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Tutorial on the theory of plasma turbulence (Part II)

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Langmuir ($\alpha = L$) and ion-sound wave ($\alpha = S$)

$$\boxed{\begin{aligned} &\omega_k^L = \omega_{pe}\left(1+\frac{3}{2}k^2\lambda_{De}^2\right) = \omega_{pe}\left(1+\frac{3}{4}\frac{k^2 v_{Te}^2}{\omega_{pe}^2}\right). \quad \omega_{-k}^L = -\omega_k^L, \\ &\omega_k^S = \frac{kc_S\left(1+3T_i/T_e\right)^{1/2}}{(1+k^2\lambda_{De}^2)^{1/2}}, \qquad \omega_{-k}^S = -\omega_k^S. \end{aligned}}$$

$$\mathrm{Im}\,\epsilon(k,\pm\omega_k^L) \approx -\pi\,\frac{\omega_{pe}^2}{k}\int dv\,\frac{\partial F_e}{\partial v}\,\delta(\pm\omega_k^L - kv),$$

$$\mathrm{Im}\,\epsilon(k,\pm\omega_k^S) = -\pi\,\frac{\omega_{pe}^2}{k}\int dv\,\left(\frac{\partial F_e}{\partial v}+\frac{m_e}{m_i}\frac{\partial F_i}{\partial v}\right)\,\delta(\pm\omega_k^S - kv),$$

$$\frac{\partial\,\mathrm{Re}\,\epsilon(k,\pm\omega_k^L)}{\partial(\pm\omega_k^L)} \approx \frac{2}{\pm\omega_k^L}, \qquad \frac{\partial\,\mathrm{Re}\,\epsilon(k,\pm\omega_k^S)}{\partial(\pm\omega_k^S)} \approx= \frac{2}{(\pm\omega_k^L)}\frac{1}{\mu_k},$$

$$\mu_k = k^3\lambda_{De}^3\left(\frac{m_e}{m_i}\right)^{1/2}\left(1+\frac{3T_i}{T_e}\right)^{1/2}.$$

Nonlinear Susceptibility:

$$\chi^{a(2)}(k_1,\omega_1|k_2,\omega_2) = -\frac{i}{2}\frac{e_a}{m_a}\frac{\omega_{pa}^2}{k_1+k_2}\int dv\,\frac{1}{\omega_1+\omega_2-(k_1+k_2)\,v+i0}$$
$$\times\frac{\partial}{\partial v}\left[\left(\frac{1}{\omega_1-k_1v+i0}+\frac{1}{\omega_2-k_2v+i0}\right)\frac{\partial F_a}{\partial v}\right]$$
$$=\frac{-i}{2}\frac{e_a}{m_a}\omega_{pa}^2\int dv\,\frac{F_a}{(\omega_1-k_1v)(\omega_2-k_2v)[\omega_1+\omega_2-(k_1+k_2)\,v]}$$
$$\times\left(\frac{k_1}{\omega_1-k_1v}+\frac{k_2}{\omega_2-k_2v}+\frac{k_1+k_2}{\omega_1+\omega_2-(k_1+k_2)\,v}\right).$$

The wave kinetic equation involves two fast waves and one slow wave. If one of the frequencies, say ω_1, is slow wave,

$$\boxed{\chi_a^{(2)}(k',\omega'|k-k',\omega-\omega') = \frac{i}{2}\frac{e_a}{m_a}\frac{k'}{\omega\,(\omega-\omega')}\,\chi_a(k',\omega').}$$

L mode wave kinetic equation

$$\frac{\partial I_k^{\sigma L}}{\partial t} = -\sigma\omega_k^L \, \mathrm{Im}\, \epsilon(k, \sigma\omega_k^L)\, I_k^{\sigma L} \qquad \text{(induced emission)}$$

$$-4\sigma\omega_k^L \sum_{\sigma'=\pm1} \sum_{\beta=L,S} \mathrm{Im} \int dk' \, \chi^{(2)}(k', \sigma'\omega_{k'}^{\beta} | k-k', \sigma\omega_k^L - \sigma'\omega_{k'}^{\beta})\}^2$$
$$\times \mathcal{P} \frac{1}{\epsilon(k-k', \sigma\omega_k^L - \sigma'\omega_{k'}^{\beta})} I_{k'}^{\sigma'\beta}\, I_k^{\sigma L} \qquad \text{(induced scattering)}$$

$$+ 2\pi\, \sigma\omega_k^L \sum_{\sigma',\sigma''=\pm1} \mathrm{Im} \int dk' \, \mu_{k-k'} \Big[\{\chi^{(2)}(k', \sigma'\omega_{k'}^L | k-k', \sigma''\omega_{k-k'}^S)\}^2$$
$$\times \left(\sigma'\omega_{k'}^L \frac{I_{k-k'}^{\sigma'' S}}{\mu_{k-k'}} I_k^{\sigma L} + \sigma''\omega_{k-k'}^L\, I_{k'}^{\sigma' L}\, I_k^{\sigma L} \right) \qquad \text{(decay)}$$
$$+ |\chi^{(2)}(k', \sigma'\omega_{k'}^L | k-k', \sigma''\omega_{k-k'}^S)|^2\, \sigma\omega_k^L\, I_{k'}^{\sigma' L} \frac{I_{k-k'}^{\sigma'' S}}{\mu_{k-k'}} \Big] \; \delta(\sigma\omega_k^L - \sigma'\omega_{k'}^L - \sigma''\omega_{k-k'}^S),$$

S mode wave kinetic equation

$$\frac{\partial}{\partial t}\frac{I_k^{\sigma S}}{\mu_k} = -\mu_k\,\sigma\omega_k^L\,\mathrm{Im}\,\epsilon(k,\sigma\omega_k^S)\,\frac{I_k^{\sigma S}}{\mu_k} \qquad \text{(induced emission)}$$

$$-4\mu_k\,\sigma\omega_k^L \sum_{\sigma'=\pm1}\sum_{\beta=L,S} \mathrm{Im}\int dk'\,\{\chi^{(2)}(k',\sigma'\omega_{k'}^\beta|k-k',\sigma\omega_k^S-\sigma'\omega_{k'}^\beta)\}^2$$
$$\times\,\mathcal{P}\,\frac{1}{\epsilon(k-k',\sigma\omega_k^S-\sigma'\omega_{k'}^\beta)}\,I_{k'}^{\sigma'\beta}\,\frac{I_k^{\sigma S}}{\mu_k} \qquad \text{(induced scattering)}$$

$$+\,\pi\,\sigma\omega_k^L \sum_{\sigma',\sigma''=\pm1} \mathrm{Im}\int dk'\,\mu_k\,\Big[\{\chi^{(2)}(k',\sigma'\omega_{k'}^L|k-k',\sigma''\omega_{k-k'}^L)\}^2$$
$$\times\left(\sigma'\omega_{k'}^L\,I_{k-k'}^{\sigma''L}\,\frac{I_k^{\sigma S}}{\mu_k}+\sigma''\omega_{k-k'}^L\,I_{k'}^{\sigma'L}\,\frac{I_k^{\sigma S}}{\mu_k}\right) \qquad \text{(decay)}$$
$$+\,|\chi^{(2)}(k',\sigma'\omega_{k'}^L|k-k',\sigma''\omega_{k-k'}^L)|^2\,\sigma\omega_k^L\,I_{k'}^{\sigma'L}\,I_{k-k'}^{\sigma''L}\Big]\;\delta(\sigma\omega_k^S-\sigma'\omega_{k'}^L-\sigma''\omega_{k-k'}^L).$$

Induced Emission: Linear Wave-Particle Resonance

$$
\begin{aligned}
\left.\frac{\partial I_k^{\sigma L}}{\partial t}\right|_{\text{ind. emiss.}} &= \pi\,\sigma\omega_k^L\,\frac{\omega_{pe}^2}{k}\int dv\;\delta(\sigma\omega_k^L - kv)\,\frac{\partial F_e}{\partial v}\,I_k^{\sigma L}, \\
\left.\frac{\partial}{\partial t}\right|_{\text{ind. emiss.}}\frac{I_k^{\sigma S}}{\mu_k} &= \pi\,\mu_k\,\sigma\omega_k^L\,\frac{\omega_{pe}^2}{k}\int dv\;\delta(\sigma\omega_k^S - kv)\,\frac{\partial}{\partial v}\left(F_e + \frac{m_e}{m_i}\,F_i\right)\,\frac{I_k^{\sigma S}}{\mu_k}.
\end{aligned}
$$

Decay/Coalescence: Nonlinear Three-Wave Resonance

$$
\begin{aligned}
\left.\frac{\partial I_k^{\sigma L}}{\partial t}\right|_{\text{decay}} &= \frac{\pi}{2}\frac{e^2}{T_e^2}\,\sigma\omega_k^L\sum_{\sigma',\sigma''=\pm1}\int dk'\,\frac{\mu_{k-k'}}{(k-k')^2}\left[\sigma\omega_k^L\,I_{k'}^{\sigma' L}\,\frac{I_{k-k'}^{\sigma'' S}}{\mu_{k-k'}}\right. \\
&\quad \left. -\left(\sigma'\omega_{k'}^L\,\frac{I_{k-k'}^{\sigma'' S}}{\mu_{k-k'}} + \sigma''\omega_{k-k'}^L\,I_{k'}^{\sigma' L}\right)\,I_k^{\sigma L}\right]\,\delta(\sigma\omega_k^L - \sigma'\omega_{k'}^L - \sigma''\omega_{k-k'}^S), \\
\left.\frac{\partial}{\partial t}\right|_{\text{decay}}\frac{I_k^{\sigma S}}{\mu_k} &= \frac{\pi}{4}\frac{e^2}{T_e^2}\,\sigma\omega_k^S\sum_{\sigma',\sigma''=\pm1}\int dk'\,\frac{\mu_k}{k^2}\left[\sigma\omega_k^L\,I_{k'}^{\sigma' L}\,I_{k-k'}^{\sigma'' L} - \left(\sigma'\omega_{k'}^L\,I_{k-k'}^{\sigma'' L}\right.\right. \\
&\quad \left.\left. + \sigma''\omega_{k-k'}^L\,I_{k'}^{\sigma' L}\right)\,\frac{I_k^{\sigma S}}{\mu_k}\right]\,\delta(\sigma\omega_k^S - \sigma'\omega_{k'}^L - \sigma''\omega_{k-k'}^L).
\end{aligned}
$$

Induced Scattering: Nonlinear Wave-Particle Resonance

It turns out that we only need to keep induced scattering term for L mode.

$$\boxed{\begin{aligned} \left.\frac{\partial I_k^{\sigma L}}{\partial t}\right|_{\text{scatt.}} &= \sigma\omega_k^L \frac{\pi}{\omega_{pe}^2}\frac{e^2}{m_e\, m_i} \sum_{\sigma'=\pm 1} \int dk' \int dv \\ &\quad \times (k-k')\,\frac{\partial F_i}{\partial v}\,\delta[\sigma\omega_k^L - \sigma'\omega_{k'}^L - (k-k')\,v]\, I_{k'}^{\sigma' L}\, I_k^{\sigma L}. \end{aligned}}$$

Tsytovich and Melrose empoy the semi-classical formalism. The present statistical and semi-classical formalism are equivalent but in semi-classical method one has to know what process to analyze ahead of time. It is not the most general method to analyze plasma turbulence.

