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Resonant MHD waves in one-dimensional equilibrium

Mikhail S. Ruderman
The University of Sheffield
UK

RESONANT MHD WAVES IN ONE-DIMENSIONAL EQUILIBRIUM

Michael S. Ruderman

University of Sheffield, UK

LECTURE CONTENT

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1. Introduction

- Resonant MHD waves can provide efficient heating of even very weakly dissipative plasmas
- Theory of resonant MHD waves can explain fast damping of global waves in weakly dissipative plasmas
- It also provides mechanisms for excitation of large-amplitude oscillations by small-amplitude incoming waves
- Resonant MHD waves were first suggested for complementary heating of fusion plasmas (e.g. Tataronis & Grossmann 1973, *Z. Phys.* **261**, 217 [199]; Chen & Hasegawa 1974, *Phys. Fluids* **17**, 1399 [219])
- Soon they were considered to explain large-amplitude oscillations observed in terrestrial magnetosphere (Lanzerotti et al. 1973, *PRL* **31**, 624 [58]; Southwood 1974, *Planet. Space Sci.* **22**, 483 [758]; Chen & Hasegava 1974, *JGR* **79**, 1024 [754])

- In a few years resonant MHD waves were suggested for explanation of solar coronal heating (Ionson 1978, *ApJ* **226**, 650 [404])
- Recently theory of resonant MHD waves successfully explained fast damping of transverse oscillations of coronal magnetic loops (Ruderman & Roberts 2002, *ApJ* **577**, 475 [107]; Goossens et al. 2002, *A&A* **394** L39 [65])

At present hundreds of papers on resonant MHD waves were published. I mentioned only 8 of them, and altogether they were cited 2564 times, more than 320 citations per paper!

2. Basic equations

We consider strongly collisional plasmas assuming that

$$l_{\rm col} \ll l_{\rm ch}, \qquad t_{\rm col} \ll t_{\rm ch}$$

where $l_{\rm col}$ and $t_{\rm col}$ are collisional distance and time of particles, and $l_{\rm ch}$ and $t_{\rm ch}$ are characteristic spatial scale and time of the problem.

Motions of such plasmas are described by MHD equations:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0 \tag{2.1}$$

$$\rho \left(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} \right) = -\nabla p + \frac{1}{\mu_0} (\nabla \times \boldsymbol{B}) \times \boldsymbol{B} + \eta \nabla^2 \boldsymbol{v} \tag{2.2}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) + \lambda \nabla^2 \boldsymbol{B} \tag{2.3}$$

$$\frac{\partial}{\partial t} \left(\frac{p}{\rho^\gamma} \right) + \boldsymbol{v} \cdot \nabla \left(\frac{p}{\rho^\gamma} \right) = 0 \tag{2.4}$$

Here ρ is density, p pressure, $\boldsymbol{v}$ velocity, $\boldsymbol{B}$ magnetic induction; μ_0 is magnetic permeability of empty space, η dynamic viscosity, λ coefficient of magnetic diffusion, and γ the ratio of specific heats.

In Eq. (2.2) expression for viscous force is written in strongly simplified form. However, it is sufficient for studying resonant MHD waves.

In spite of presence of dissipation we use entropy conservation equation (2.2). To be precise we have to write two dissipative terms on right-hand side of Eq. (2.2). However, in static equilibria with potential magnetic field, this terms are quadratic with respect to perturbations. Since we are going to study only linear theory of resonant MHD waves, we neglected these terms.

In what follows we use either Cartesian coordinates x, y, z, or cylindrical coordinates r, φ, z, and consider one-dimensional static equilibria where ρ, p and $\boldsymbol{B}$ are functions of either x or r, and $\boldsymbol{v} = 0$. We mark equilibrium quantities with subscript '0'. It follows from solenoidal condition for magnetic field, $\nabla \cdot \boldsymbol{B} = 0$, that either $B_x = 0$ or $B_r = 0$. Equilibrium quantities satisfy condition of total pressure balance,

$$p_0 + \frac{B_0^2}{\mu_0} = 0 \tag{2.5}$$

Now we consider small perturbations to equilibrium and take $\rho = \rho_0 + \rho'$, $p = p_0 + p'$ and $\boldsymbol{B} = \boldsymbol{B}_0 + \boldsymbol{b}$. In what follows we drop prime at ρ and p. Then we linearised MHD equations to obtain

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho_0 \boldsymbol{v}) = 0, \tag{2.6}$$

$$\rho_0 \frac{\partial \boldsymbol{v}}{\partial t} = -\nabla \left(p + \frac{\boldsymbol{B}_0 \cdot \boldsymbol{b}}{\mu_0} \right) + \frac{1}{\mu_0} (\boldsymbol{B}_0 \cdot \nabla) \boldsymbol{b} + \eta \nabla^2 \boldsymbol{v}, \tag{2.7}$$

$$\frac{\partial \boldsymbol{b}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B}_0) + \lambda \nabla^2 \boldsymbol{b}, \tag{2.8}$$

$$\frac{\partial}{\partial t}(p - c_S^2 \rho) + \rho_0^\gamma \boldsymbol{v} \cdot \nabla \frac{p_0}{\rho_0^\gamma} = 0, \tag{2.9}$$

where $c_S^2 = \gamma p_0 / \rho_0$ is square of sound speed.

3. MHD waves in ideal homogeneous plasmas

Let us assume that ρ_0, p_0 and $\boldsymbol{B}_0$ are constant. We also assume that the plasma is ideal, i.e. $\eta = \lambda = 0$. Then we take perturbations of all variables proportional to $\exp[i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)]$, where $\boldsymbol{k} = (k_x, k_y, k_z)$ and $\boldsymbol{r} = (x, y, z)$. Then Eqs. (2.6)–(2.9) reduce to

$$\omega\rho - \rho_0 \boldsymbol{k} \cdot \boldsymbol{v} = 0 \tag{3.1}$$

$$\rho_0 \omega \boldsymbol{v} = \boldsymbol{k} \left(p + \frac{\boldsymbol{B}_0 \cdot \boldsymbol{b}}{\mu_0} \right) - \frac{1}{\mu_0} (\boldsymbol{k} \cdot \boldsymbol{B}_0) \boldsymbol{b} \tag{3.2}$$

$$\omega \boldsymbol{b} = \boldsymbol{B}_0 (\boldsymbol{k} \cdot \boldsymbol{v}) - \boldsymbol{v} (\boldsymbol{k} \cdot \boldsymbol{B}_0) \tag{3.3}$$

$$p = c_S^2 \rho \tag{3.4}$$

Eliminating all variables in favour of $\boldsymbol{v}$ we obtain

$$(\omega^2 - k^2 V_A^2 \cos^2\alpha)\boldsymbol{v} = \boldsymbol{k}(c_S^2 + V_A^2)(\boldsymbol{k}\cdot\boldsymbol{v})$$

$$- \; V_A^2 \boldsymbol{b}_0 k(\boldsymbol{k}\cdot\boldsymbol{v})\cos\alpha - \boldsymbol{k}kV_A^2(\boldsymbol{b}_0\cdot\boldsymbol{v})\cos\alpha \qquad (3.5)$$

where $V_A^2 = B_0^2/\mu_0\rho_0$ is the square of Alfvén speed, α is the angle between $\boldsymbol{k}$ and $\boldsymbol{B}_0$, and $\boldsymbol{b}_0 = \boldsymbol{B}_0/B_0$ is the unit vector in the equilibrium magnetic field direction. Taking scalar product of this equation with $\boldsymbol{k}$, $\boldsymbol{b}_0$ and $\boldsymbol{k}\times\boldsymbol{b}_0$ yields

$$[\omega^2 - k^2(c_S^2 + V_A^2)](\boldsymbol{k}\cdot\boldsymbol{v}) + k^3 V_A^2(\boldsymbol{b}_0\cdot\boldsymbol{v})\cos\alpha = 0 \qquad (3.6)$$

$$kc_S^2(\boldsymbol{k}\cdot\boldsymbol{v})\cos\alpha - \omega^2(\boldsymbol{b}_0\cdot\boldsymbol{v}) = 0 \qquad (3.7)$$

$$(\omega^2 - k^2 V_A^2\cos^2\alpha)(\boldsymbol{k}\times\boldsymbol{b}_0)\cdot\boldsymbol{v} = 0 \qquad (3.8)$$

