\perp|)$ and $\mathbf{A}_\perp$

$$(\beta_{ph} p_{||} - m_e c \gamma)'' - \frac{\omega_{pe}^2 p_{||}}{(m_e c \beta_{ph} \gamma - p_{||}) c^2} = 0, \tag{22}$$

$$\mathbf{A}_\perp'' + \frac{\omega_{pe}^2 \beta_{ph} \gamma_{ph}^2}{(\beta_{ph} \gamma - p_{||}) c^2} \mathbf{A}_\perp = 0 \, . \tag{23}$$

Here $\beta_{ph} = v_{ph}/c$, $\gamma_{ph}^2 = (1 - \beta_{ph}^2)^{-1}$ (γ_{ph}^2 is negative for superluminous waves) and a prime denotes a differentiation with respect to the variable X.

Longitudinal Relativistically Strong Waves in Cold Plasmas

Assuming the transverse components of electron momentum to be zero which implies $\mathbf{A}_\perp = 0$ and $\gamma = \gamma(p_{||})$, we obtain from equation (22):

$$[(\beta_{ph} p_{||}/m_e c - \gamma)']^2/2 = (\omega_{pe}^2/c^2)\,(\gamma_m - \gamma)\,, \tag{24}$$

where $\gamma_m = [1 + (p_m/m_e c)^2]^{1/2} = 1/(1 - \beta_m^2)^{1/2}$ is an integration constant, and $\beta_m c$ the maximum value of the electron velocity in the longitudinal wave: $-\beta_m \leq \beta = p_{||}/m_e c\gamma \leq \beta_m$. Integrating equation (24) we obtain an implicit solution for $X = X(\gamma)$

$$(2\beta_{ph})^{1/2}(\omega_{pe}/c)X = (\gamma_m - \gamma)^{1/2} - 2\beta_{ph}\left[\mathcal{F}\left(\Psi, \kappa\right) - (\gamma_m + 1)\mathcal{E}\left(\Psi, \kappa\right)\right], \tag{25}$$

where $\mathcal{F}\left(\Psi, \kappa\right)$ and $\mathcal{E}\left(\Psi, \kappa\right)$ are the incomplete elliptic integrals of the first and second kind, $\Psi = \mathrm{arcsinh}[(\gamma_m - \gamma)^{1/2}/(\gamma_m + 1)^{1/2}]$ their argument and $\kappa = [(\gamma_m - 1)/(\gamma_m + 1)]^{1/2}$ their modulus.

In a relativistically strong Langmuir wave the electric field depends on the coordinate X through the relationship

$$E = \frac{m_e \omega_{pe} c}{e} \left[2(\gamma_m - \gamma)\right]^{1/2}. \tag{26}$$

The maximum electric field is at the point where $p_{||}(X) = 0$. The expression for γ given by equation (25) is periodic in X. For the wave frequency we obtain

$$\omega = \pi \omega_{pe} / \left[2\left(K\left(\kappa\right) - (\gamma_m + 1) E\left(\kappa\right)\right)\right], \tag{27}$$

where $\mathcal{K}\left(\kappa\right)$ and $\mathcal{E}\left(\kappa\right)$ are the complete elliptic integrals.
The wave frequency does not depend on the phase velocity of the wave (cold plasma).

Simple formulae for the frequency: small and large amplitudes of the wave. In the small amplitude case when $p_m \ll m_e c$, the frequency is

$$\omega = \omega_{pe}\left[1 - 3(p_m/4m_e c)^2\right], \tag{28}$$

which corresponds to the nonlinear shift of the frequency. In the large amplitude case case when $\gamma_m \gg 1$ the frequency is[23]

$$\omega = \pi\omega_{pe}/(\gamma_m^{1/2} 2^{3/2}). \tag{29}$$

Rewriting this expression in terms of the maximum value of the electron momentum, $p_m = m_e c(\gamma_m^2 - 1)^{1/2}$, we obtain that the period of the wave is $T = 2\pi/\omega = 4(p_m/2m_e c)^{1/2}/\omega_{pe}$.

[23]Note the ω_{pe}^2/γ_m behaviour which is responsible for the "parabolic" shape od the Langmuir wake waves mentioned before.

The expression for the electron density given by equation (21) becomes singular when *the maximum velocity of the electrons in the wave becomes equal to the wave phase velocity*, i.e., when $\gamma = \gamma_{ph} = (1 - \beta_{ph}^2)^{-1/2}$.
This corresponds to the so called **wave breaking** (crossing of the fluid element trajectories).

Close to the wave-breaking limit when, $c\beta_m \to v_{ph}$ *the maximum of the electron density tends to infinity while the width of the density spike tends to zero*.

For $c\beta_m = v_{ph}$, from equations (21) and (25) we obtain that the electron density in the spike tends to infinity as $X \to 0$ as

$$n(X)/n_0 = 2^{1/3}\gamma_m(3\omega_{pe}|X/|c\beta_m)^{-2/3} + \dots . \tag{30}$$

Note the characteristic cusp-like pattern that appears in phase plane:

$$p_{||}(X)/m_e c \simeq \beta_m \gamma_m [1 - (3\omega_{pe}|X|/c\beta_m)^{2/3}] + \dots . \tag{31}$$

Integrating the electron density (30) in the neighbourhood of the singularity we find that the total number of particles in the density spike is finite.

The wave breaking imposes a constraint on the maximum value of the electric field in the wave:

$$E_m = \frac{m_e \omega_{pe} c}{e} [2(\gamma_{ph} - 1)]^{1/2} \tag{32}$$

which is *the Akhiezer–Polovin limiting electric field*[24].

[24]This electric field is used in the experiments of electron laser wake field acceleration. This subject is presently of great interest but is not treated in these lectures.