Adding effects of spontaneous thermal fluctuation

In this tutorial I have no time to discuss spontaneous thermal effects, but in general, induced processes must be balanced by spontaneous processes. The equations of plasma turbulence that include spontaneous thermal effects are given below:

Electron Particle Kinetic Equation

$$
\begin{aligned}
\frac{\partial F_e}{\partial t} &= \frac{\partial}{\partial v_i}\left(A_i\, F_e + D_{ij}\,\frac{\partial F_e}{\partial v_j}\right), \\
A_i &= \frac{e^2}{4\pi m_e}\int d\mathbf{k}\,\frac{k_i}{k^2}\sum_{\sigma=\pm1}\sigma\omega_{\mathbf{k}}^L\,\delta(\sigma\omega_{\mathbf{k}}^L - \mathbf{k}\cdot\mathbf{v}), \quad \text{(spont. drag coeff.)} \\
D_{ij} &= \frac{\pi e^2}{m_e^2}\int d\mathbf{k}\,\frac{k_i\,k_j}{k^2}\sum_{\sigma=\pm1}\delta(\sigma\omega_{\mathbf{k}}^L - \mathbf{k}\cdot\mathbf{v})\,I_{\mathbf{k}}^{\sigma L}. \quad \text{(diffusion coeff.)}
\end{aligned}
$$

Forward/backward-Langmuir Wave Kinetic Equation

$$
\begin{aligned}
\frac{\partial I_{\mathbf{k}}^{\sigma L}}{\partial t} \;=\;& \frac{\pi\omega_{pe}^2}{k^2}\int d\mathbf{v}\,\delta(\sigma\omega_{\mathbf{k}}^L-\mathbf{k}\cdot\mathbf{v})\left(\underbrace{\frac{\hat{n}\,e^2}{\pi}F_e}_{\text{spont. emission}}+\underbrace{\sigma\omega_{\mathbf{k}}^L\,I_{\mathbf{k}}^{\sigma L}\,\mathbf{k}\cdot\frac{\partial F_e}{\partial\mathbf{v}}}_{\text{induced emission}}\right)\\
&+2\sum_{\sigma',\sigma''=\pm1}\sigma\omega_{\mathbf{k}}^L\int d\mathbf{k}'\,\frac{\pi}{4}\frac{e^2}{T_e^2}\frac{\mu_{\mathbf{k}-\mathbf{k}'}\,(\mathbf{k}\cdot\mathbf{k}')^2}{k^2\,k'^2\,|\mathbf{k}-\mathbf{k}'|^2}\,\delta(\sigma\omega_{\mathbf{k}}^L-\sigma'\omega_{\mathbf{k}'}^L-\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^S)\\
&\times\left(\underbrace{\sigma\omega_{\mathbf{k}}^L\,I_{\mathbf{k}'}^{\sigma' L}\,I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' S}}_{\text{spont. decay}}\underbrace{-\sigma'\omega_{\mathbf{k}'}^L\,I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' S}\,I_{\mathbf{k}}^{\sigma L}-\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^L\,I_{\mathbf{k}'}^{\sigma' L}\,I_{\mathbf{k}}^{\sigma L}}_{\text{induced decay}}\right)\\
&-\frac{\pi e^2}{m_e^2\,\omega_{pe}^2}\sum_{\sigma'=\pm1}\int d\mathbf{k}'\int d\mathbf{v}\,\frac{(\mathbf{k}\cdot\mathbf{k}')^2}{k^2\,k'^2}\,\delta[\sigma\omega_{\mathbf{k}}^L-\sigma'\omega_{\mathbf{k}'}^L-(\mathbf{k}-\mathbf{k}')\cdot\mathbf{v}]\\
&\times\left[\frac{\hat{n}\,e^2}{\pi\omega_{pe}^2}\sigma\omega_{\mathbf{k}}^L\,(\sigma'\omega_{\mathbf{k}'}^L\,I_{\mathbf{k}}^{\sigma L}-\sigma\omega_{\mathbf{k}}^L\,I_{\mathbf{k}'}^{\sigma' L})\,(F_e+F_i)\qquad\text{spont. scattering}\right.\\
&\left.+\underbrace{I_{\mathbf{k}'}^{\sigma' L}\,I_{\mathbf{k}}^{\sigma L}\,(\mathbf{k}-\mathbf{k}')\cdot\frac{\partial}{\partial\mathbf{v}}\left((\sigma\omega_{\mathbf{k}}^L-\sigma'\omega_{\mathbf{k}'}^L)\,F_e-\frac{m_e}{m_i}(\sigma\omega_{\mathbf{k}}^L)\,F_i\right)}_{\text{induced scattering}}\right].
\end{aligned}
$$

Forward/backward-Ion-sound Wave Kinetic Equation

$$\frac{\partial I_{\mathbf{k}}^{\sigma S}}{\partial t} = \frac{\pi \mu_{\mathbf{k}} \omega_{pe}^2}{k^2} \int d\mathbf{v}\, \delta(\sigma\omega_{\mathbf{k}}^S - \mathbf{k}\cdot\mathbf{v})$$

$$\times \left[\underbrace{\frac{\hat{n}\, e^2}{\pi} (F_e + F_i)}_{\text{spont. emission}} + \underbrace{\sigma\omega_{\mathbf{k}}^L\, I_{\mathbf{k}}^{\sigma S} \left(\mathbf{k}\cdot\frac{\partial}{\partial \mathbf{v}}\right)\left(F_e + \frac{m_e}{m_i} F_i\right)}_{\text{induced emission}} \right]$$

$$\sum_{\sigma',\sigma''=\pm 1} \sigma\omega_{\mathbf{k}}^L \int d\mathbf{k}' \, \frac{\pi}{4} \frac{e^2}{T_e^2} \frac{\mu_{\mathbf{k}}\, [\mathbf{k}'\cdot(\mathbf{k}-\mathbf{k}')]^2}{k^2\, k'^2\, |\mathbf{k}-\mathbf{k}'|^2}\, \delta(\sigma\omega_{\mathbf{k}}^S - \sigma'\omega_{\mathbf{k}'}^L - \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^L)$$

$$\times \Bigg(\underbrace{\sigma\omega_{\mathbf{k}}^L\, I_{\mathbf{k}'}^{\sigma' L}\, I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' L}}_{\text{spont. decay}} \underbrace{- \sigma'\omega_{\mathbf{k}'}^L\, I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' L}\, I_{\mathbf{k}}^{\sigma S} - \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^L\, I_{\mathbf{k}'}^{\sigma' L}\, I_{\mathbf{k}}^{\sigma S}}_{\text{induced decay}} \Bigg) .$$

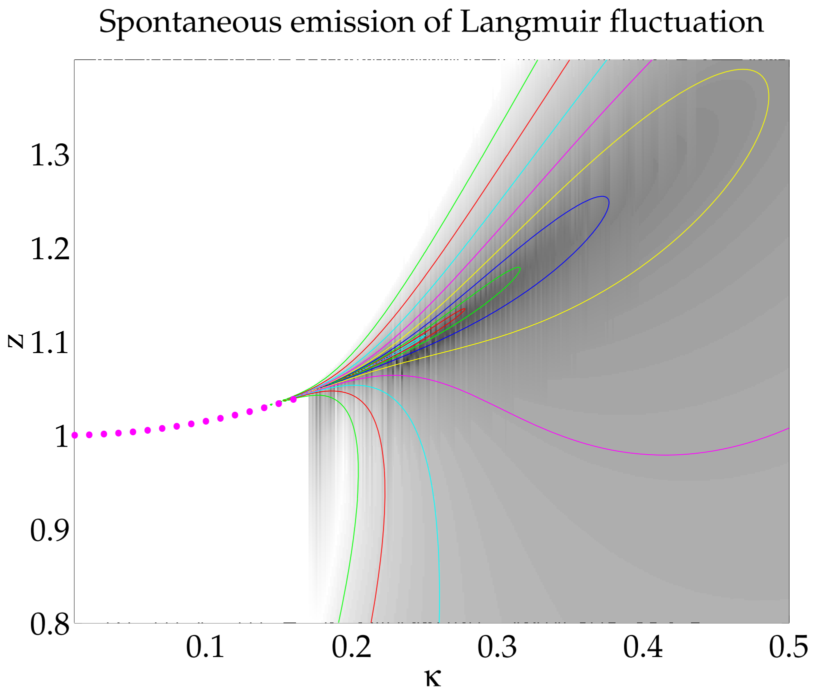
Adding collisional effects to particle kinetic equation: Balescu-Lenard collision integral

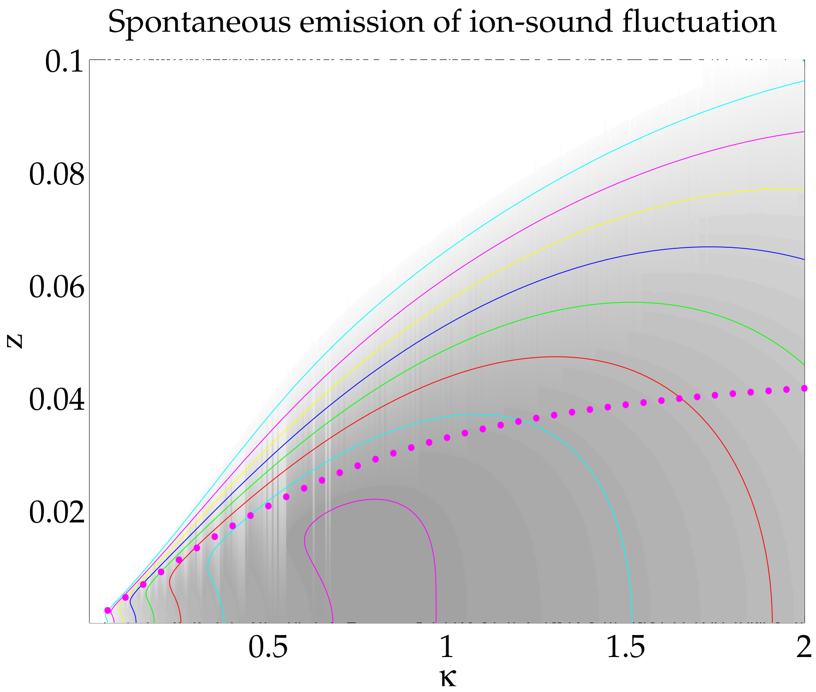
One could add collision integral, i.e., Balescu-Lenard/Landau collision integal, to the particle kinetic equation.