Equations (3.6) and (3.7) constitute the system of two linear homogeneous algebraic equations for $\boldsymbol{k}\cdot\boldsymbol{v}$ and $\boldsymbol{b}_0\cdot\boldsymbol{v}$. It has non-trivial solution only when its determinant is zero, which gives the dispersion equation

$$\omega^4 - \omega^2 k^2(c_S^2 + V_A^2) + c_S^2 V_A^2 \cos^2\alpha = 0 \qquad (3.9)$$

Equations (3.8) is an equation for $(\boldsymbol{k} \times \boldsymbol{b}_0) \cdot \boldsymbol{v}$. It has a non-trivial solution only when

$$\omega^2 = \omega_A^2 \equiv k^2 V_A^2 \cos^2 \alpha \tag{3.10}$$

(3.9) is the dispersion equation for **magnetosonic waves**. Its solutions are

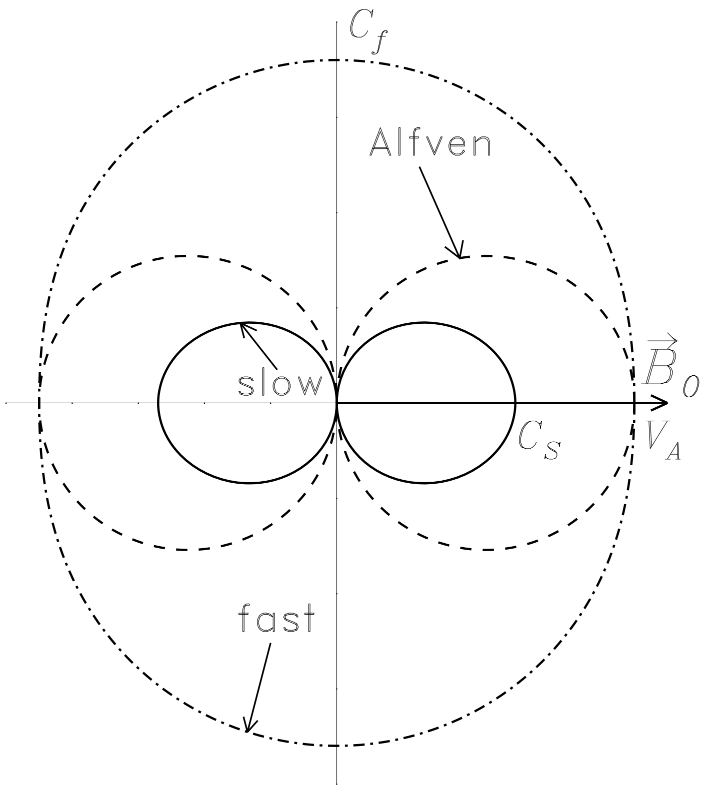
$$\omega_\pm^2 = \frac{1}{2}\left(c_S^2 + V_A^2 \pm \sqrt{(c_S^2 + V_A^2)^2 - c_S^2 V_A^2 \cos^2 \alpha}\right) \tag{3.11}$$

The signs '$+$' and '$-$' correspond to **fast** and **slow** magnetosonic waves.

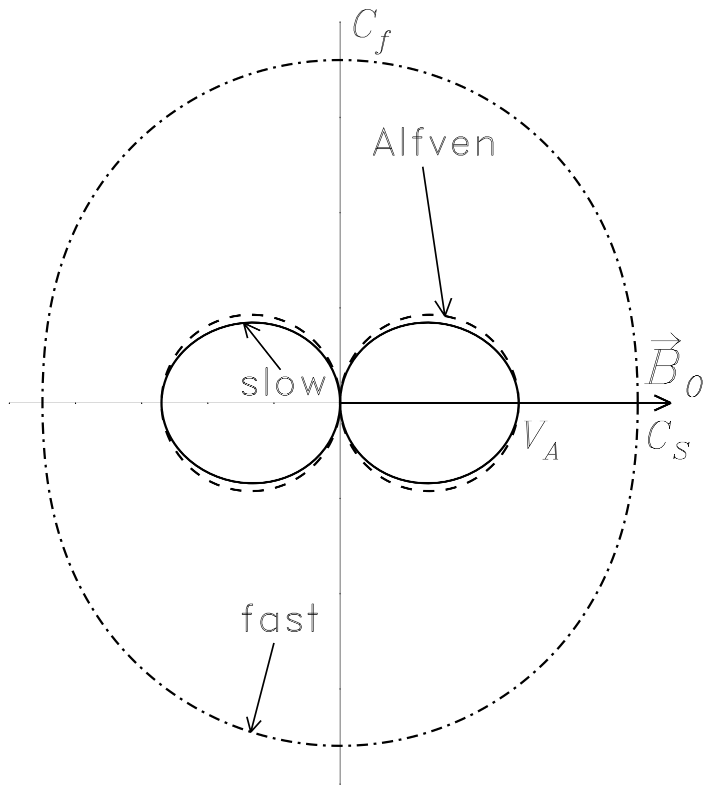
The dispersion relation (3.10) corresponds to **Alfvén waves**. When $\alpha \neq 0$ and $\alpha \neq \frac{\pi}{2}$, $\omega_\pm^2 \neq \omega_A^2$. Hence, in magnetosonic waves $(\boldsymbol{k} \times \boldsymbol{b}_0) \cdot \boldsymbol{v} = 0$, so that these waves are polarised in the $\boldsymbol{k}\boldsymbol{b}_0$-plane (i.e. the vectors $\boldsymbol{k}$, $\boldsymbol{b}_0$, $\boldsymbol{v}$ and $\boldsymbol{b}$ are coplanar.

In Alfvén waves $\boldsymbol{k} \cdot \boldsymbol{v} = \boldsymbol{b}_0 \cdot \boldsymbol{v} = 0$, so that $\boldsymbol{v}$ and $\boldsymbol{b}$ are perpendicular to $\boldsymbol{k}$ and $\boldsymbol{b}_0$.