Transverse Relativistically Strong Electromagnetic Waves

For a purely transverse **circularly polarized** electromagnetic wave, from equation (23) with $p_{\|} = 0$ we find that the amplitude of the transverse component of the vector potential $A_\perp$, $\mathbf{A}_\perp = A_y + iA_z = A_\perp \exp(i\omega X/v_{ph})$[25], is constant and the frequency is

$$\omega^2 = -\frac{\omega_{pe}^2 \beta_{ph}^2 \gamma_{ph}^2}{[1 + (eA_\perp/m_e c^2)^2]^{1/2}}. \tag{33}$$

In the x, t-coordinates this corresponds to the dispersion equation for the frequency and wavenumber[26]

$$\omega^2 = k^2 c^2 + \frac{\omega_{pe}^2}{[1 + (eA_\perp/m_e c^2)^2]^{1/2}}. \tag{34}$$

[25] $\exp(i\omega X/v_{ph}) \equiv \exp(i\omega x/v_{ph} - i\omega t) \equiv \exp(ikx - i\omega t)$

[26] This result proves the nonlinear dispersion relation used in the previous lecture to illustrate the phenomenon of relativistic self-focusing: $[1 + (eA_\perp/m_e c^2)^2]^{1/2} = (1 + a^2)^{1/2} = \gamma(a)$.

The dispersion equation (34) can be rewritten in the form

$$k = \frac{[\omega^2(1 + (eA_\perp/m_e c^2)^2)^{1/2} - \omega_{pe}^2]^{1/2}}{c[1 + (eA_\perp/m_e c^2)^2]^{1/4}}.$$

The electromagnetic wave can propagate (*relativistic transparency*) in an overdense plasma, where $\omega \ll \omega_{pe}$, provided

$$\omega \gg \omega_{pe}/[1 + (eA_\perp/m_e c^2)^2]^{1/4}. \tag{35}$$

The e.m. fields in the wave are $E = \omega A_\perp/c$ and $B = c\omega A_\perp/v_{ph}$.
The wave velocity is greater than the speed of light in vacuum:

$$v_{ph} = c\,[1 - \omega_{pe}^2/\omega^2(1 + (eA_\perp/m_e c^2)^2)^{1/2}]^{-1/2}. \tag{36}$$

Langmuir Wave Excitation

Consider a circularly polarized laser pulse with dimensionless amplitude a, propagating in an underdense plasma ($\omega_0 \gg \omega_{pe}$) along the x-axis.
Write the dimensionless vector potential a in the form

$$a(X,t)\exp(-i\omega_0 t - ik_0 x) + c.c..$$

The complex amplitude $a(X,t)$ is a function of the variables t and $X = x - v_g t$ with v_g the group velocity $v_g = c^2 k_0/\omega_0$. Assume the ions to be at rest.
Assume that the change in time of $a(X,t)$ and of the dimensionless electrostatic potential $\phi(X,t)$ are slow ($\partial/\partial t \ll c\partial/\partial X$) and that $v_g \approx c$.
From the relativistic hydrodynamic equations and from Maxwell's equations (14-16) we obtain a system of coupled equations[27]

[27] See e.g., S.V. Bulanov, *et al*, *JETP Lett.* , **50**, 176 (1989); S. V. Bulanov, *et al*, *Sov. J. Plasma Phys.* **16**, 543 (1990).

$$2i\omega_0\frac{\partial a}{\partial t}+\left(\frac{\omega_{pe}}{\omega_0}\right)^2 c^2\frac{\partial^2 a}{\partial X^2}+2v_g\frac{\partial^2 a}{\partial X\partial t}=\left(\frac{\omega_{pe}}{\omega_0}\right)^2\frac{\phi}{1+\phi}a, \tag{37}$$

$$\frac{\partial^2\phi}{\partial X^2}=\frac{k_p^2}{2}\left[\frac{1+|a|^2}{(1+\phi)^2}-1\right], \tag{38}$$

where $k_p=\omega_{pe}/v_g$.
Here the vector potential a is normalized on $m_e\omega_0 c/e$ and the electrostatic potential ϕ is normalized on $m_e c^2/e$.
If $a=0$, equation (38) describes free Langmuir oscillations in the limit $v_{ph}=c$ considered above (see equation (24)).

If for the sake of reasoning, the laser pulse is assumed to be given as a square-pulse profile with amplitude a_0 ($|a|^2=a_0^2$ at $L<X<0$, and $|a|^2=0$ at $X<L, X>0$), equation (38) can be solved analytically.

In the region occupied by the pulse in terms of elliptic functions we have

$$k_pX = -2(1+a_0^2)^{1/2}E\left\{\arcsin\left[\left(\frac{(1+a_0^2)\phi}{a_0^2(1+\phi)}\right)^{1/2}\right], \frac{a_0}{(1+a_0^2)^{1/2}}\right\}$$

$$+2\left(\frac{\phi(a_0^2-\phi)}{1+\phi}\right)^{1/2}. \tag{39}$$

By matching this solution with the solution for the free plasma wave we obtain that the typical value of the electrostatic potential in the plasma wave is

$$\varphi_{max} \approx m_e c^2 a_0^2/e. \tag{40}$$

The optimal pulse length is $L = 2(1+a_0^2)^{1/2}E\left(a_0/(1+a_0^2)^{1/2}\right)$.

The corresponding wake field has a wavelength equal to $\lambda = 2^{3/2}|a_0|k_p^{-1}$, and the maximum electric field (before breaking, it can be higher in a breaking wave) is

$$E = m_e c^2 \omega_{pe} a_0^2/(1 + a_0^2)^{1/2} e. \tag{41}$$

The maximum electron energy in the wake wave is given by

$$\frac{1}{2}\left(\frac{d\phi}{dX}\right)^2 = \frac{k_p^2}{2}\frac{\varphi(a_0^2 - \phi)}{1 + \phi}. \tag{42}$$

Here the constant of integration is chosen such that there is no wake wave before the laser pulse. We see that the potential inside the laser pulse varies between zero and $m_e c^2 a_0^2/e$. Behind the laser pulse, for an optimal pulse length, the electrostatic potential also scales as a_0^2.

In the case of a laser pulse with amplitude $a > (m_i/m_e)^{1/2}$, which corresponds to the petawatt power range, ions can no longer be considered to remain at rest. The modifications of the wake field generated by a sufficiently short laser pulse with $a \approx (m_i/m_e)^{1/2}$ propagating in an underdense plasma (see for comparison equations (22) and (38)) are given by

$$\frac{d^2\phi}{dX^2} = \frac{\gamma_{ph}^3 \beta_{ph} (1+\phi)}{\left[\gamma_{ph}^2 (1+\phi)^2 - (1 + a^2(X))\right]^{1/2}} \qquad (43)$$

$$- \frac{\gamma_{ph}^3 \beta_{ph} (\mu - \phi)}{\left[\gamma_{ph}^2 (\mu-\phi)^2 - (\mu^2 + a^2(X))\right]^{1/2}},$$

where $\mu = m_i/m_e$.