$$
\boxed{
\begin{aligned}
\frac{\partial F_a}{\partial t} &= \sum_b \frac{2\hat{n}e_a^2 e_b^2}{m_a^2} \int d\mathbf{k} \int d\mathbf{v}' \left(\frac{\mathbf{k}}{k}\cdot\frac{\partial}{\partial \mathbf{v}}\right) \mathcal{P}\,\frac{\delta(\mathbf{k}\cdot\mathbf{v}-\mathbf{k}\cdot\mathbf{v}')}{k^2\,|\epsilon(\mathbf{k},\mathbf{k}\cdot\mathbf{v})|^2} \\
&\quad \times \left[F_b(\mathbf{v}')\left(\frac{\mathbf{k}}{k}\cdot\frac{\partial F_a(\mathbf{v})}{\partial \mathbf{v}}\right) - \frac{m_a}{m_b}F_a(\mathbf{v})\left(\frac{\mathbf{k}}{k}\cdot\frac{\partial F_b(\mathbf{v}')}{\partial \mathbf{v}'}\right)\right] \\
&\quad + \frac{\partial}{\partial v_i}\left(A_i\,F_e + D_{ij}\,\frac{\partial F_e}{\partial v_j}\right), \\
A_i &= \frac{e^2}{4\pi m_e}\int d\mathbf{k}\,\frac{k_i}{k^2}\sum_{\sigma=\pm 1}\sigma\omega_{\mathbf{k}}^L\,\delta(\sigma\omega_{\mathbf{k}}^L-\mathbf{k}\cdot\mathbf{v}), \\
D_{ij} &= \frac{\pi e^2}{m_e^2}\int d\mathbf{k}\,\frac{k_i\,k_j}{k^2}\sum_{\sigma=\pm 1}\delta(\sigma\omega_{\mathbf{k}}^L-\mathbf{k}\cdot\mathbf{v})\,I_{\mathbf{k}}^{\sigma L}.
\end{aligned}
}
$$

Adding collisional integral to the particle kinetic equation implies that electric field is made of two parts, one is the usual normal mode contribution and the other is the fluctuation,

$$\begin{aligned}
\left\langle \delta E^2 \right\rangle_{\mathbf{k},\omega} &= \left\langle \delta E^2 \right\rangle^0_{\mathbf{k},\omega} + \sum_{\sigma=\pm1} \sum_{\alpha=L,S} I^{\sigma\alpha}_{\mathbf{k}} \delta(\omega - \sigma\omega^\alpha_{\mathbf{k}}), \\
\left\langle \delta E^2 \right\rangle^0_{\mathbf{k},\omega} &= \frac{2}{\pi} \mathcal{P} \frac{\hat{n}}{k^2 \, |\epsilon(\mathbf{k},\omega)|^2} \sum_a e_a^2 \int d\mathbf{v} \; \delta(\omega - \mathbf{k}\cdot\mathbf{v}) \, F_a(\mathbf{v}).
\end{aligned}$$





Langmuir Turbulence by Beam-Plasma Interaction

Quasi-stationary ions

$$\boxed{F_i = \frac{e^{-v^2/v_{Ti}^2}}{\pi^{1/2}\, v_{Ti}}.}$$

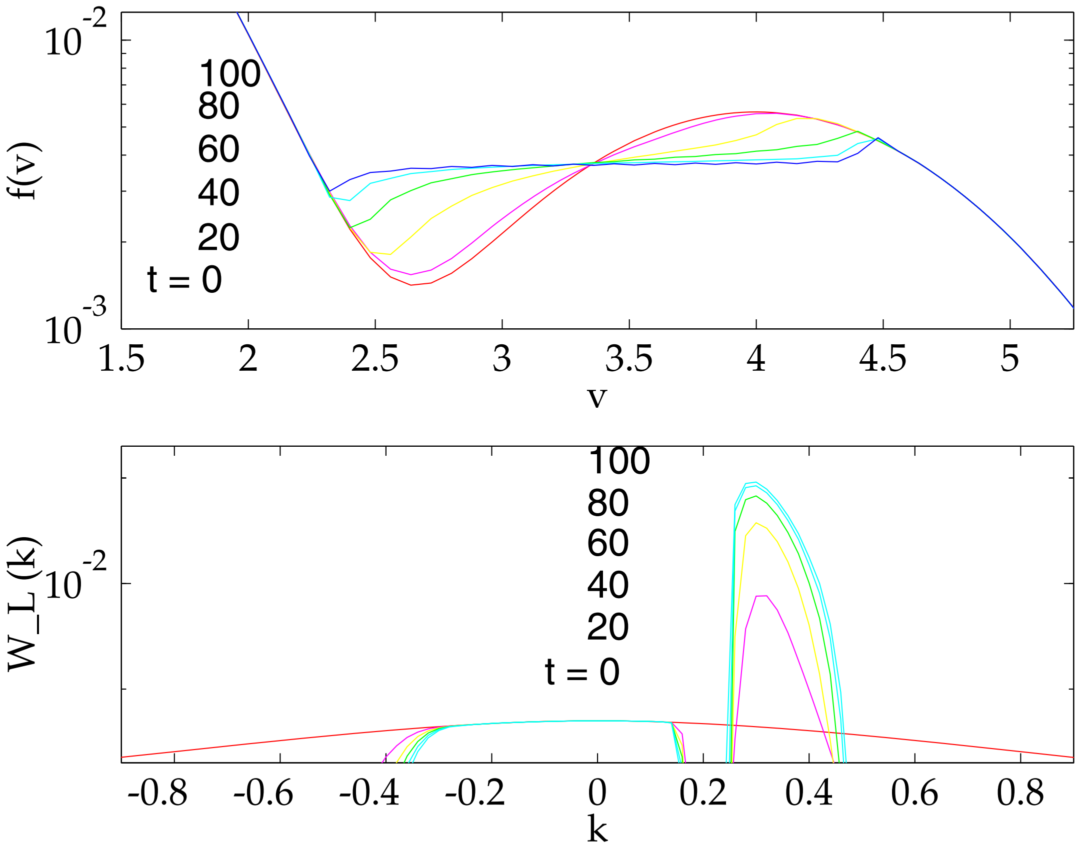
Initial electron distribution

$$\boxed{F_e(v,0) = \frac{(1-\delta)\, e^{-v^2/v_{Te}^2}}{\pi^{1/2}\, v_{Te}} + \frac{\delta\ e^{-(v-v_0)^2/v_{Te}^2}}{\pi^{1/2}\, v_{Te}}.}$$

Quasilinear theory without spontaneous emission

$$\frac{\partial F_e}{\partial t} = \frac{\partial}{\partial v_i}\left(D_{ij}\frac{\partial F_e}{\partial v_j}\right), \qquad D_{ij} = \frac{\pi e^2}{m_e^2}\int d\mathbf{k}\,\frac{k_i\,k_j}{k^2}\sum_{\sigma=\pm1}\delta(\sigma\omega_{\mathbf{k}}^L - \mathbf{k}\cdot\mathbf{v})\,I_{\mathbf{k}}^{\sigma L},$$

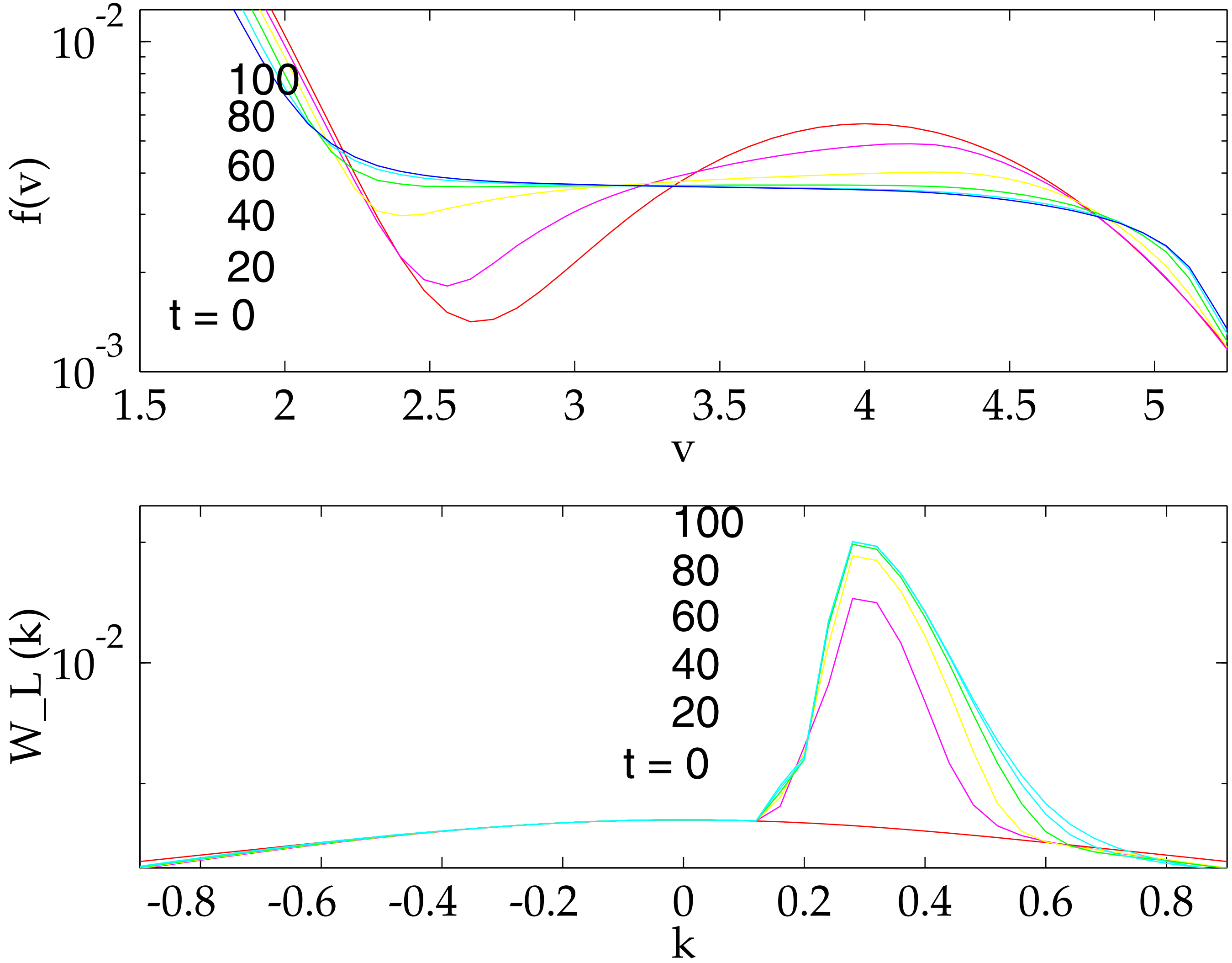
$$\frac{\partial I_{\mathbf{k}}^{\sigma L}}{\partial t} = \frac{\pi\omega_{pe}^2}{k^2}\int d\mathbf{v}\,\delta(\sigma\omega_{\mathbf{k}}^L - \mathbf{k}\cdot\mathbf{v})\left(\sigma\omega_{\mathbf{k}}^L\,I_{\mathbf{k}}^{\sigma L}\,\mathbf{k}\cdot\frac{\partial F_e}{\partial \mathbf{v}}\right).$$