Polar diagrams for phase speed ω/k



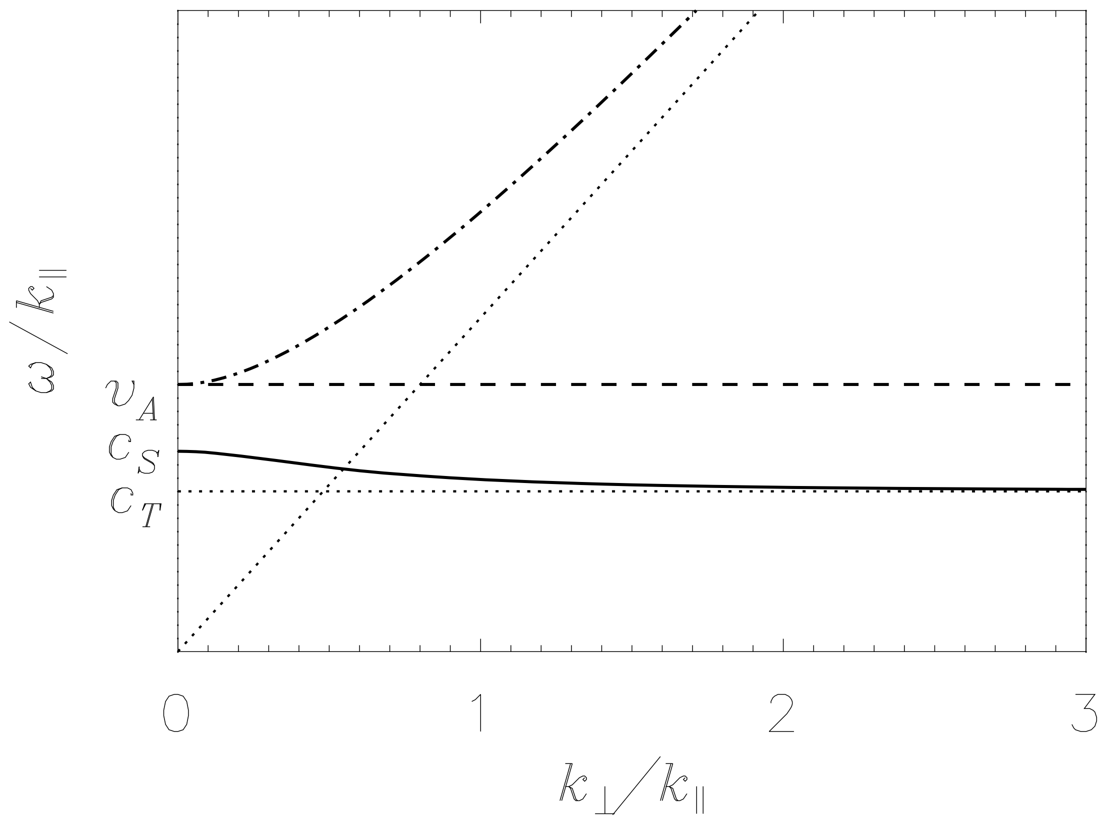
$$V_A > C_S$$

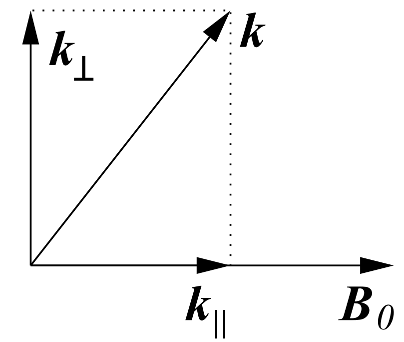


$$V_A < C_S$$

$$C_f = (C_S^2 + V_A^2)^{1/2}$$

Phase speed along magnetic field





$$C_T = \frac{C_S V_A}{\sqrt{C_S^2 + V_A^2}}$$

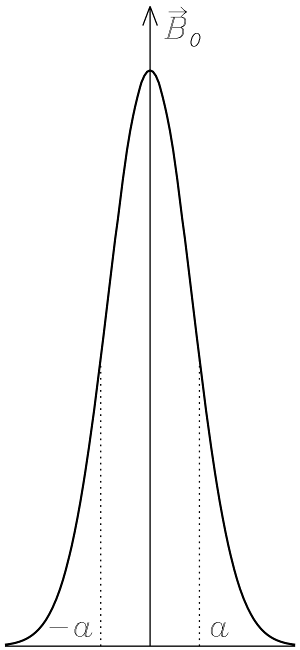
fast slow Alfvén

Group velocity $\dfrac{\partial \omega}{\partial \boldsymbol{k}}$:

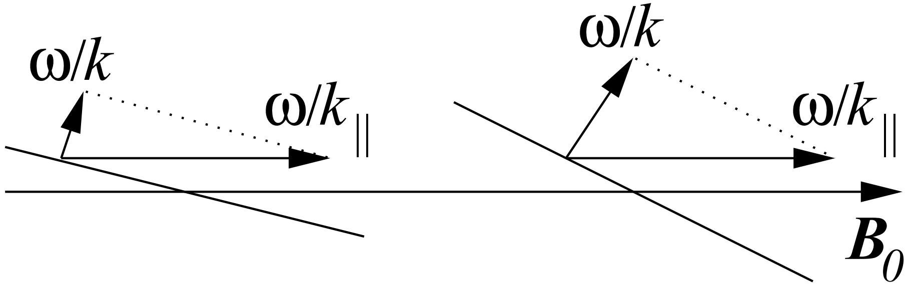
For Alfvén waves: $\dfrac{\partial \omega}{\partial \boldsymbol{k}} = V_A \boldsymbol{b}_0$

For slow waves: $\omega \approx C_T(\boldsymbol{b}_0 \cdot \boldsymbol{k})$ for $k_\perp \gg k_\parallel \implies \dfrac{\partial \omega}{\partial \boldsymbol{k}} \approx C_T \boldsymbol{b}_0$

Strongly localised Alfvén and slow waves $(ak_\parallel \ll 1)$



In all harmonics of Fourier expansion $k_\perp \gg k_\parallel$



Let $\boldsymbol{k}_\perp$ be in x-direction, $\boldsymbol{B}_0$ be in z-direction. Then

$$f(x,z,t) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} dk_\perp \int_{-\infty}^{\infty} \hat{f}(k_\perp, k_\parallel)\exp(ik_\perp + ik_\parallel - i\omega t)\, dk_\parallel$$

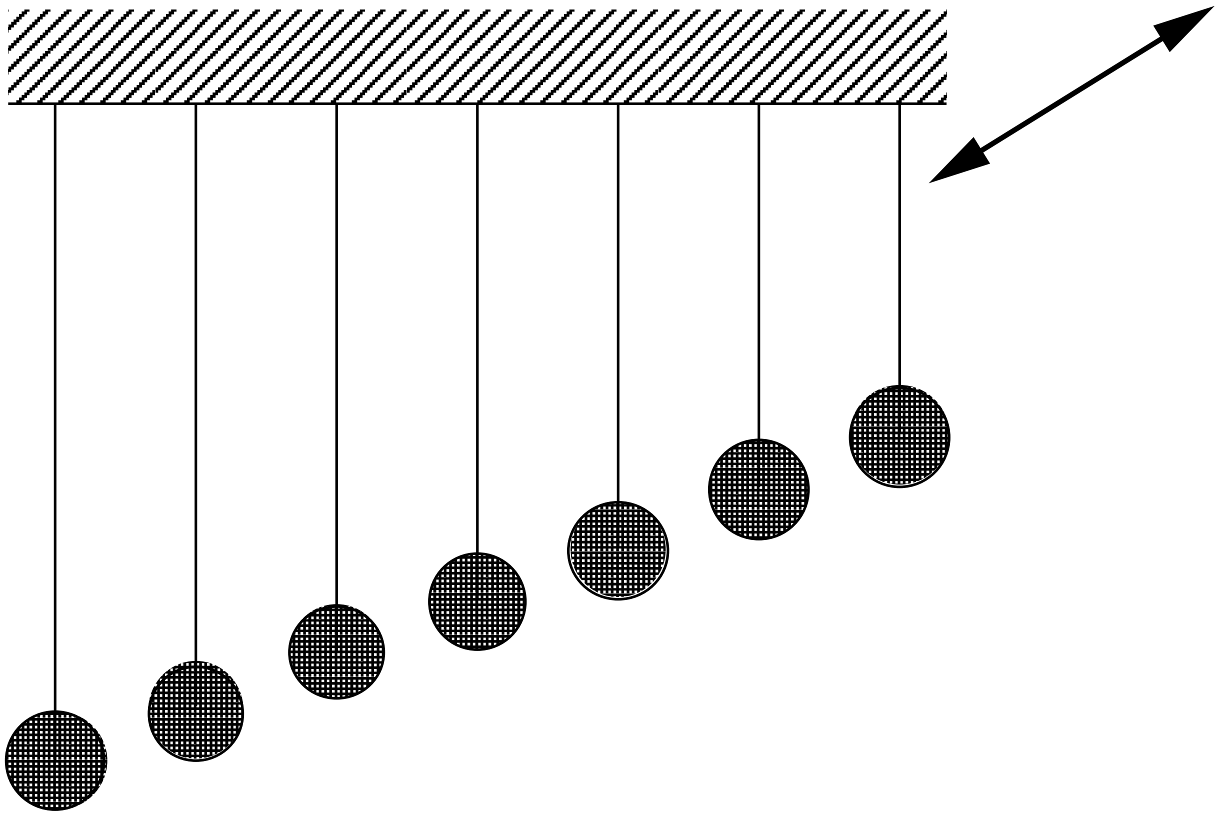
$$= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} dk_\perp \int_{-\infty}^{\infty} \hat{f}(k_\perp, k_\parallel)\exp[ik_\perp x + ik_\parallel (z - V_A t)]\, dk_\parallel$$