We assume the (circularly polarized) laser pulse to be given. The effect of the ion motion restricts the potential ϕ between the two bounds (in these estimates we assume $\beta_g \to 1$)

$$-1 < \phi < \min\{\mu, a_m^2\}. \tag{44}$$

From equation (43) we can also find that behind a short laser pulse with optimal length $l = 2^{1/2}/a_m$, λ_w and the maximum value of the electric field E_w and of the potential ϕ_w scale for $1 < a_m < \mu^{1/2}$, as

$$\lambda_w = 2^{3/2} a_m, \quad E_w = a_m/2^{1/2}, \quad \phi_w = a_m^2,$$

and, for $a_m > \mu^{1/2}$, as

$$\lambda_w = 2^{1/2}\mu/a_m, \quad E_w = a_m/2^{1/2}, \quad \phi_w = \mu$$

For $a_m > \mu^{1/2}$ the wake field wavelength decreases with increasing laser pulse amplitude while the value of the electrostatic potential does not change.

Relativistic Solitons

Solitary structures (solitons for short) arise from the balance between nonlinear and dispersion effects. *Among nonlinear modes, solitons are of fundamental importance for basic nonlinear science in a wide variety of fields.*

Nonlinear one-dimensional (1-D) relativistic solitons in a plasma have been studied analytically and numerically[28]

[28] J.H. Marburger and R.F. Tooper, *Phys. Rev. Lett.*, **35**, 1001 (1975); N.L. Tsintsadze, D.D. Tskhakaya, *JETP*, **45**, 252 (1977); V.A. Kozlov, A.G. Litvak, and E.V. Suvorov, *Sov. Phys. JETP*, **76**, 148 (1979); M.Y. Yu, P.K. Shukla and N.L. Tsintsadze, *Phys. Fluids*, **25**, 1049 (1982); R.N. Sudan, Ya.S. Dimant, and O.B. Shiryaev, *Phys. Plasmas*, **4**, 1489 (1997); T.Zh. Esirkepov, F.F. Kamenets, S.V. Bulanov, N.M. Naumova, *JETP Lett.* . **68**, 36, (1998); S.V. Bulanov, T.Zh. Esirkepov, F.F. Kamenets, and N.M. Naumova, *Plasma Phys. Rep.*, **21,** 600 (1995); F. Cattani *et al.*, *Phys. Rev.*, **E 64**, 016412 (2001); D. Farina, S.V. Bulanov, *Plasma Phys. Contr. Fusion*, **47**, A73 (2005); V. Saxena, A. Das, S. Sengupta, P. Kaw, A. Sen, *Phys. Plasmas*, **14**, 072307 (2007), and references therein.

2D and 3D (subcycle) solitons have been found with PIC simulations[29] and "detected" experimentally[30] (+ analytical solutions expanding in powers of a). Let us return to the relativistic electron equations (14-16), assume the electromagnetic wave to be circularly polarized, introduce the new coordinates $X = x - v_s t$, and $\tau = t$, and look for solutions of the form[31]

$$\mathbf{A}_\perp = A_y + iA_z = A(X) \exp\{i\omega[(1 - v_s^2/c^2)\tau - v_s X/c^2]\}, \tag{45}$$

$$p_{||}/m_e c = \beta_s b(X)\,. \tag{46}$$

[29] S. V. Bulanov, T. Zh. Esirkepov, N. M. Naumova, *et al., Phys. Rev. Lett.* **82**, 3440 (1999); ,S.V. Bulanov, F. Califano, T.Zh. Esirkepov, *et al., J. Plasma Fusion Research* **75**, 506 (1999); Y. Sentoku, T.Zh. Esirkepov, K. Mima, *et al., Phys. Rev. Lett.* **83**, 3434 (1999); N.M. Naumova, *et al., Phys. Rev. Lett.* **87**, 185004 (2001); T. Esirkepov, K. Nishihara, S.V. Bulanov, F. Pegoraro, *Phys. Rev. Lett.*, **89**, 275002 (2002).

[30] M. Borghesi, *et al., Phys. Rev. Lett.* **88**, 135002 (2002).

[31] $\exp\{i\omega[(1 - v_s^2/c^2)\tau - v_s X/c^2]\} = \exp[i\omega(\tau - v_s x/c^2)]$.
Note that for linear transverse e.m. waves in a plasma $v_g v_{ph} = c^2$, with v_g the group velocity.

Inserting expressions (45,46) into (17-20) and assuming the ion density to be homogeneous we obtain

$$\left(\gamma - \beta_s^2 b\right)'' = \frac{\omega_{pe}^2 b}{(\gamma - b)c^2}, \tag{47}$$

$$a'' + \frac{\omega^2}{c^2} a = \frac{\omega_{pe}^2 \gamma_s^2}{(\gamma - b)c^2} a, \tag{48}$$

where $\gamma = \left(1 + a^2 + \beta_s^2 b^2\right)^{1/2}$, $\gamma_s^2 = \left(1 - \beta_s^2\right)^{-1}$, $\beta_s = v_s/c$, $a = eA/m_e c^2$ and a prime denotes a differentiation with respect to the variable X.

The system of equations (47,48), with boundary conditions[32] $a(\infty) = b(\infty) = 0\,, \quad a(X) < \infty\,, \quad b(X) < \infty$, describes a one-dimensional relativistic electromagnetic soliton propagating through a cold collisionless plasma.

[32] Constant amplitude e.m. transverse waves are recovered by taking $a'' = b = 0$.