Quasilinear theory with spontaneous emission

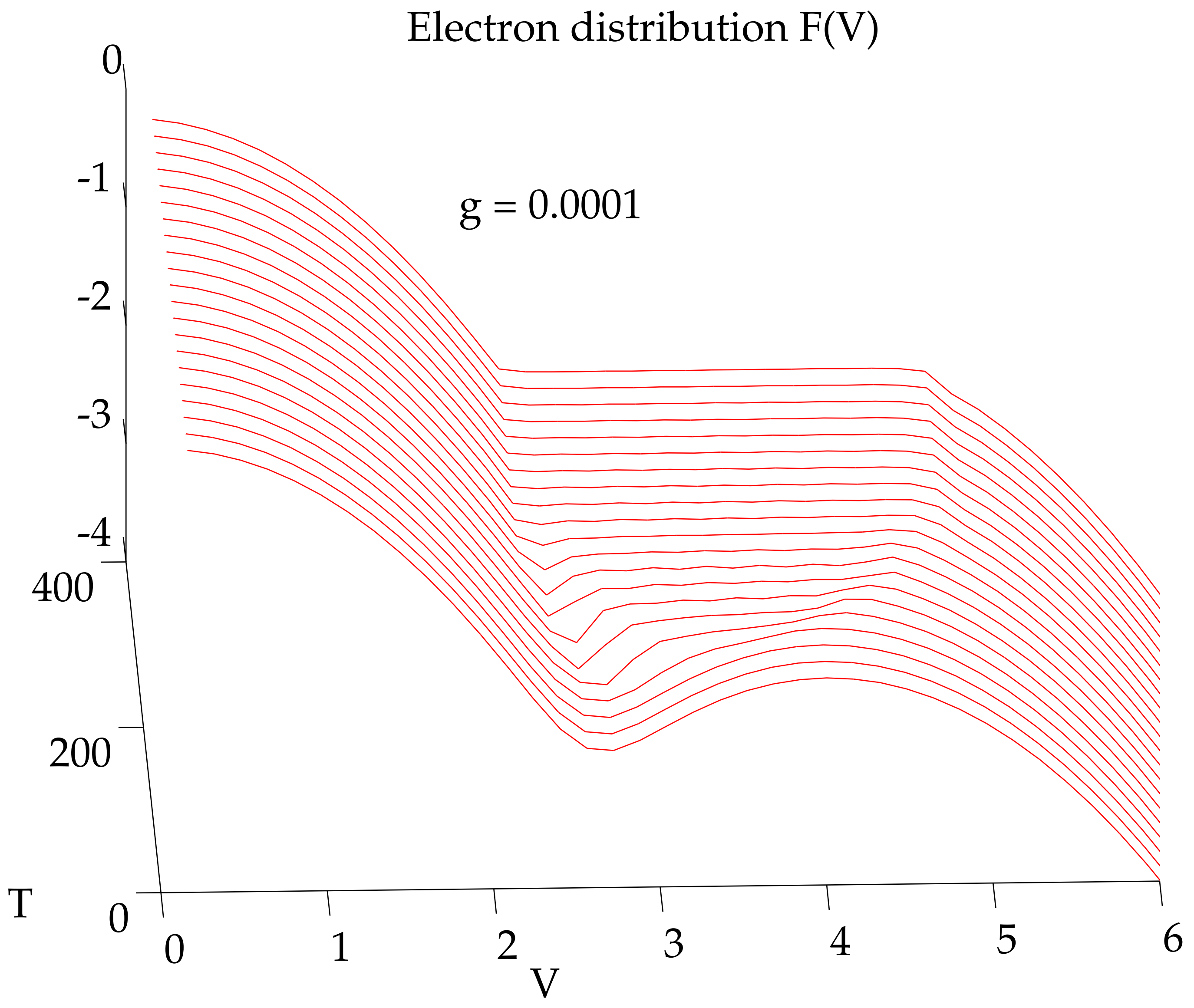
$$\begin{aligned}
\frac{\partial F_e}{\partial t} &= \frac{\partial}{\partial v_i}\left(A_i\,F_e + D_{ij}\,\frac{\partial F_e}{\partial v_j}\right), \\
A_i &= \frac{e^2}{4\pi m_e}\int d\mathbf{k}\,\frac{k_i}{k^2}\sum_{\sigma=\pm1}\sigma\omega_{\mathbf{k}}^L\,\delta(\sigma\omega_{\mathbf{k}}^L-\mathbf{k}\cdot\mathbf{v}), \\
D_{ij} &= \frac{\pi e^2}{m_e^2}\int d\mathbf{k}\,\frac{k_i\,k_j}{k^2}\sum_{\sigma=\pm1}\delta(\sigma\omega_{\mathbf{k}}^L-\mathbf{k}\cdot\mathbf{v})\,I_{\mathbf{k}}^{\sigma L},
\end{aligned}$$

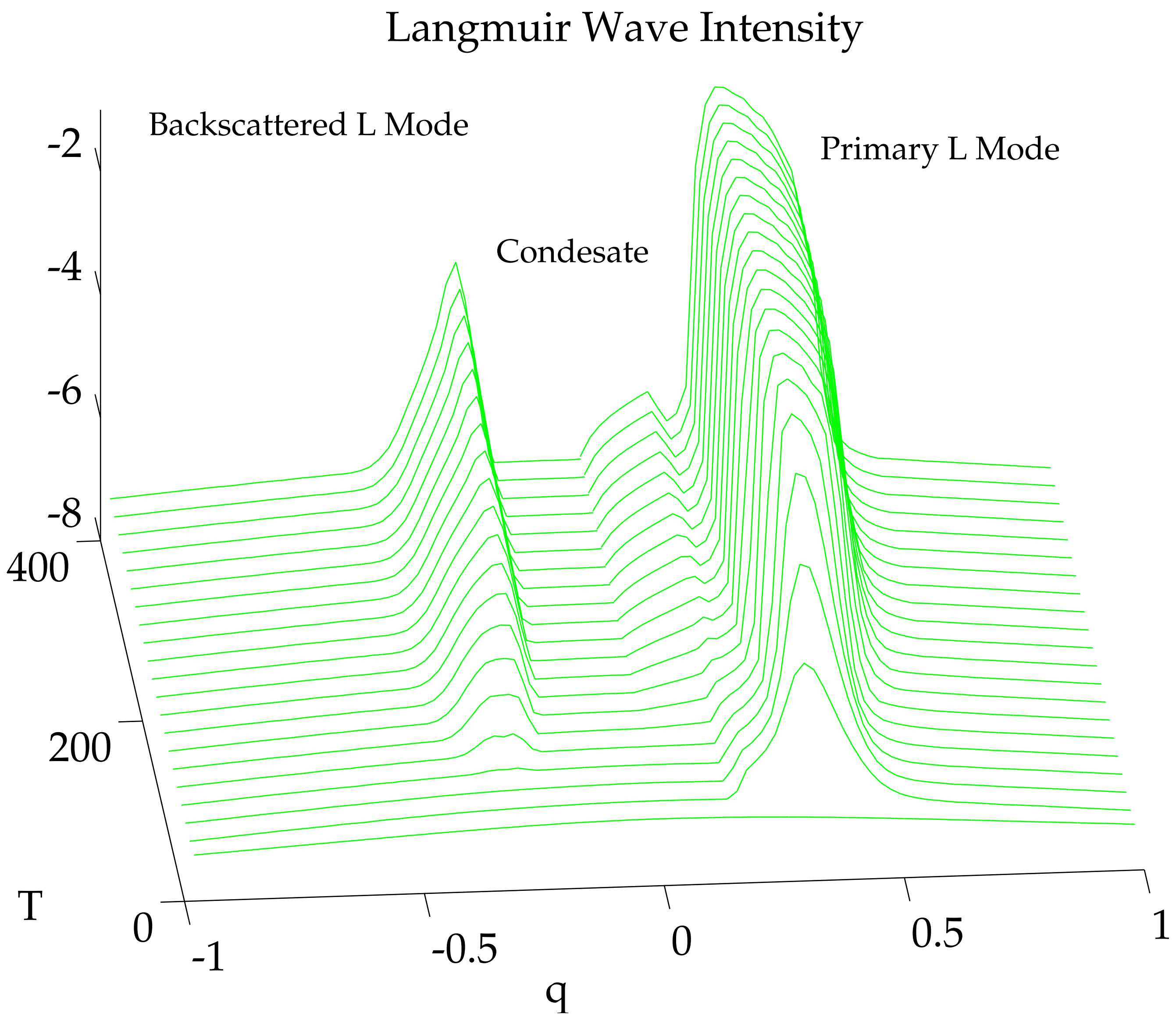
$$\frac{\partial I_{\mathbf{k}}^{\sigma L}}{\partial t} = \frac{\pi\omega_{pe}^2}{k^2}\int d\mathbf{v}\,\delta(\sigma\omega_{\mathbf{k}}^L-\mathbf{k}\cdot\mathbf{v})\left(\frac{\hat{n}\,e^2}{\pi}\,F_e + \sigma\omega_{\mathbf{k}}^L\,I_{\mathbf{k}}^{\sigma L}\,\mathbf{k}\cdot\frac{\partial F_e}{\partial \mathbf{v}}\right).$$