$$= f(x, z - V_A t)$$

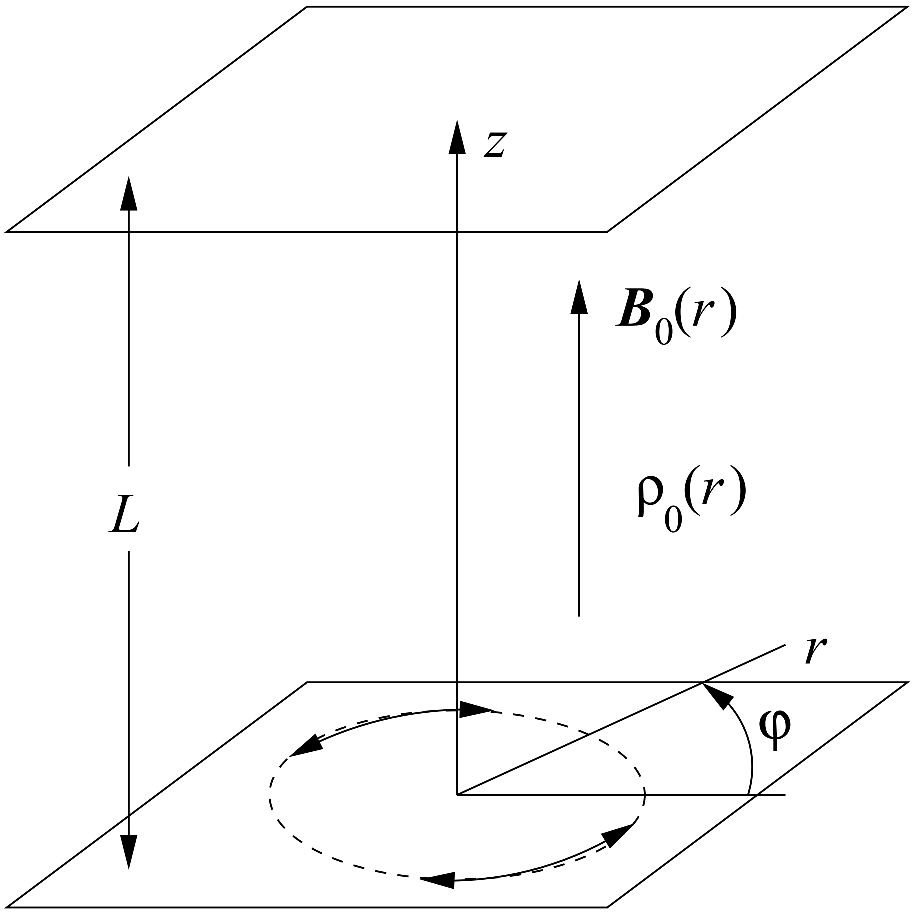
This is also approximately valid for slow waves:

$$f(x,z,t) = f(x, z - C_T t)$$

4. Torsional waves in inhomogeneous plasmas



The sketch of a system consisting of N oscillators with different frequencies. The bar is harmonically shaken in the y-direction.



Magnetic field lines are frozen in dense plasma at $z = 0, L$. Plasma is at rest at $z = L$, so that

$$\boldsymbol{v} = 0 \quad \text{at} \quad z = L \quad (4.1)$$

There is driver exciting rotational oscillations at $z = 0$, so that

$$v_r = v_z = 0 \quad \text{at} \quad z = 0 \quad (4.2)$$

$$v_\varphi = v_0(r, t) \quad \text{at} \quad z = 0 \quad (4.3)$$

Now we assume that perturbations of all variables but the v_φ and b_φ are zero, and v_φ and b_φ are independent of φ. Then all linearised MHD equations, Eqs. (2.6)–(2.9), but the φ-components of the linearised momentum and induction equation, are satisfied identically, while the φ-components of the linearised momentum and induction equation reduce to

$$\frac{\partial v_\varphi}{\partial t} = \frac{B_0}{\mu_0 \rho_0} \frac{\partial b_\varphi}{\partial z} + \nu \nabla^2 v_\varphi \tag{4.4}$$

$$\frac{\partial b_\varphi}{\partial t} = B_0 \frac{\partial b_\varphi}{\partial z} + \lambda \nabla^2 b_\varphi \tag{4.5}$$

where $\nu = \eta/\rho_0$ is kinematic viscosity. We assume that dissipation is weak. Then we will see that it is only important in a very thin annulus embracing a cylindrical surface called the resonant surface. In this annulus the derivatives with respect to r are much larger than the derivatives with respect to φ and z, so we can take

$$\nabla^2 \approx \frac{\partial^2}{\partial r^2}$$

Then, eliminating b_φ from Eqs. (4.4) and (4.5), we arrive at

$$\frac{\partial^2 v_\varphi}{\partial t^2} - V_A^2 \frac{\partial^2 v_\varphi}{\partial z^2} = (\nu + \lambda) \frac{\partial^3 v_\varphi}{\partial t \partial r^2} \tag{4.6}$$

Assume that

$$v_\varphi = \tilde{v}_1(r, z), \quad \frac{\partial v_\varphi}{\partial t} = \tilde{v}_2(r, z) \quad \text{at} \quad t = t_0 \tag{4.7}$$

Introduce new variable

$$v(r, z, t) = v_\varphi(r, z, t) - v_0(r, t) \left(1 - \frac{z}{L}\right) \tag{4.8}$$

Then Eq. (4.6) is transformed to

$$\frac{\partial^2 v}{\partial t^2} - V_A^2 \frac{\partial^2 v}{\partial z^2} - (\nu + \lambda) \frac{\partial^3 v}{\partial t \partial r^2} = \left(1 - \frac{z}{L}\right) g(r, t) \tag{4.9}$$

and the initial and boundary conditions, Eqs. (4.7) and (4.1), (4.3) reduce to

$$v = v_1(r, z), \quad \frac{\partial v}{\partial t} = v_2(r, z) \quad \text{at} \quad t = t_0 \tag{4.10}$$

$$v = 0 \quad \text{at} \quad z = 0, L \tag{4.11}$$

where

$$g(r,t) = (\nu + \lambda)\frac{\partial^3 v_0}{\partial t \partial r^2} - \frac{\partial^2 v_0}{\partial t^2}$$

$$v_1(r,z) = \tilde{v}_1(r,z) - v_0(r,t)\left(1 - \frac{z}{L}\right)$$

$$v_2(r,z) = \tilde{v}_2(r,z) - \frac{\partial v_0}{\partial t}\left(1 - \frac{z}{L}\right)$$