The soliton speed and frequency are smaller than the speed of light and of the electron plasma frequency: $\beta_s < 1$ and $\omega < \omega_{pe}$. In the case $v_s = 0$, $p_{||}$ vanishes and we have

$$a'' + k_p^2[(\omega/\omega_{pe})^2 - (1 + k_p^2\gamma'')/\gamma]a = 0\,, \tag{49}$$

where $k_p = \omega_{pe}/c$. With the help of the substitution $a = \mathrm{sh}u$, $\gamma = \mathrm{ch}u$ this equation can be transformed into

$$u'' = k_p^2 \mathrm{sh}\, u\; [1 - (\omega/\omega_{pe})^2\, \mathrm{ch}u]. \tag{50}$$

which leads to a subcycle structure

$$a(X,\tau) = \frac{2[1-(\omega/\omega_{pe})^2]^{1/2}\ \cosh\left[k_p^2 X\left(1-(\omega/\omega_{pe})^2\right)^{1/2}\right]\exp(i\omega\tau)}{\cosh^2\left[k_p^2 X\,(1-(\omega/\omega_{pe})^2)^{1/2}\right] + 1 - (\omega/\omega_{pe})^2}\quad . \tag{51}$$

The relationship between the soliton amplitude a_m, which is equal to $a_m = a(0,0)$, and the soliton frequency ω is given by

$$a_m = 2\omega_{pe}\left(\omega_{pe}^2 - \omega^2\right)^{1/2}/\omega^2 \ . \tag{52}$$

Ion motion has important effect on the propagating envelope (multi-humped) solitons and single cycle solitons. It can be shown[33] (in the framework of two fluid cold equations) that no solution can be found for propagation velocities v_s smaller than a critical value $v_{s,cr}$
The non propagating solution given in Eq.(51) is not continuously connected to those with $v_s \neq 0$: its structure will change on the ion dynamical time.

[33] D. Farina, S.V. Bulanov, Phys. Rev. Lett. 86, 5289 (2001);
D. Farina, S.V. Bulanov, Plasma Phys. Rep. 27, 641 (2001)

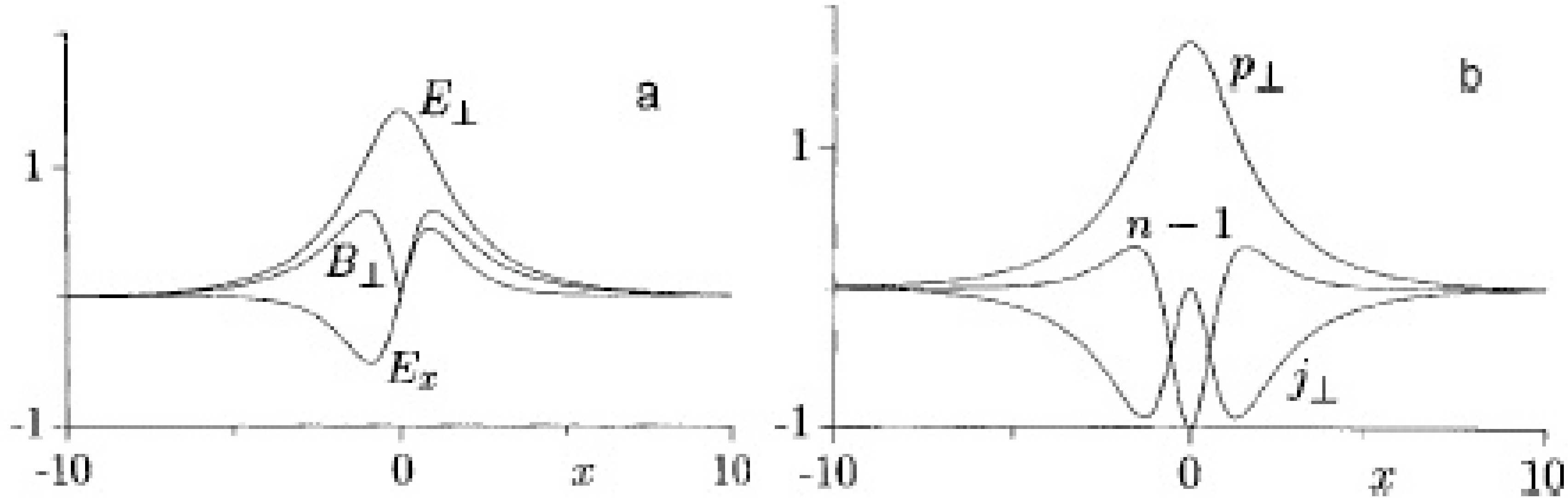


FIG. 1. Electromagnetic fields and the momentum and density of electron fluid in the soliton. a — Fields $E_\perp$, $B_\perp$, E_x. b — Transverse momentum $p_\perp$ and density n of the electron fluid and the transverse current $j_\perp$.

Figure 3: One dimensional, sub-cycle soliton

Higher dimensions: numerical simulations

These results have been derived using idealized low-dimensionality models. At higher dimensions these investigations are technically and sometimes conceptually (example: topology of field lines in 3D solitons) much more difficult and exact solutions are seldom available. *Numerical simulations play a fundamental role in the analysis of regimes that are outside the reach of most analytical developments because of their high dimensionality and because of their full nonlinear dynamics.*

Simulations here are not only used for validating analytical models, but also as an investigative tool for discovering new phenomena. This simulation analysis must be accompanied by the development of appropriate semantics that can only be obtained from a physical understanding based on the extrapolation of simplified, lower dimensionality, models.