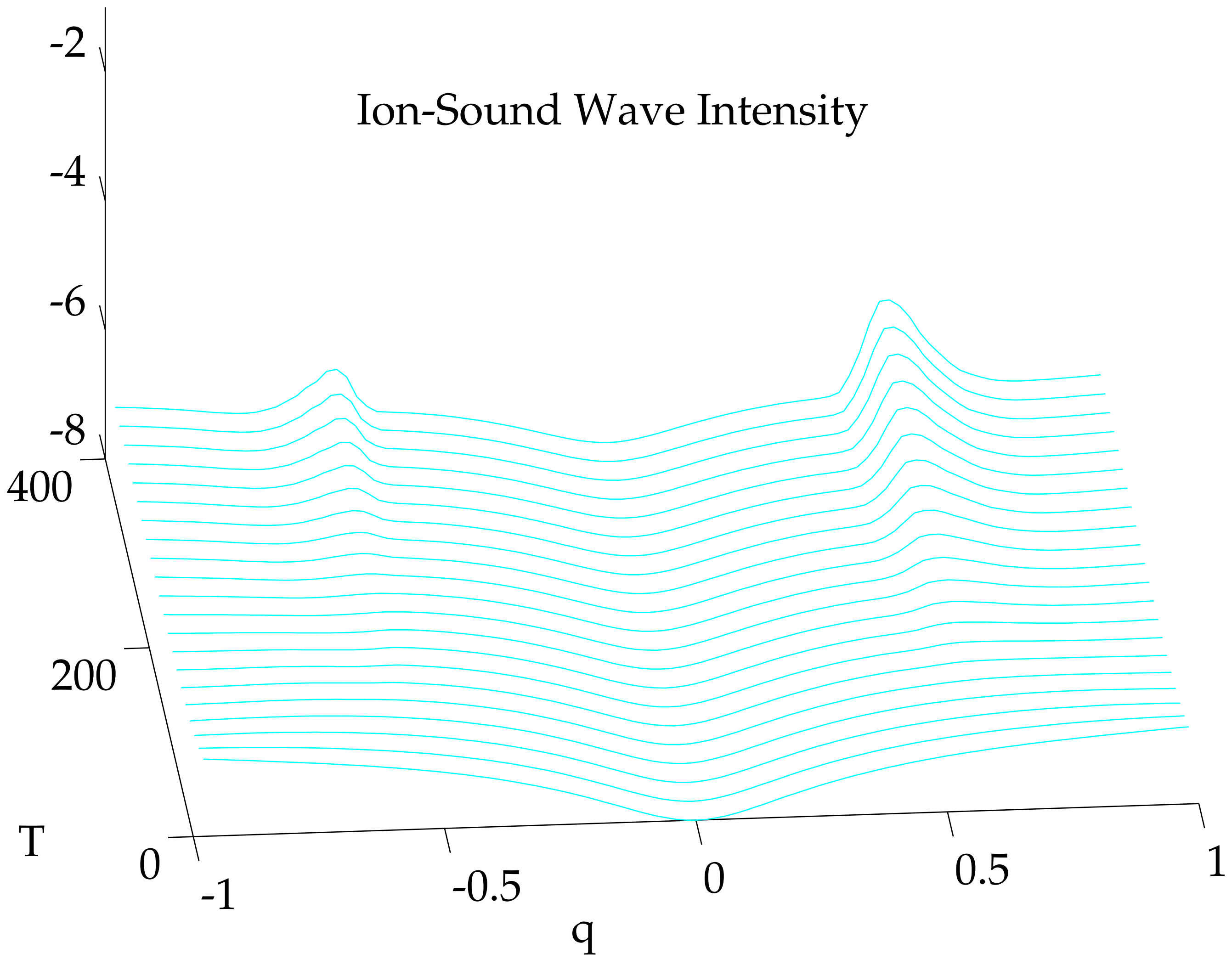


f(v)
10-2
10-3
100
80
60
40
20
t = 0
1.5
2
2.5
3
3.5
4
4.5
5
v
W_L (k)
10-2
100
80
60
40
20
t = 0
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
k

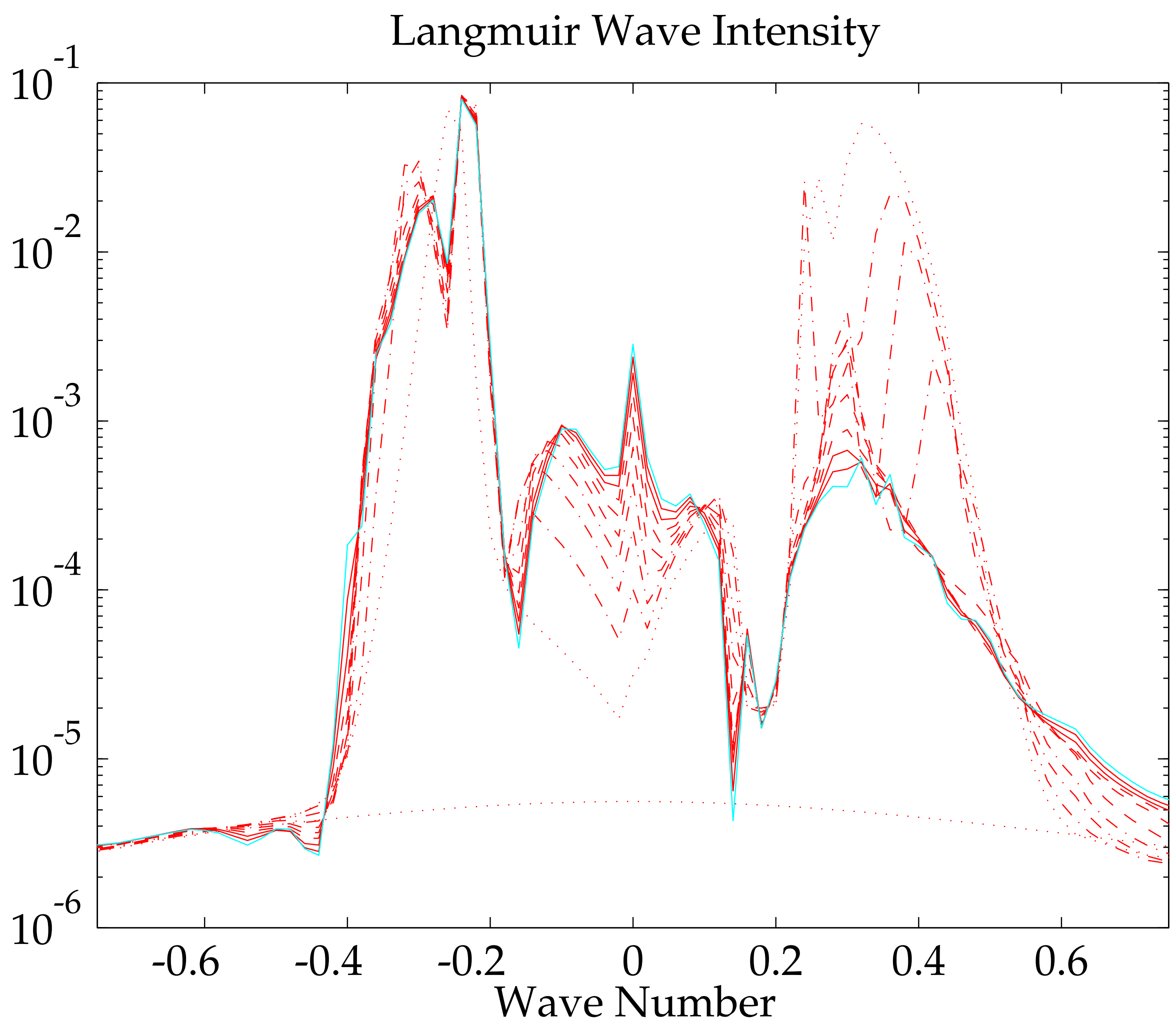
Fully nonlinear theory (weak turbulence)



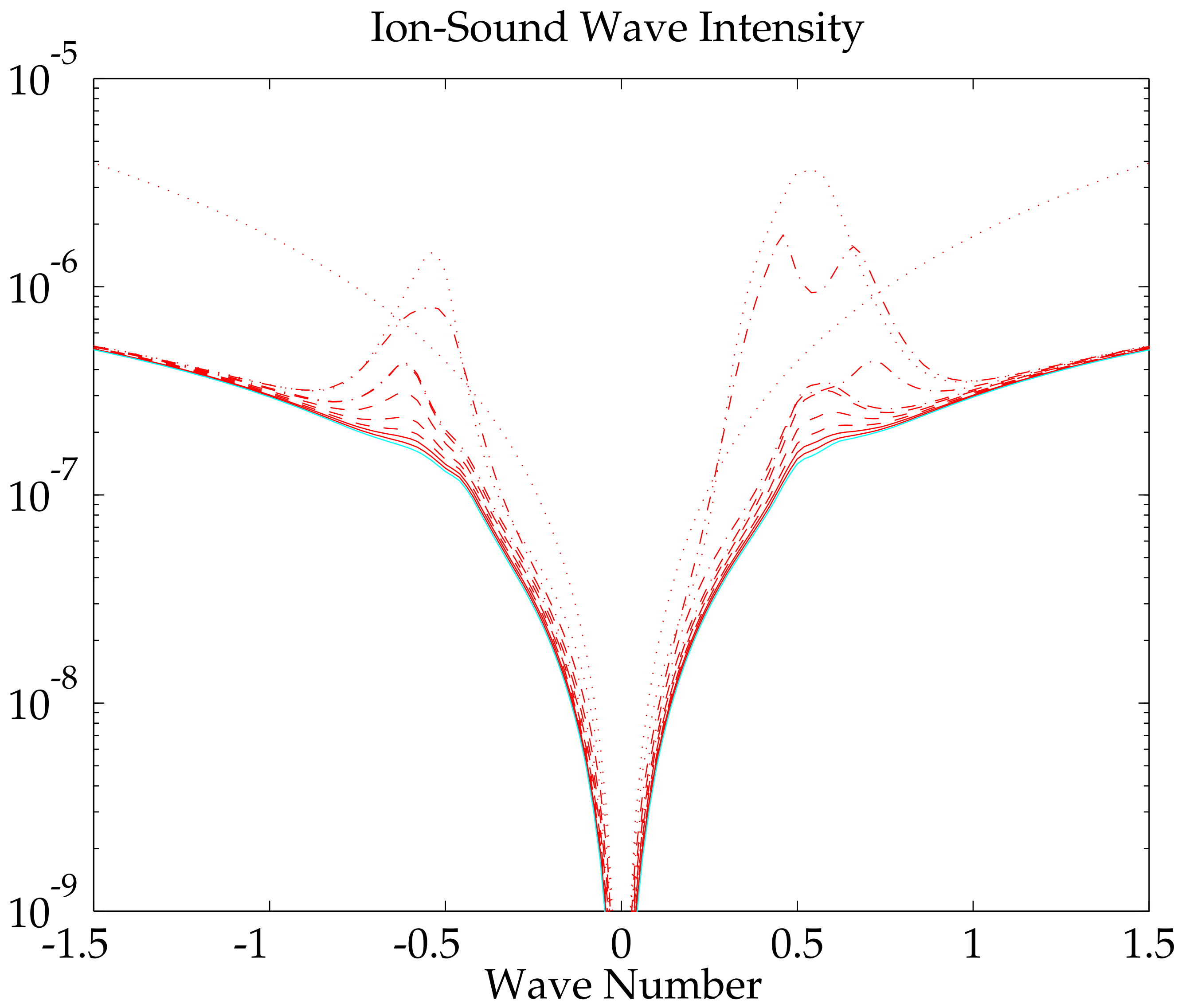

Langmuir Wave Intensity
Backscattered L Mode
Condesate
Primary L Mode
-2
-4
-6
-8
400
200
T
0
-1
-0.5
0
0.5
1
q