We assume that $v_1(r,z)$ and $v_2(r,z)$ also satisfy the boundary conditions

$$v_1 = v_2 = 0 \quad \text{at} \quad z = 0, L \tag{4.12}$$

Now we expand v, v_1 and v_2 in Fourier series of the form

$$f = \sum_{n=1}^{\infty} f_n \sin\frac{\pi n z}{L}$$

Then, using

$$1 - \frac{z}{L} = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{\pi n z}{L}$$

we obtain

$$\frac{\partial^2 v_n}{\partial t^2} + n^2\omega_A^2 v_n - (\nu+\lambda)\frac{\partial^3 v_n}{\partial t \partial r^2} = \frac{2g(r,t)}{\pi n} \tag{4.13}$$

$$v_n = v_{1n}(r), \quad \frac{\partial v_n}{\partial t} = v_{2n}(r) \quad \text{at} \quad t = t_0 \tag{4.14}$$

We assume that $v_{1n}(r)$ and $v_{1n}(r)$ are smooth functions with the characteristic variation scale $r_{\rm ch}$ satisfying $R = r_{\rm ch}V_{A0}/(\nu+\lambda) \gg 1$, where $V_{A0} = V_A(0)$. Then, for "not large values of $t - t_0$", the ratio of the third term in Eq. (4.13) to the second one is of the order of $1/R \ll 1$, so that the third term can be neglected. Then Eq. (4.13) becomes an ordinary differential equation, and its solution satisfying Eq. (4.14) is

$$v_n = A_c(t)\cos[n\omega_A(t-t_0)] + A_s(t)\sin[n\omega_A(t-t_0)] \tag{4.15}$$

$$A_c(t) = v_{1n} - \frac{2}{\pi n^2\omega_A}\int_{t_0}^{t} g(\tau)\sin(n\omega_A\tau)\,d\tau \tag{4.16}$$

$$A_s(t) = \frac{v_{2n}}{n\omega_A} - \frac{2}{\pi n^2\omega_A}\int_{t_0}^{t} g(\tau)\cos(n\omega_A\tau)\,d\tau \tag{4.17}$$

If we now use this solution to calculate the third term in Eq. (4.13), then we find that it grows as t^2. This growth is caused by dependence of ω_A on r. Due to this dependence neighbouring magnetic field lines oscillate with difference frequencies and become more and more out of phase. As a result the spatial gradient in the r-direction grows. This process is called the **phase-mixing**.

Let us obtain solution to Eq. (4.13) valid for any moment of time. First we consider its homogeneous counterpart and take $g = 0$. We introduce stretched time $T = \epsilon(t - t_0)$ and look for solution to Eq. (4.13) in the form

$$v_n = \Re\{Q(r,T)\exp[i\epsilon^{-1}\Theta(r,T)]\}$$

Substituting this expression in Eq. (4.13) we obtain

$$Q\left(\frac{\partial\Theta}{\partial T}\right)^2 - n^2\omega_A^2 Q - 2i\epsilon\frac{\partial Q}{\partial T}\frac{\partial\Theta}{\partial T} - i\epsilon^{-2}(\nu+\lambda)\frac{\partial Q}{\partial T}\left(\frac{\partial\Theta}{\partial r}\right)^2$$
$$= \mathcal{O}(\epsilon^2) + \mathcal{O}[(\epsilon^{-1}(\nu+\lambda)] \qquad (4.18)$$

Now we demand that the third and fourth term on the left-hand side be of the same order. Then $\epsilon = R^{-1/3}$.

First-order approximation:

$$\frac{\partial \Theta}{\partial T} = \pm n\omega_A \tag{4.19}$$

For simplicity we take $\Theta = 0$ at $T = 0$.

Second-order approximation:

$$\frac{\partial \Theta}{\partial T} = -3\Lambda n^2 T^2, \qquad \Lambda = \frac{aV_{A0}}{6}\left(\frac{d\omega_A}{dr}\right)^2 \tag{4.20}$$

Solution to this equation is

$$Q = Q_0(r)\exp(-\Lambda n^2 T^3)$$

Then the general solution to homogeneous counterpart of Eq. (4.13) valid for any time is

$$v_n = \exp[-\Lambda n^2(t-t_0)^3/R]\{\widetilde{A}_c(r)\cos[n\omega_A(t-t_0)] + \widetilde{A}_s(r)\sin[n\omega_A(t-t_0)]\} \tag{4.21}$$

To find solution to Eq. (4.13) we need to construct the Green's function $G_n(t,\tau)$. It is determined by the following conditions:

(i) $G_n(t,\tau)$ is continuous;

(ii) $G_n(t,\tau) = 0$ for $t < \tau$;

(iii) $\dfrac{\partial G_n(t,\tau)}{\partial t} = 1$ at $t \to \tau + 0$.

It is easily verified that $G_n(t-\tau)$ determined by

$$G_n(t) = H(t)\exp(-\Lambda n^2 t^3/R)\frac{\sin(n\omega_A t)}{n\omega_A} \tag{4.22}$$

where $H(t)$ is the Heaviside function, satisfies all these conditions.

The convolution of $G_n(t)$ and $2g(t)/\pi n$ gives a particular solution to Eq. (4.13). The sum of this particular solution and the general solution to the homogeneous equation given by Eq. (4.21) give the general solution to Eq. (4.13). Recalling that it also has to satisfy the initial conditions we eventually obtain that the solution to Eq. (4.13) satisfying Eq. (4.14) is

$$v_n = \exp[-\Lambda n^2 (t-t_0)^3/R] \Big\{ v_{1n} \cos[n\omega_A (t-t_0)] + \frac{v_{2n}}{n\omega_A} \sin[n\omega_A (t-t_0)] \Big\} + \frac{2}{\pi n} \int_{t_0}^{t} G_n(t-\tau) g(\tau)\, d\tau \qquad (4.23)$$

The system "forgets" initial conditions after the time $t_{\text{mix}} = R^{1/3}/\omega_A$. To get read off the initial conditions and obtain the steady state of driven oscillations we take $t_0 \to -\infty$. Then

$$v_n = \frac{2}{\pi n} \int_{-\infty}^{t} G_n(t-\tau) g(\tau)\, d\tau \qquad (4.24)$$

Let us now assume that the driving is harmonic: $v_0 = f(r)\cos(\omega t)$. Then

$$g = \omega^2 f(r)\cos(\omega t) + \mathcal{O}(R^{-1})$$

We assume that $\omega_A(r)$ is a monotonically growing function. The range of frequencies

$$[n \min \omega_A, n \max \omega_A]$$

is called the **Alfvén continuum** corresponding to fundamental mode with respect to z for $n = 1$, and to the $(n-1)$st overtone for $n > 1$. We assume that ω is in the fundamental continuum, and not in any of the overtone continua,

$$\omega \in [\min \omega_A, \max \omega_A], \qquad \omega \notin [n \min \omega_A, n \max \omega_A] \;\; (n > 1)$$

Using integration by parts we obtain from Eqs. (4.22) and (4.24)

$$v_n = \frac{2\omega^2 f(r)\cos(\omega t)}{\pi n(n^2\omega_A^2 - \omega^2)} + \mathcal{O}(R^{-1/3}) \qquad (4.25)$$

Equation (4.25) is valid for $n > 1$. It is also valid for $n = 1$ if r is not very close to the Alfvén resonant position r_A determined by $\omega_A(r_A) = \omega$.