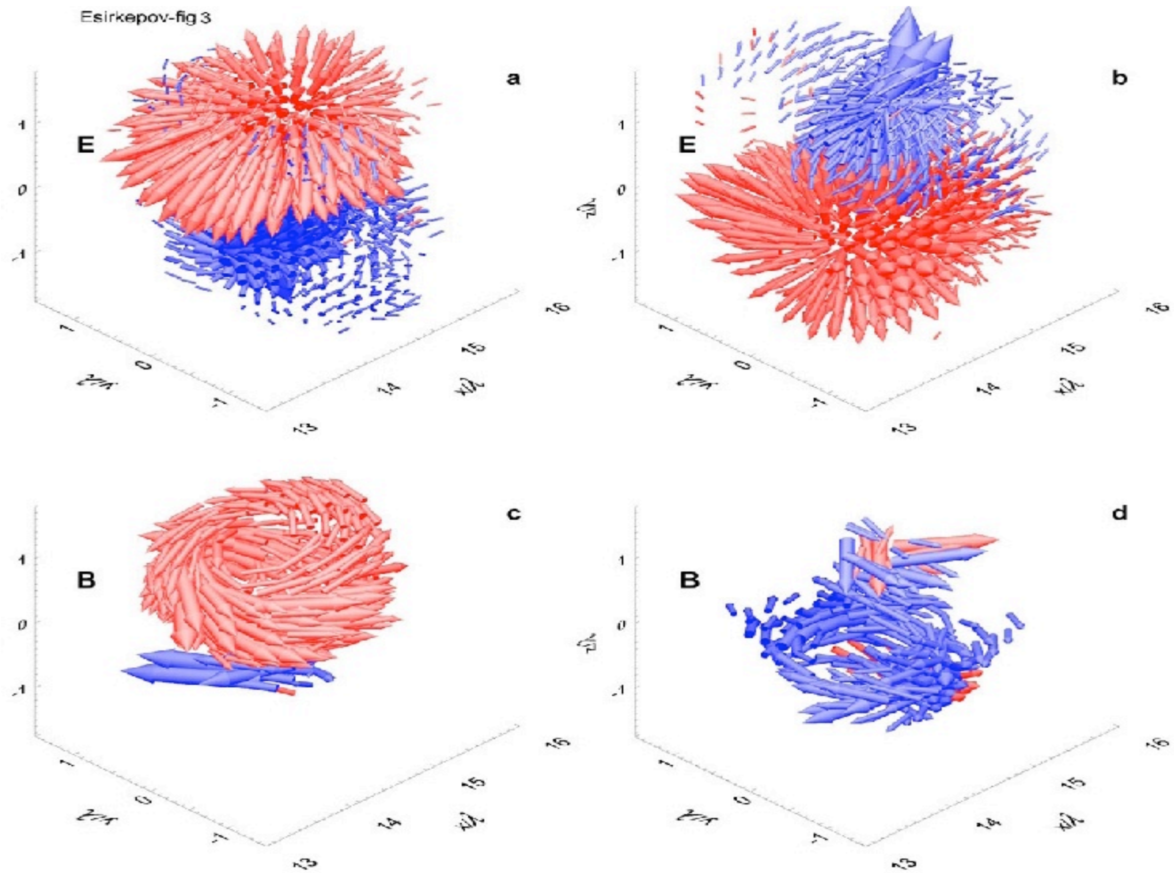


Figure 4: Three dimensional soliton topology

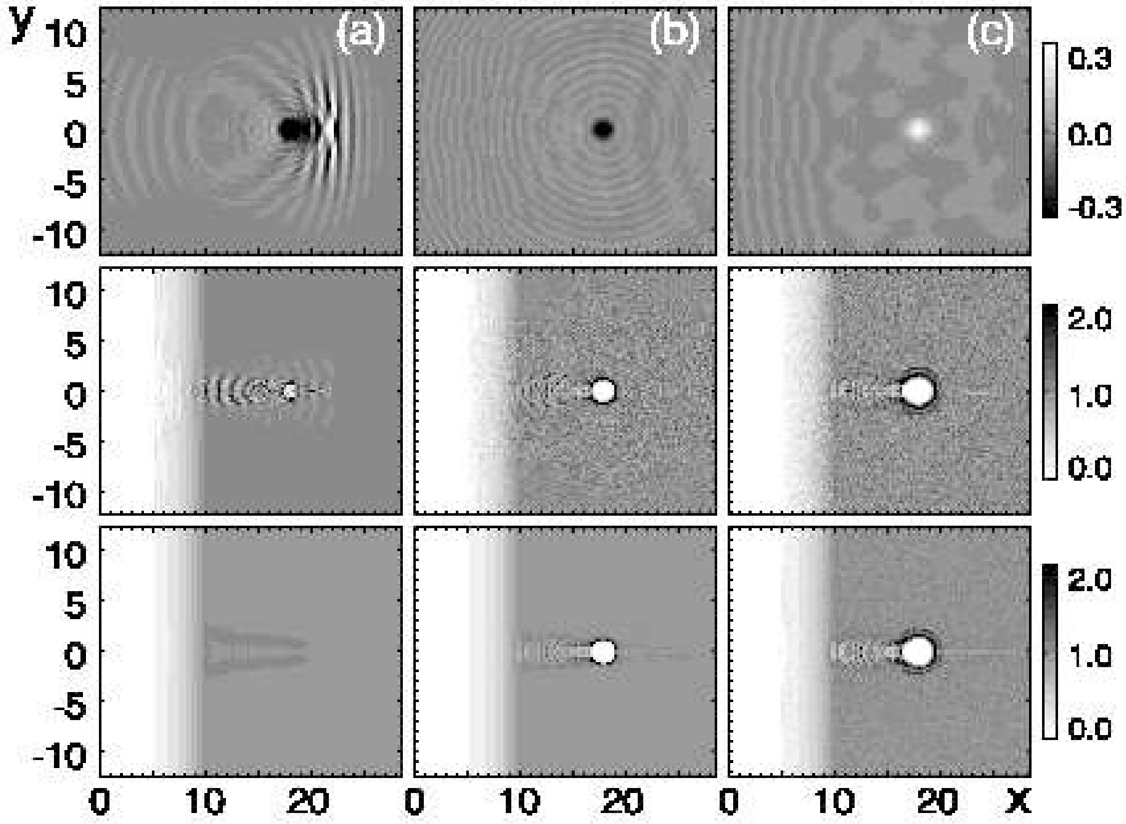
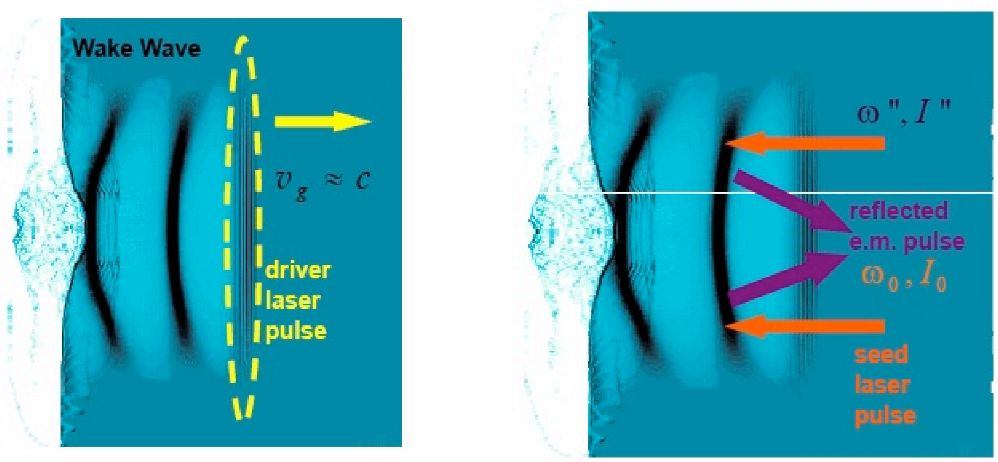