Ion-Sound Wave Intensity
-2
-4
-6
-8
400
200
T
0
-1
-0.5
0
0.5
1
q

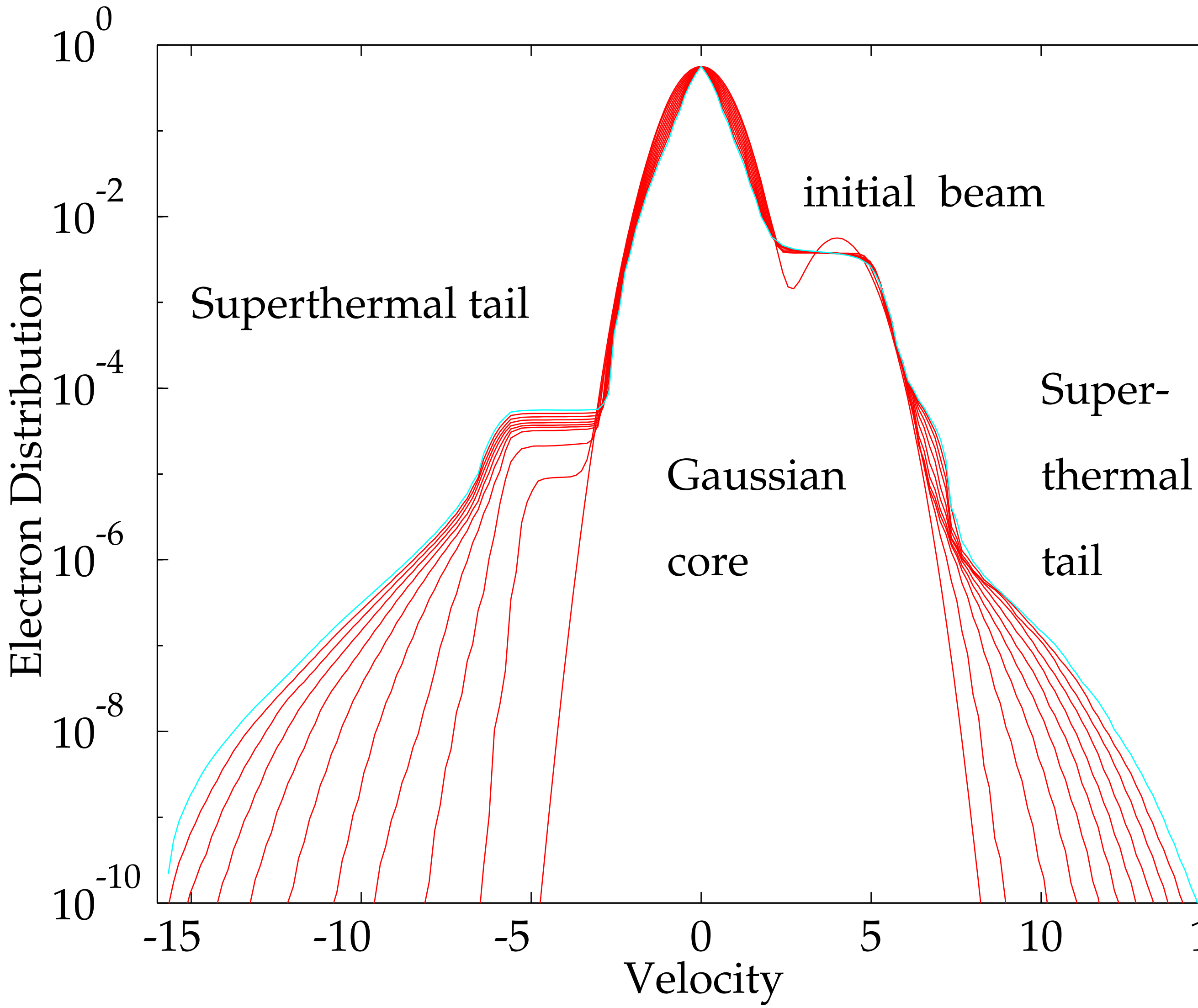
Formation of kappa-like distribution by Langmuir turbulence



Final time $= 2 \times 10^4 \omega_{pe}^{-1}$. Time step $\Delta t = 2 \times 10^3 \omega_{pe}^{-1}$.

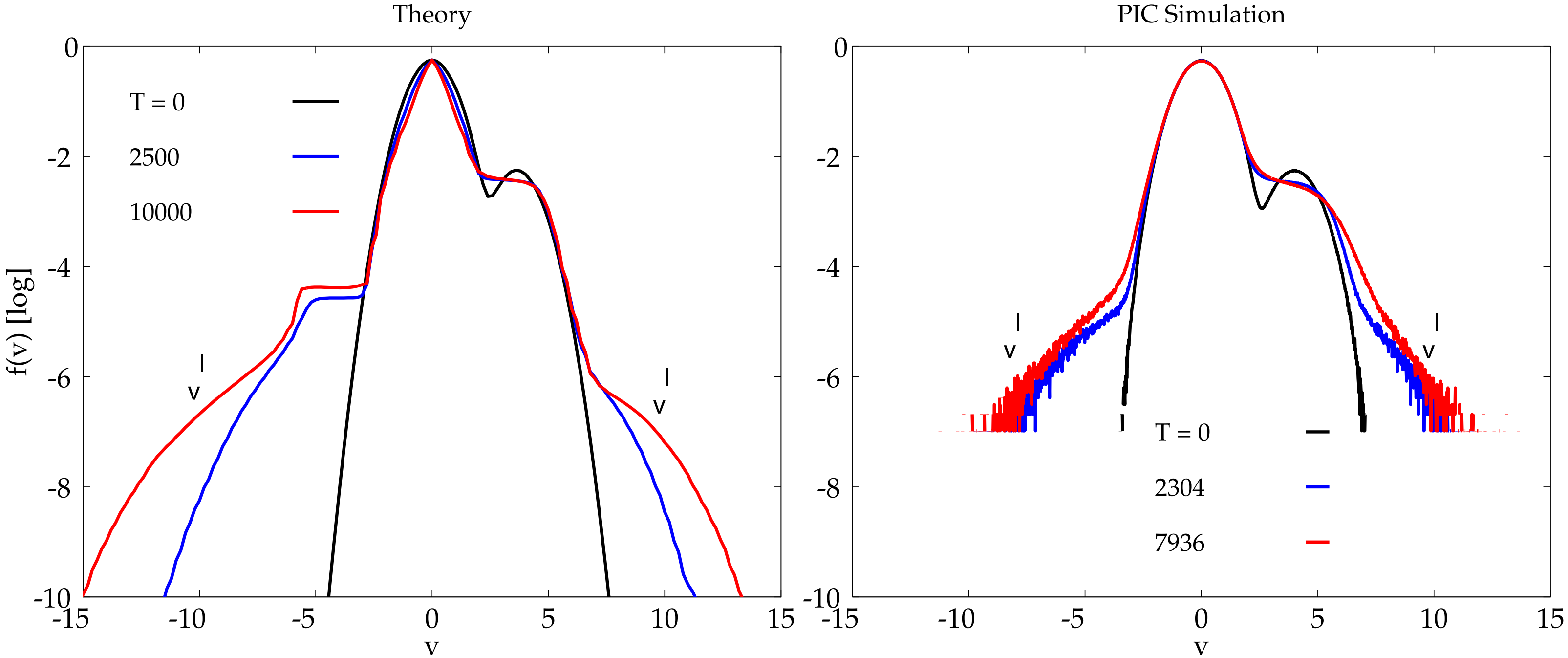


Ion-Sound Wave Intensity
10^{-5}
10^{-6}
10^{-7}
10^{-8}
10^{-9}
-1.5
-1
-0.5
0
0.5
1
1.5
Wave Number

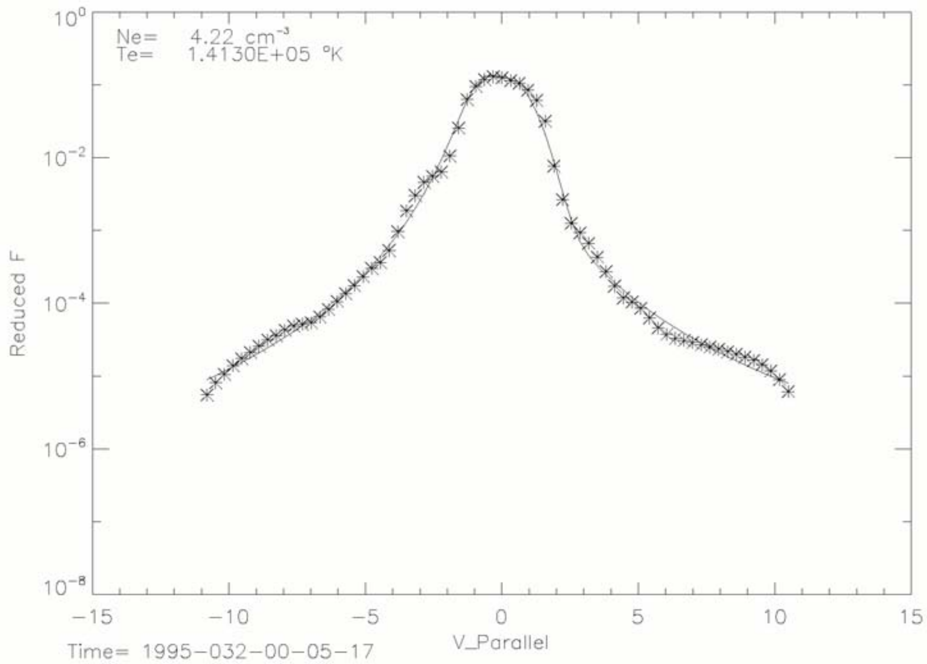


initial beam
Superthermal tail
Gaussian
core
Super-
thermal
tail
Electron Distribution
Velocity
10^{0}
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
-15
-10
-5
0
5
10
15

Particle simulation of kappa-like distribution by Langmuir turbulence



Typical solar wind electron VDF at 1AU



Conclusion

- Weak turbulence theory developed by the pioneers of modern plasma physics is a powerful tool for nonlinear plasma research.

- However, this was not recognized until quite recently. The reason is twofold.

- First, most younger generation of plasma physicists are not trained in this theory, so they do not understand the theory very well.

- Second, numerical solutions of the fundamental equations of weak turbulence theory were not obtained until recently.