Consider the vicinity of r_A, $|r - r_A| \ll r_{\rm ch}$. Then

$$\omega_A - \omega \approx \frac{\Delta}{2\omega}(r - r_A), \qquad \Delta = \left.\frac{d\omega_A^2}{dr}\right|_{r=r_A} \tag{4.26}$$

Using Eqs. (4.22) and (4.24) we obtain

$$\begin{aligned} v_1 &= \frac{\omega^2 f(r)}{\pi\omega_A}\int_{-\infty}^{t} \exp[-\Lambda(t-\tau)^3/R]\{\sin[\omega_A t + \tau(\omega - \omega_A)] \\ &\qquad + \sin[\omega_A t - \tau(\omega + \omega_A)]\}\, d\tau \\ &= \frac{\omega^2 f(r)}{\pi\omega_A}\int_{-\infty}^{t} \exp[-\Lambda(t-\tau)^3/R]\sin[\omega_A t + \tau(\omega - \omega_A)]\, d\tau + \mathcal{O}(1) \end{aligned} \tag{4.27}$$

Now we use Eq. (4.26) and make substitution $\tau = t - \tau'$. As a result we obtain the approximate expression

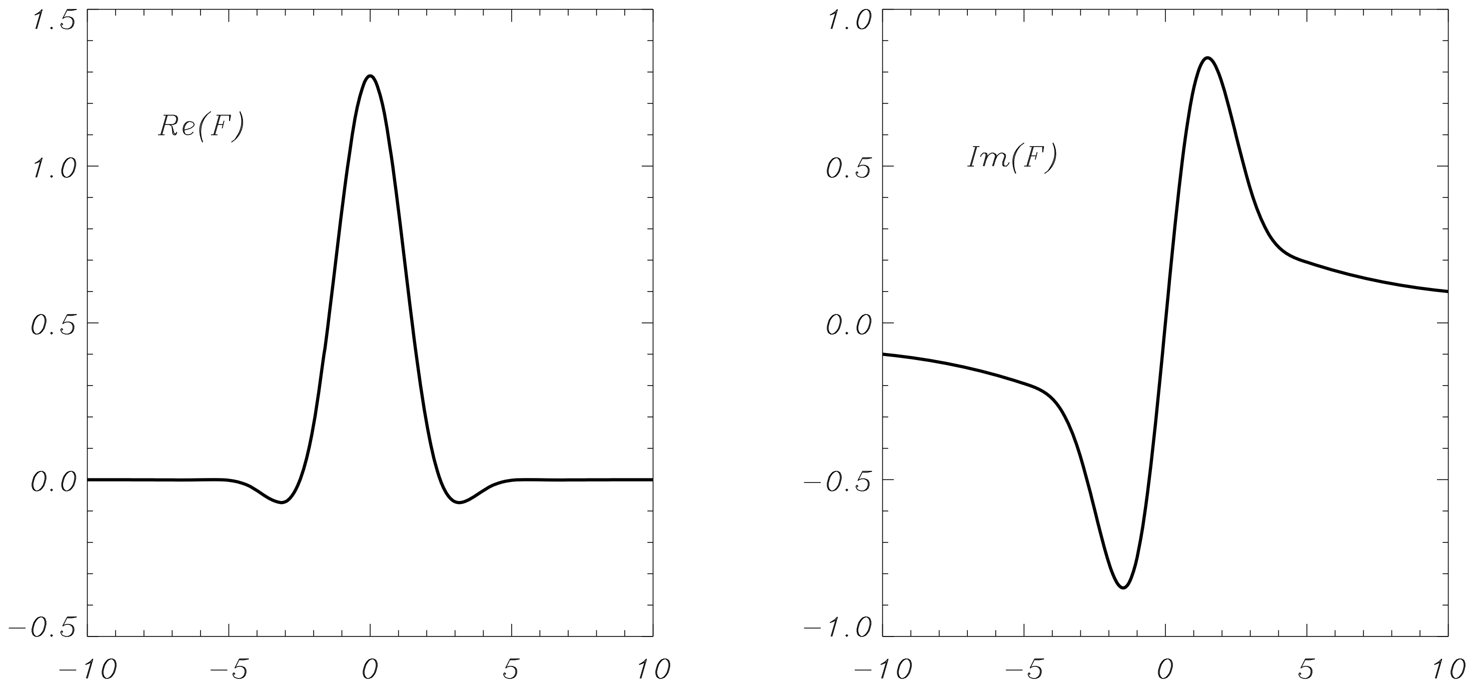
$$v_1 = \frac{2\omega^2 f_A}{\pi\delta_A\Delta}\Re\{ie^{-i\omega t}F[(r - r_A)/\delta_A]\} \tag{4.28}$$

where $f_A = f(r_A)$ and

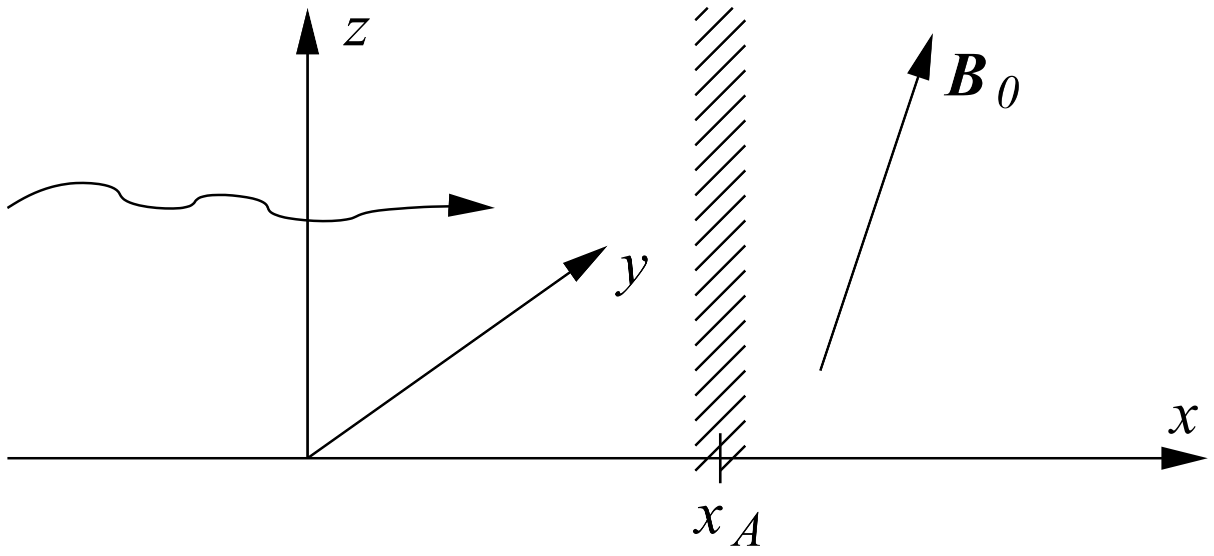
$$F(x) = \int_0^\infty \exp(-ix\sigma - \sigma^2/3)\, d\sigma \qquad (4.29)$$

$$\delta_A = R^{-1/3} \left(\frac{\omega r_{\rm ch} V_{A0}}{\Delta} \right)^{1/3} = \left(\frac{\omega(\nu + \lambda)}{\Delta} \right)^{1/3} \qquad (4.30)$$

The layer $|r - r_A| \lesssim \delta_A$ is called the **dissipative layer.**



5. Laterally driven resonant waves



$\boldsymbol{B}_0 \perp \boldsymbol{e}_x$

All equilibrium quantities depend on x only, incoming wave is harmonic.