Figure 5: 2D (post)soliton

EM Pulse Intensification and Shortening in Flying Mirror Light Intensification Process



$$\omega'' = \frac{1+v_{ph}/c}{1-v_{ph}/c}\,\omega \approx 4\gamma_{ph}^2\omega_0$$

$$I''_{\max} \approx \kappa(\gamma_{ph})\,\gamma_{ph}^6\left(\frac{D}{\lambda}\right)^2 I_0$$

$$\kappa(\gamma_{ph}) \sim \gamma_{ph}^{-3}$$

Figure 6: 2D wake

Soliton reflection by a relativistic mirror

Non propagating relativistic solitons trap e.m. energy by surrounding it by plasma walls $\omega < \omega_{pe}$: the radiation pressure of the trapped e.m. fields acts on the electrons that are kept in place by the charge separation field on time scales short with respect to the ion dynamics. On longer time scales the ions start to move and the solitons expand (post-solitons) as shown in Fig.(5).
It possible to manipulate the energy density inside the soliton by exploiting the same mechanism of energy density enhancement described above where we have shown that when a laser pulse interacts with a breaking wake plasma wave part of the pulse is reflected in the form of a highly compressed and focused e.m. pulse with an up-shifted carrier frequency.

The idea is simply to substitute a soliton for the incoming pulse[34].

[34] A.V. Isanin, *et al.*, *Phys. Lett.* **A 337**, 107 (2005); S.S. Bulanov, *et al.*, *Phys. Rev.* E **73**, 036408 (2006).

One-dimensional analytical model

As already mentioned, the reflection of one-dimensional coherent structures can be derived by performing a Lorentz transformation to the reference frame where the wake plasma wave is at rest. In this frame the e.m. fields are Fourier transformed with respect to time and a frequency dependent reflection coefficient is used[35]. The form and amplitude of the reflected pulse in the laboratory frame can obtained by adding the reflected Fourier components and by performing the inverse Lorentz transformation of the resulting e.m. fields.

The main conclusion of this one dimensional analysis is that in a tenuous plasma the frequency up-shift of the reflected pulse, and its related compression, would be so large that it could lead to the generation of attosecond pulses[36].

[35]In this model the mirror is taken not to be deformed by the interaction with the coherent structure in a sort of "test particle approach" where the energy of the mirror, of the wake wave that produces it, is as large as required.

[36]attosecond $= 10^{-18}$ seconds. Note that ~ 24 attoseconds is the time taken by an electron to travel from "one side of a hydrogen atom to the other".

Numerical documentation

To take into account the effects of multi-dimensional geometry and strongly nonlinear plasma dynamics, as well as the influence of kinetic effects, two dimensional simulations were performed[37].

The grid mesh size is $\lambda_d/20$; space and the time unit is λ_d and $2\pi/\omega_d$. Here λ_d and ω_d is the driver laser wavelength and frequency, respectively. The electric and magnetic field components are normalized to $m_e\omega_d c/e$ and the electron density is normalized to the critical density $n_{cr} = m_e\omega_d^2/4\pi e^2$.
The ions are assumed to form an immobile neutralizing background and thus only the electron motion is taken into account. This approximation is applicable because the typical interaction period is much shorter than the ion response time.
The boundary conditions are absorbing for the e. m. field and the quasi-particles. The interaction of a wake wave with a soliton is simulated in a box with size $60\lambda_d \times 40\lambda_d$, including the absorbing edges of thickness $3\lambda_d$.

[37] S.S. Bulanov, *et al.*, *Phys. Rev.* E **73**, 036408 (2006).

A single relativistic e. m. sub-cycle soliton is generated by an auxiliary laser pulse with wavelength $\lambda_a = 2\lambda_d$ and dimensionless amplitude $a_a = 0.5$, corresponding to the peak intensity $a_a^2 \times I_1$, where $I_1 = 1.37 \times 10^{18}$ W/cm^2 $\times$ $(1\ \mu m/\lambda_d^2)$. The pulse is Gaussian with FWHM size (length×waist) $4\lambda_d \times 6\lambda_d$. The auxiliary laser pulse is linearly polarized with its electric field along the z-axis; it is generated at the bottom boundary at $t = 0$ and propagates along the y-axis at $x = 20$.

The plasma wakefield, which interacts with the soliton, is formed by a Gaussian laser pulse, the driver pulse, with amplitude $a_d = 1.5$ and FWHM size $2\lambda_d \times 12\lambda_d$, starting at time $t = 45$ from the left boundary and propagating along the x-axis. The driver laser pulse is linearly polarized, its electric field is directed along the y-axis. The plasma slab occupies the region $5 \leq x \leq 35$, $5 \leq y \leq 35$; it is homogeneous in the direction of the y-axis and it has convex parabolic slopes along the x-axis from $x = 5$ to 11 and from 29 to 35. This plasma-vacuum interface profile is chosen so as to make the laser pulse entrance into the plasma smoother and to avoid a fast wake wave breaking. The electron density at the center of the plasma slab is $n_e = 0.09 n_{cr}$, corresponding to the Langmuir frequency $\omega_{pe} = 0.3$. The number of quasi-particles is 3.24×10^6. The phase velocity of the wakefield when it starts to interact with the soliton is $v_{ph} \approx 0.925$, corresponding to the Lorentz factor $\gamma_{ph} \approx 2.63$.