Appendix: Weak Turbulence Theory of Ion-Cyclotron Instability

$$\frac{\partial F_a}{\partial t} = \frac{\partial}{\partial \mathbf{v}} \cdot (A_\perp F_a) + \frac{\partial}{\partial \mathbf{v}} \cdot \left(\vec{\vec{D}} \cdot \frac{\partial F_a}{\partial \mathbf{v}} \right), \qquad (1)$$

$$\begin{aligned}
A_\perp &= \frac{e_a^2 B^2 v_\perp}{16\pi^2 n m_a m_i c^2} \sum_{\sigma=\pm 1} \sum_{+,-} \int dk \, \frac{\mp \Omega_a \, g_\pm^\sigma(k)}{\sigma \omega_k^\pm} \, r_\pm, \\
A_\parallel &= \frac{e_a^2 B^2}{16\pi^2 n m_a m_i c^2} \sum_\sigma \int dk \left(\sum_{+,-} \frac{k v_\perp^2 \, g_\pm^\sigma(k)}{\sigma \omega_k^\pm} \, r_\pm + \frac{4\pi n m_i c^2}{B^2} \, v_\parallel \, h(k) \, r_\parallel \right), \\
D_{\perp\perp} &= \frac{\pi e_a^2}{m_a^2} \sum_\sigma \sum_{+,-} \int dk \, \frac{\Omega_a^2 \, I_\pm^\sigma(k)}{(\omega_k^\pm)^2} \, r_\pm, \qquad (2) \\
D_{\perp\parallel} &= D_{\parallel\perp} = \frac{\pi e_a^2}{m_a^2} \sum_\sigma \sum_{+,-} \int dk \, \frac{\mp \Omega_a \, k v_\perp \, I_\pm^\sigma(k)}{(\omega_k^\pm)^2} \, r_\pm, \\
D_{\parallel\parallel} &= \frac{\pi e_a^2}{m_a^2} \sum_\sigma \int dk \left(\sum_{+,-} \frac{k^2 v_\perp^2 \, I_\pm^\sigma(k)}{(\omega_k^\pm)^2} \, r_\pm + I_\parallel^\sigma(k) \, r_\parallel \right),
\end{aligned}$$

where $r_\pm = \delta(\sigma\omega_k^\pm - k v_\parallel \pm \Omega_a)$, $r_\parallel = \delta(\sigma\omega_k^\parallel - k v_\parallel)$, $h(k) = k^2 c_S^2/\omega_{pi}^2$ and $g_\pm^\sigma(k) = (\Omega_i \pm \sigma\omega_k^\pm)^2/[\Omega_i \, (2\Omega_i \pm \sigma\omega_k^\pm)]$.

$$
\begin{aligned}
\frac{\partial I_\parallel^\sigma(k)}{\partial t} &= \pi\sigma c_S\, h(k) \sum_a \omega_{pa}^2 \int d\mathbf{v}\; r_\parallel \left(\frac{m_a}{4\pi^2}\, \sigma c_S\, h(k)\, F_a + \frac{\partial F_a}{\partial v_\parallel}\, I_\parallel^\sigma(k) \right) \\
&+ \frac{\pi\, e^2}{m_i^2}\, \sigma k c_S\, h(k) \sum_{+,-} \sum_{\sigma',\sigma''} \int dk'\, t_\parallel \Big[v_\pm(k,k')\, I_\mp^{\sigma'}(k')\, I_\pm^{\sigma''}(k-k') \\
&- u_\pm(k,k')\, I_\pm^{\sigma''}(k-k')\, I_\parallel^\sigma(k) + w_\pm(k,k')\, I_\mp^{\sigma'}(k') I_\parallel^\sigma(k) \Big], \qquad (3)
\end{aligned}
$$

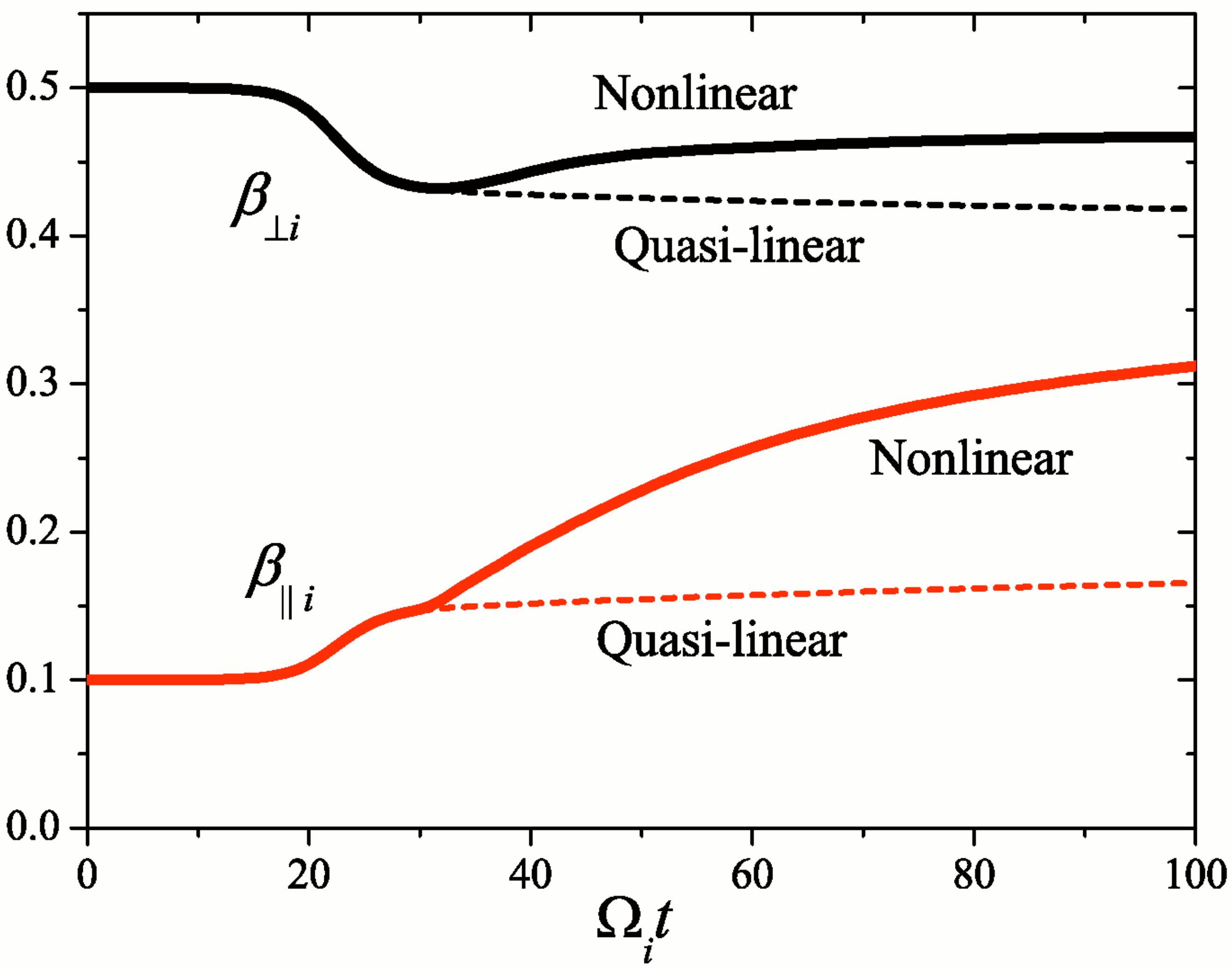
where $t_\parallel = \delta(\sigma\omega_k^\parallel - \sigma'\omega_{k'}^\mp - \sigma''\omega_{k-k'}^\pm)$, and

$$
\begin{aligned}
v_\pm(k,k') &= \frac{\sigma\, \omega_{pi}^4\, h(k)\, M(k,k')^2}{(\omega_{k'}^\mp)^2\, (\omega_{k-k'}^\pm)^2\, k^3 c_S^3}, \\
u_\pm(k,k') &= \frac{\omega_{pi}^2\, \Omega_i^2\, g_\mp^{\sigma'}(k')\, M(k,k')}{(\sigma'\omega_{k'}^\mp)\, (\sigma''\omega_{k-k'}^\pm)\, (\Omega_i \pm \sigma''\omega_{k-k'}^\pm)\, k^3 c_S^4}, \\
w_\pm(k,k') &= \frac{\omega_{pi}^2\, \Omega_i^2\, g_\pm^{\sigma''}(k-k')\, M(k,k')}{(\sigma'\omega_{k'}^\mp)\, (\sigma''\omega_{k-k'}^\pm)\, (\Omega_i \mp \sigma'\omega_{k'}^\mp)\, k^3 c_S^4}, \\
M(k,k') &= \frac{\sigma'' k'\, \omega_{k-k'}^\pm}{\Omega_i \pm \sigma''\omega_{k-k'}^\pm} - \frac{\sigma'(k-k')\, \omega_{k'}^\mp}{\Omega_i \mp \sigma'\omega_{k'}^\mp}. \qquad (4)
\end{aligned}
$$

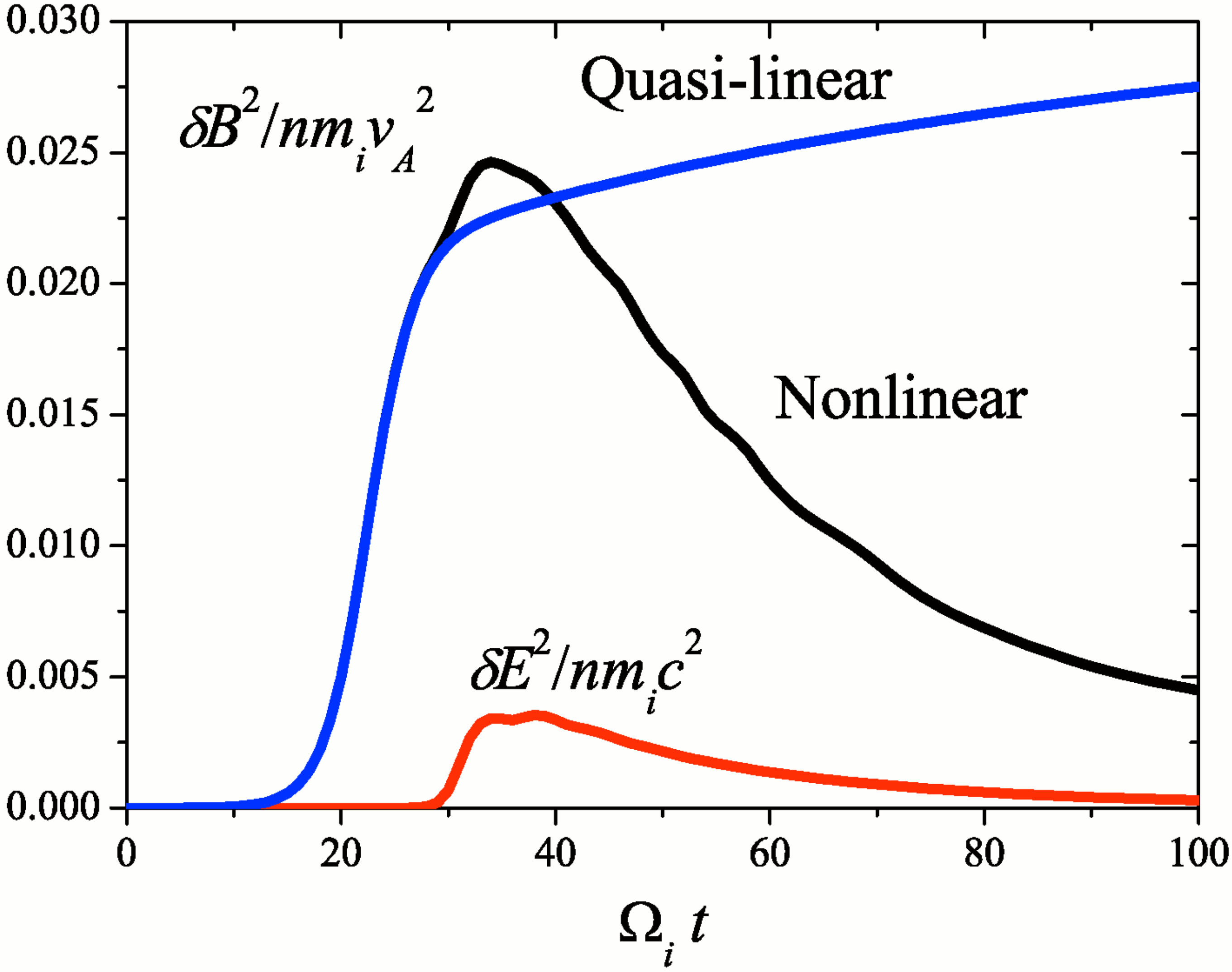
$$
\begin{aligned}
\frac{\partial I_{\pm}^{\sigma}(k)}{\partial t} &= \pi \frac{\Omega_i^2}{\omega_{pi}^2} g_{\pm}^{\sigma}(k) \sum_a \frac{\omega_{pa}^2}{\sigma\omega_k^{\pm}} \int d\mathbf{v}\, v_\perp^2\, r_\pm \left[\frac{m_a}{4\pi^2} \frac{\Omega_i^2}{\omega_{pi}^2} \sigma\omega_k^{\pm}\, g_{\pm}^{\sigma}(k)\, F_a \right. \\
&+ \left. \left(\mp\Omega_a \frac{\partial F_a}{v_\perp \partial v_\perp} + k \frac{\partial F_a}{\partial v_\parallel} \right) I_{\pm}^{\sigma}(k) \right] \\
&+ \frac{2\pi\, e^2}{m_i^2} \frac{\Omega_i^2}{(\omega_k^{\pm})^2} \sigma\omega_k^{\pm}\, g_{\pm}^{\sigma}(k) \sum_{\sigma',\sigma''} \int dk'\, t_\pm \left[V_\pm(k,k')\, I_{\pm}^{\sigma'}(k')\, I_{\parallel}^{\sigma''}(k-k') \right. \\
&- \left. U_\pm(k,k')\, I_{\parallel}^{\sigma''}(k-k')\, I_{\pm}^{\sigma}(k) - W_\pm(k,k')\, I_{\pm}^{\sigma'}(k')\, I_{\pm}^{\sigma}(k) \right], \qquad (5)
\end{aligned}
$$