Alfvén resonance: $\dfrac{\omega}{k_\parallel} = V_A(x_A)$ **slow resonance:** $\dfrac{\omega}{k_\parallel} = C_T(x_c)$

Alfvén continuum: $\left[\min(V_A^2 k_\parallel^2), \max(V_A^2 k_\parallel^2)\right]$

Slow continuum: $\left[\min(C_T^2 k_\parallel^2), \max(C_T^2 k_\parallel^2)\right]$

We look for solution to Eqs. (2.6)–(2.9) where perturbations of all quantities are proportional to $\exp[i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]$, where $\boldsymbol{k}=(0,k_y,k_z)$. Then Eqs. (2.6)–(2.9) can be reduced to

$$D\frac{d\xi_x}{dx}+\left[\omega^4-C_f^2k^2(\omega^2-\omega_c^2)\right]P=0 \tag{5.1}$$

$$\frac{dP}{dx}-\rho_0(\omega^2-\omega_A^2)\xi_x=0 \tag{5.2}$$

where $\boldsymbol{\xi}$ is plasma displacement related to velocity by $\boldsymbol{v}=-i\omega\boldsymbol{\xi}$, and

$$D=\rho_0C_f^2(\omega^2-\omega_A^2)(\omega^2-\omega_c^2),\quad \omega_A=V_Ak_\parallel,\quad \omega_c=C_Tk_\parallel$$

P is perturbation of total pressure given by

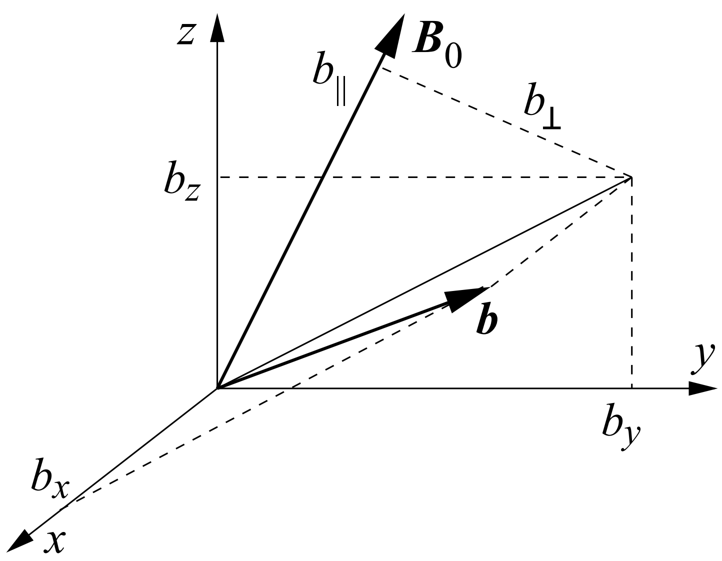
$$P=p+\frac{\boldsymbol{B}_0\cdot\boldsymbol{b}}{\mu_0} \tag{5.3}$$

We see that $D=0$ at $x=x_A$, where $\omega_A^2=\omega^2$, and at $x=x_c$, where $\omega_c^2=\omega^2$, so that these points are singular point of system (5.1), (5.2).

The singular points x_A and x_c are regular singular points, so that the solution in their vicinities can be found in the form of Frobenius series. We assume that there is only one Alfvén resonant position x_A, and no slow resonant positions, and ω_A is monotonically increasing. Omitting the details we only give the result. The expansions for P and ξ_x are given by

$$P = P_1 + \mathcal{O}((x - x_A)), \qquad \xi_x = \frac{P_1 k_\perp^2}{\rho_A \Delta} \ln |x - x_A| + \mathcal{O}(1) \tag{5.4}$$

where P_1 is constant. Using the linearised MHD equations (2.6)–(2.9) with



$\eta = \lambda = 0$ we now can easily find the expansions for other variables. It turns out that expansions for p, ρ, $\xi_\parallel$ and $b_\parallel$ start from constant terms. Expansion for b_x starts from the term proportional to $\ln |x - x_A|$. Finally, expansions for $\xi_\perp$ and $b_\perp$ are given by

$$\xi_\perp = \frac{iP_1 k_\perp}{\rho_A \Delta (x - x_A)} + \mathcal{O}(1), \quad b_\perp = \frac{-P_1 B_0 k_\perp k_\parallel}{\rho_A \Delta (x - x_A)} + \mathcal{O}(1) \tag{5.5}$$

These two variables are most singular, so that they are called **large variables**. Note that near x_c the large variables are $\xi_\parallel$, $b_\parallel$, ρ and p.

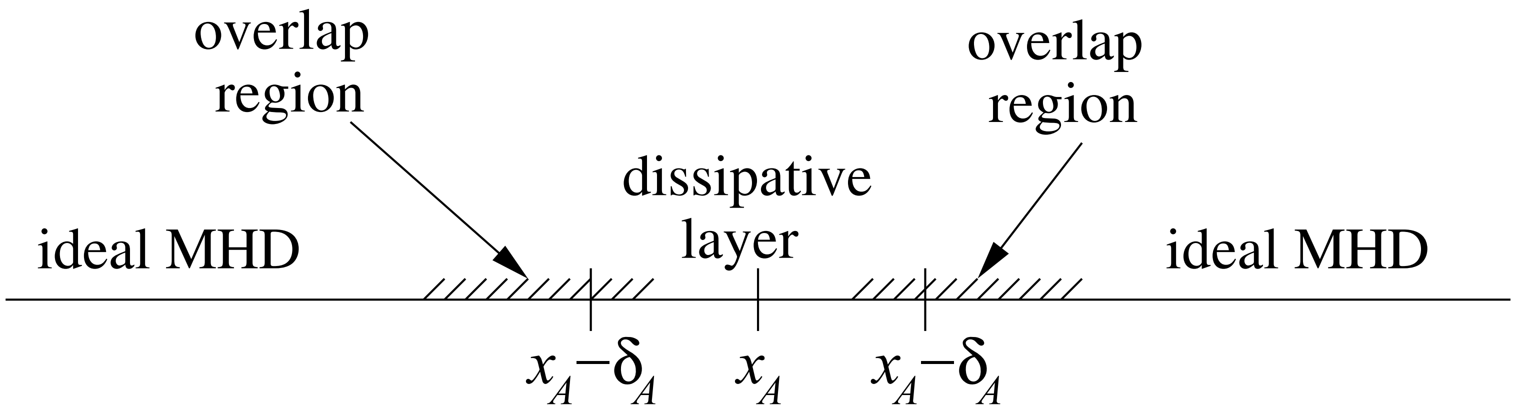
Near x_A there are large spatial gradient, so that dissipation becomes important, and we have to use dissipative linearised MHD equations (2.6)–(2.9). On the other hand, we need to use dissipative equations only in a narrow **dissipative layer** embracing x_A. In this layer we can approximate all equilibrium quantities by the first non-zero terms of their expansions in Taylor series with respect to $x - x_A$. In particular we use the approximate expression (4.26). Then Eqs. (2.6)–(2.9) reduce to

$$s\Delta \frac{d\xi_x}{ds} + i\omega(\nu + \lambda)\frac{d^3\xi_x}{ds^3} = -\frac{Pk_\perp^2}{\rho_A} \tag{5.6}$$

$$s\Delta \xi_\perp + i\omega(\nu + \lambda)\frac{d^2\xi_\perp}{ds^2} = -\frac{iPk_\perp}{\rho_A} \tag{5.7}$$

where $s = x - x_A$. In addition we obtain $P = \mathrm{const}$. This implies that, inside the dissipative layer, P is completely defined by its value outside this layer.

Matched asymptotic expansions



Thickness of the dissipative layer is determined by condition that the two terms on the left hand side of Eq. (5.6) or Eq. (5.7) are of the same order. We easily obtain that it is equal to δ_A given by Eq. (4.30).