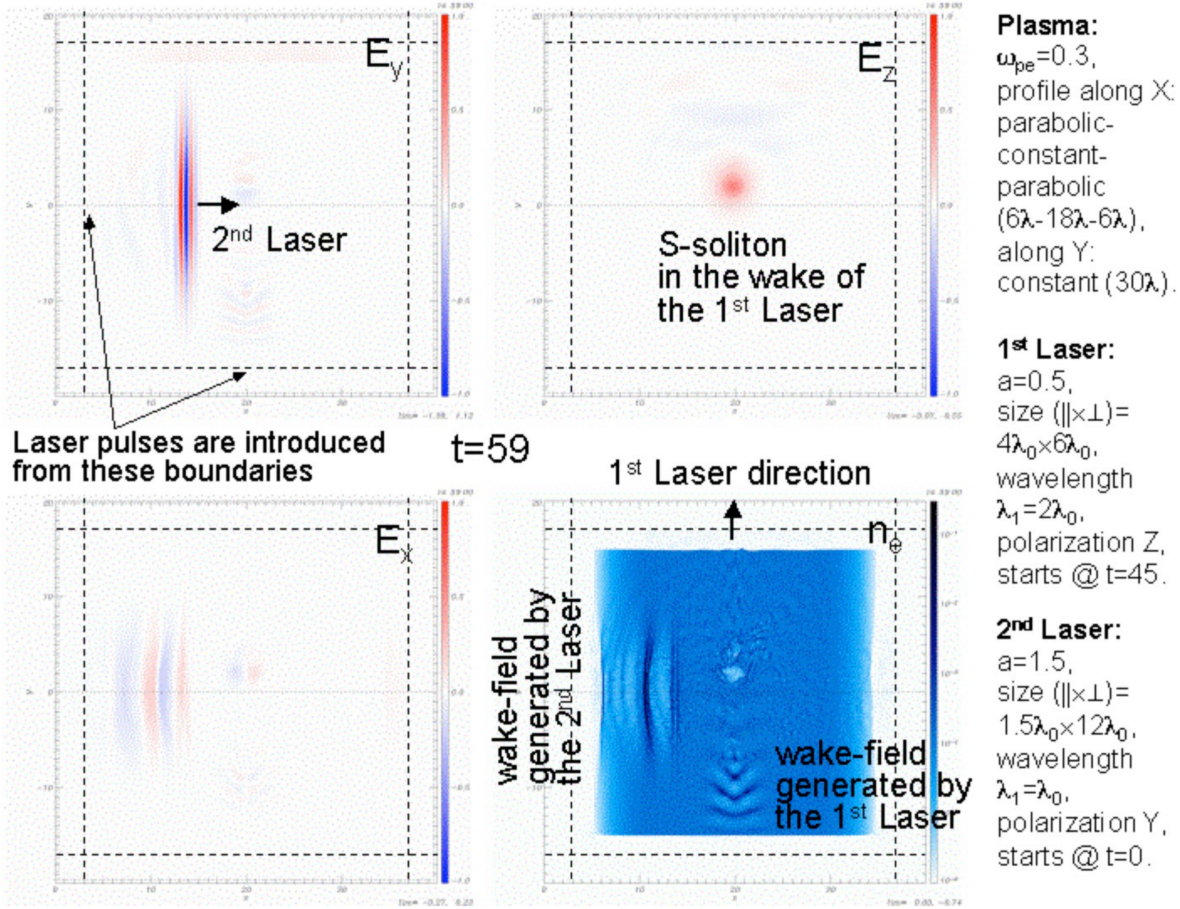


Figure 7: Mirror-soliton interaction geometry

Fig.(7) shows a portion of the simulation box shortly before the interaction.

The auxiliary laser pulse has already gone through the box: in its wake we see a single s-polarized relativistic e. m. sub-cycle soliton and remnants of a broken wakefield at the bottom of the window. The soliton frequency is well below the unperturbed plasma frequency, $\Omega_S \approx 0.25\omega_d < \omega_{pe}$. The soliton appears as a region of low electron density. Since the driver and the auxiliary laser pulses have different polarizations and the soliton inherits its polarization from the auxiliary laser, it is easy to distinguish the e. m. field reflected from the soliton in the distribution of the E_z component. The driver laser pulse induces a strong wakefield which is seen in the electron density distribution as a series of wide regions of rarefaction alternating with thin horseshoe-shaped regions of compression.
Regions of compression correspond to spikes (cusps) in the longitudinal profile of the electron density.

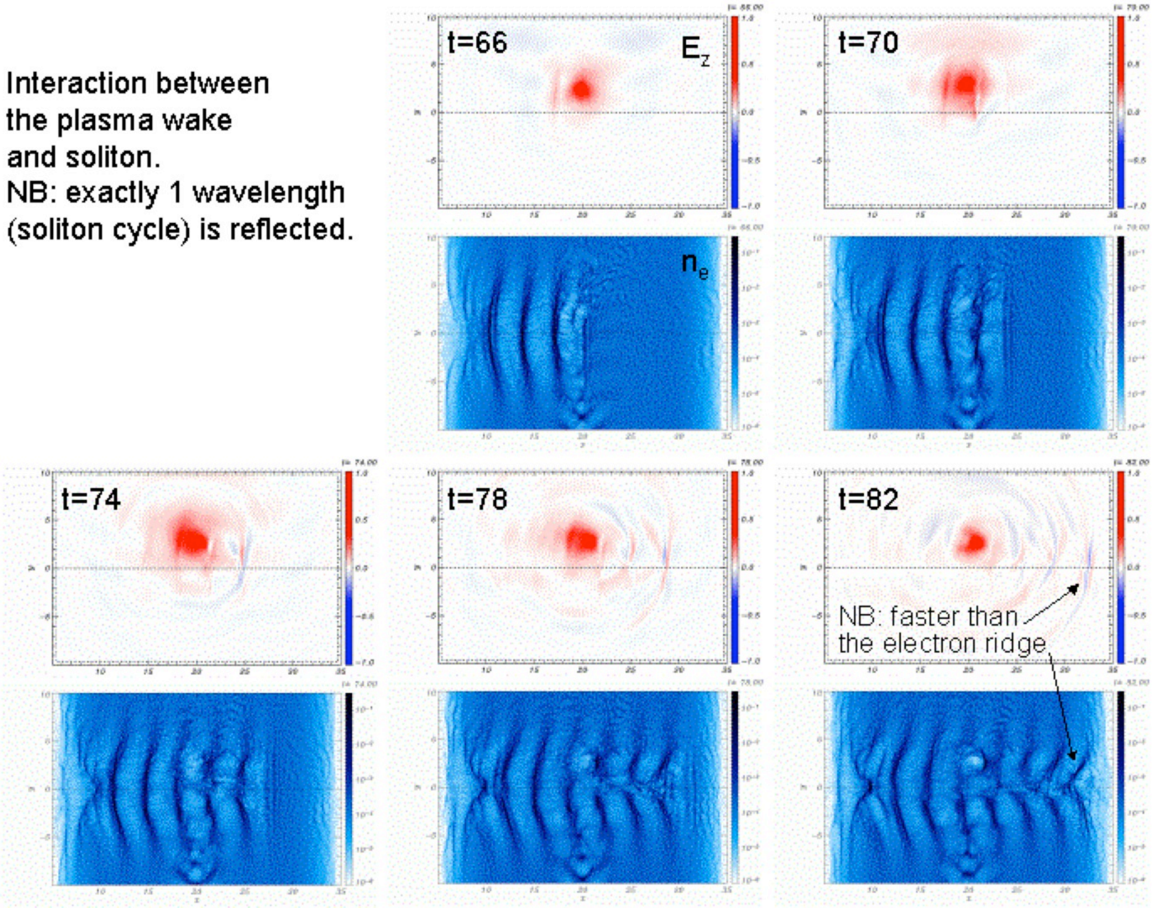


Figure 8: Mirror-soliton interaction

Fig.(8) shows the interaction of the density cusps in the wake of the driver pulse with the soliton.