Introducing $\tau = s/\delta_A$ we rewrite Eqs. (5.6) and Eq. (5.7) as

$$\frac{d^3\xi_x}{d\tau^3} - i\tau\frac{d\xi_x}{d\tau} = \frac{iPk_\perp^2}{\rho_A\Delta} \tag{5.8}$$

$$\frac{d^2\xi_\perp}{d\tau^2} - i\tau\xi_\perp = -\frac{Pk_\perp}{\rho_A\delta_A\Delta} \tag{5.9}$$

First we solve Eq. (5.9). For this we introduce the Fourier transform with respect to τ,

$$\hat{\xi}_\perp(\sigma) = \int_{-\infty}^{\infty}\xi_\perp(\tau)e^{i\sigma\tau}d\tau, \quad \xi_\perp(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{\xi}_\perp(\sigma)e^{-i\sigma\tau}d\sigma \tag{5.10}$$

Taking the Fourier transform of Eq. (5.9) we obtain

$$\frac{d\hat{\xi}_\perp}{d\sigma} + \sigma^2\hat{\xi}_\perp = \frac{2\pi Pk_\perp}{\rho_A\delta_A\Delta}\delta(\sigma) \tag{5.11}$$

$\hat{\xi}_\perp$ has to vanish as $\sigma \to \infty$. The solution to Eq. (5.11) satisfying this condition is

$$\hat{\xi}_\perp = \begin{cases} 0, & \sigma < 0, \\ \dfrac{2\pi Pk_\perp}{\rho_A\delta_A\Delta}e^{-\sigma^3/3}, & \sigma > 0 \end{cases} \tag{5.12}$$

Then the inverse Laplace transform gives

$$\xi_\perp = \frac{P k_\perp}{\rho_A \delta_A \Delta} F(\tau) \tag{5.13}$$

where $F(\tau)$ is given by Eq. (4.29). Integrating by parts we obtain

$$F(\tau) \approx \frac{i}{\tau}, \quad \text{as} \quad |\tau| \to \infty \tag{5.14}$$

which is in agreement with the asymptotics of the ideal solution as $|s| \to 0$ given by Eq. (5.5).

Now we can immediately write down the solution to Eq. (5.8):

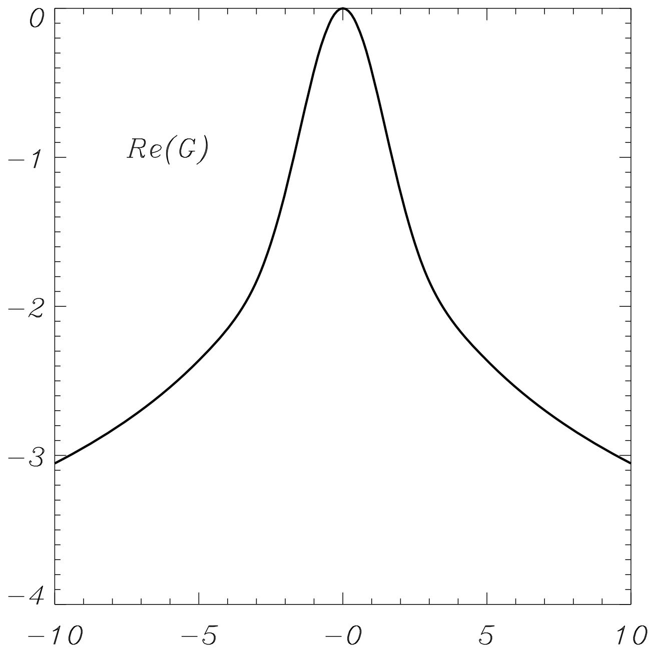
$$\frac{d\xi_x}{d\tau} = -\frac{i P k_\perp^2}{\rho_A \Delta} F(\tau) \tag{5.15}$$

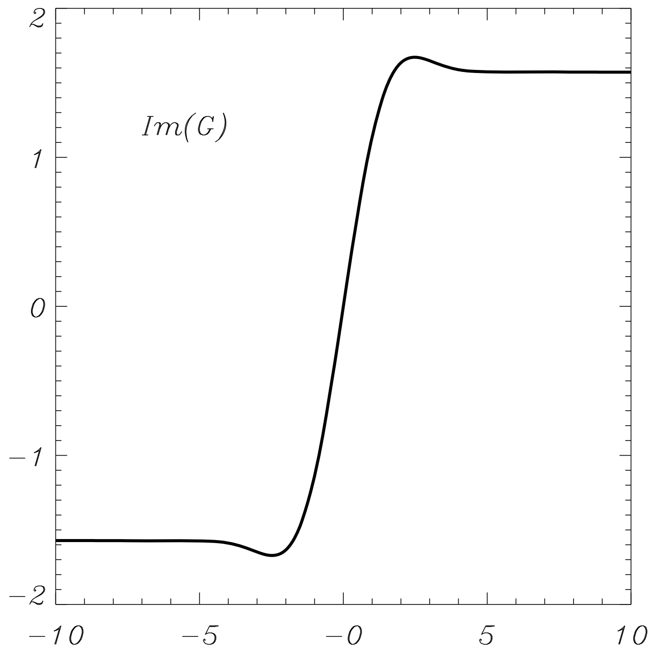
Integrating with respect to τ we obtain

$$\xi_x = \frac{P k_\perp^2}{\rho_A \Delta} G(\tau) + \mathrm{const} \tag{5.16}$$

where

$$G(\tau) = \int_0^\infty \left[\exp(i\sigma\tau) - 1\right] \frac{e^{-\sigma^3/3}}{\sigma}\, d\sigma \qquad (5.17)$$





It can be shown that

$$G(\tau) \approx -\ln|\tau| + \mathrm{const}, \quad \text{as} \quad |\tau| \to \infty \qquad (5.18)$$

which is in agreement with the asymptotics of the ideal solution as $|s| \to 0$ given by Eq. (5.4).

Let us introduce the jump of a quantity across the dissipative layer,

$$[\![f(\tau)]\!] = \lim_{\tau\to\infty} \{f(\tau) - f(-\tau)\}$$

Since P is constant in the dissipative layer, we immediately obtain

$$[\![P]\!] = 0 \tag{5.19}$$

Now

$$[\![G(\tau)]\!] = \lim_{\tau\to\infty} \int_0^\infty [\exp(i\sigma\tau) - \exp(-i\sigma\tau)] \frac{e^{-\sigma^3/3}}{\sigma} d\sigma$$

$$= 2i \lim_{\tau\to\infty} \int_0^\infty \frac{\sin(\sigma\tau)}{\sigma} e^{-\sigma^3/3} d\sigma = 2i \lim_{\tau\to\infty} \int_0^\infty \frac{\sin\sigma}{\sigma} e^{-(\sigma/\tau)^3/3} d\sigma$$

$$= 2i \int_0^\infty \frac{\sin\sigma}{\sigma} d\sigma = 2i\frac{\pi}{2} = \pi i$$

so that

$$[\![\xi_x]\!] = \frac{\pi i P k_\perp^2}{\rho_A \Delta} \tag{5.20}$$

Equations (5.19) and (5.20) are called the **connection formulae**.

The connection formulae connect the solutions at the two sides of the dissipative layer. Using the connection formulae we can consider the dissipative layer as a surface of discontinuity, solve the ideal MHD equations (5.1) and (5.2) at the two sides of these surface, and then connect the obtained solution.

6. Summary and conclusions

- In inhomogeneous plasmas the resonance between global wave/oscillation and local waves/oscillations can occur at resonant magnetic surface where the driver frequency matches either local Alfvén or local slow frequency.
- Resonant MHD waves are characterised by large amplitudes and large spatial gradients in the vicinity of resonant magnetic surface.
- In weakly dissipative plasmas dissipation is only important in a narrow dissipative layer embracing the resonant surface, while the wave motion can be described by ideal MHD equations far from the resonant surface.
- In case of driven torsional waves the wave motion in the dissipative layer is described in terms of F-function.
- In case of laterally driven waves the wave motion in the dissipative layer is described in terms of F and G-function.

- In case of lateral driving there are connection formulae determining the jumps across the dissipative layer of the total pressure and the plasma displacement in the direction perpendicular to the dissipative layer.
- The connection formulae enables us to consider the dissipative layer as a surface of discontinuity. We solve the ideal MHD equations at the two sides of the dissipative layer, and then connect the obtained solutions using the connection formulae.