The z-component of electric field and the electron density are shown. The wake wave of the driver is close to the wave breaking regime.
Each electron density maximum (each cusp) in the wake acts as a fast moving semitransparent parabolic mirror that partially reflects the e. m. fields of the soliton as it propagates through the soliton.
The process is repeated when the subsequent cusps of the electron density propagate through the soliton. Thus a set of short e. m. pulses is formed.
Even though the electron density cusp is substantially distorted as it moves through the soliton, it recovers after leaving the soliton. This transient distortion of the cusp when crossing the soliton does not prevent the formation of well pronounced single-cycle pulses.
We also note that the single cycle pulses move faster than the electron ridge.

The frequency of the fields in the reflected single-cycle e. m. pulses is up-shifted and their longitudinal size is much smaller than the size of the soliton.
Since the soliton is not exactly positioned at the crossing of laser pulse axes, the reflected pulse is not exactly directed along the x-axis.
This is a consequence of the parabolic profile of the wakefield: as the pulse is reflected by the upper wing of parabola it propagates at an angle with respect to the x-axis.

The reflection of coherent structures such as solitons by a wake wave can be exploited in order to produce ultra-short intense e.m. pulses. The modulations of electron density in a strong wake wave close to the wave-breaking regime have the shape of spikes and each spike acts as a semi-transparent mirror moving with a relativistic velocity. Such a mirror partially reflects the electromagnetic field of a coherent nonlinear structure and thus generates an electromagnetic pulse. For each spike, the reflected pulse consists of a single cycle oscillation, which results in a train of single oscillation pulses.

Conclusions

Analytical and numerical investigations show that in the complex physics of the interaction of high intensity ultrashort laser pulses with plasmas, fundamental physical mechanisms can be identified that form the basic blocks of the nonlinear physics of continuous media such as breaking waves and solitons.

In addition these investigation show that these nonlinear processes can be harnessed in order to concentrate the e.m. radiation in space and in time and produce e.m. pulses of unprecedented high intensity or short duration that can be used to explore collective ultra-high energy density effects in the laboratory.

Acknowledgments

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Appendix

In the investigation of relativistic plasmas a new basic tool, besides experiments and analytical modelling, has taken a major role: multi-dimensional, fully relativistic Particle in Cell (PIC) numerical codes have made it possible[38] to reproduce, in many important plasma regimes, the kinetic plasma behaviour that shapes the plasma and the laser pulse dynamics.
PIC codes solve the system of the coupled Maxwell's and Vlasov equations along an appropriate sample of particle characteristics (orbits in phase space).

PIC simulations indeed are not only used for validating analytical models or for reproducing experimental results, but can also play the vital role of an investigative tool for discovering new phenomena and new interaction regimes.

[38] See e.g., J.M. Dawson, *Phys. Plasmas*, **6**, 4436 (1999)

Hodograph transformation

1-D gasdynamics in coordinate free form and the hodograph transformation

For a 1-D fluid configuration, where quantities depend on x and t only, the continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} = 0 \tag{53}$$

and the Euler equation

$$\rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} \right) + \frac{\partial p}{\partial x} = 0 \tag{54}$$

with density ρ, pressure $p = p(\rho)$ and enthalpy $h = h(\rho)$, can be written in the notation of differential forms as

$$d\rho \wedge dx = d(\rho u) \wedge dt \tag{55}$$

$$du \wedge dx = d(h + u^2/2) \wedge dt. \tag{56}$$

Equations (53, 54) follow by taking x, t as the independent variables and by expressing the dependent variables ρ, u as functions of x, t. Taking instead ρ, u as independent variables and expressing x, t as functions of ρ, u we obtain the hodograph transformed equations

$$\frac{\partial x}{\partial u} - u\frac{\partial t}{\partial u} + \rho\frac{\partial t}{\partial \rho} = 0 \tag{57}$$

$$\frac{\partial x}{\partial \rho} + \frac{dh}{d\rho}\frac{\partial t}{\partial u} - u\frac{\partial t}{\partial \rho} = 0 \tag{58}$$

which are linear in the dependent variables x and t.

Diffraction spreading

Formulae: Numerical factors are neglected

$$\exp\left(-[x^2/L^2 + y^2/D^2] + ik_0x\right) = \tag{59}$$

$$\int\int_{-\infty}^{+\infty} \exp\left(-[(k_x - k_0)^2L^2/4 + k_y^2D^2/4]\right) \exp\left[+i(k_xx + k_yy)\right]\, dk_x\, dk_y$$

$$\int\int_{-\infty}^{+\infty} \exp\left(-[(k_x - k_0)^2L^2 + k_y^2D^2]/4 + ik_xx + ik_yy - i\omega(k)t\right)\, dk_x\, dk_y \tag{60}$$

with $k_0D \gg 1$ and

$$\omega(k) = c[k_x^2 + k_y^2]^{1/2} \approx ck_x\, [1 + k_y^2/(2k_x^2)] \approx ck_x + k_y^2/(2k_0). \tag{61}$$

The integrals over k_x and k_y are factorized. Then

$$\int_{-\infty}^{+\infty} \exp\left(-[k_y^2 D^2/4 + ictk_y^2/(2k_0)]\right) \exp\left(+ik_y y\right)\, dk_y \tag{62}$$
$$\propto \frac{\exp\left(-[y^2/(D^2 + 2ict/k_0)]\right)}{(D^2 + 2ict/k_0)^{1/2}}$$

The effective width of this complex Gaussian is

$$[D^2 + 4(ct/k_0)^2]^{1/2} = [D^2 + 4(X/k_0)^2]^{1/2} \tag{63}$$

where $X(t) = ct$.