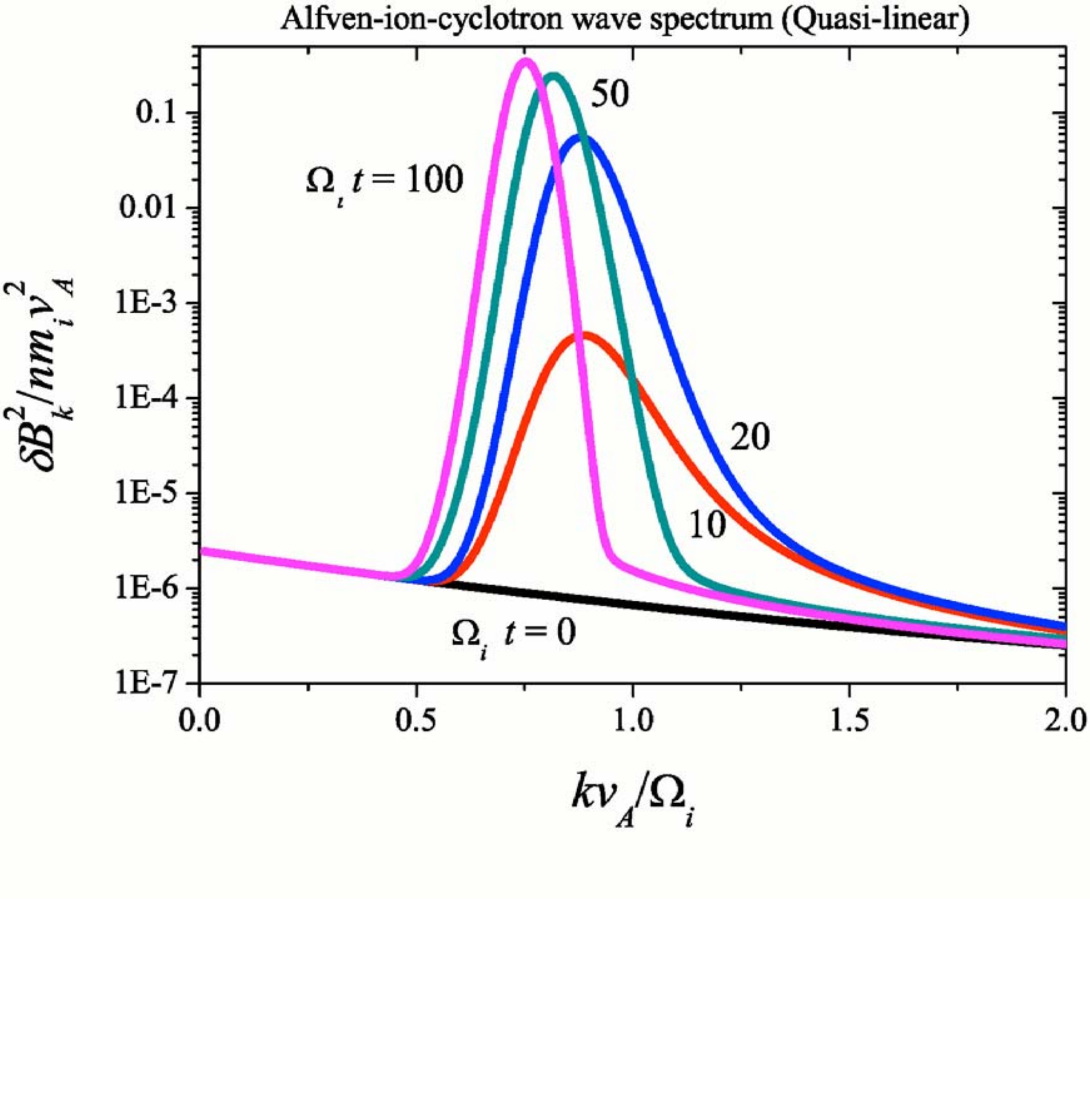
where $t_\pm = \delta(\sigma\omega_k^{\pm} - \sigma'\omega_{k'}^{\pm} - \sigma''\omega_{k-k'}^{\parallel})$, and

$$
\begin{aligned}
V_\pm(k,k') &= \frac{\sigma\omega_k^{\pm}\, \Omega_i^2\, g_{\pm}^{\sigma}(k)}{(\Omega_i \pm \sigma'\omega_{k'}^{\pm})^2\, (k-k')^2 c_S^4}, \\
U_\pm(k,k') &= \frac{\sigma\omega_k^{\pm}\, \Omega_i^2\, g_{\pm}^{\sigma'}(k')}{(\Omega_i \pm \sigma'\omega_{k'}^{\pm})\, (\Omega_i \pm \sigma\omega_k^{\pm})\, (k-k')^2\, c_S^4}, \\
W_\pm(k,k') &= \frac{\sigma''\, \omega_{pi}^2\, h(k-k')}{(\sigma'\omega_{k'}^{\pm})\, (\Omega_i \pm \sigma'\omega_{k'}^{\pm})\, (k-k')^2\, c_S^3} \left(\frac{\sigma' k\, \omega_{k'}^{\pm}}{\Omega_i \pm \sigma'\omega_{k'}^{\pm}} - \frac{\sigma k'\, \omega_k^{\pm}}{\Omega_i \pm \sigma\omega_k^{\pm}} \right). \qquad (6)
\end{aligned}
$$

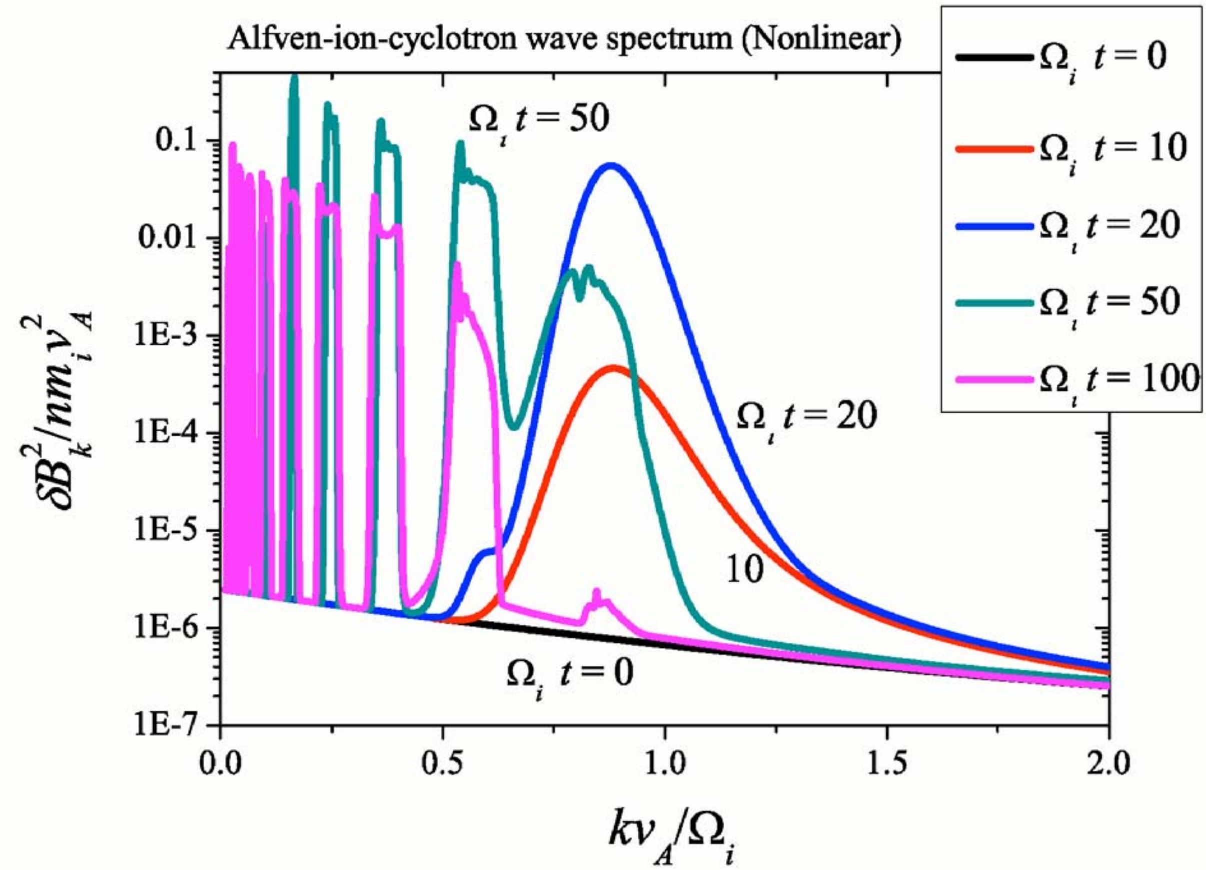


0.5
0.4
0.3
0.2
0.1
0.0
0
20
40
60
80
100
$\Omega_i t$
Nonlinear
$\beta_{\perp i}$
Quasi-linear
Nonlinear
$\beta_{\parallel i}$
Quasi-linear

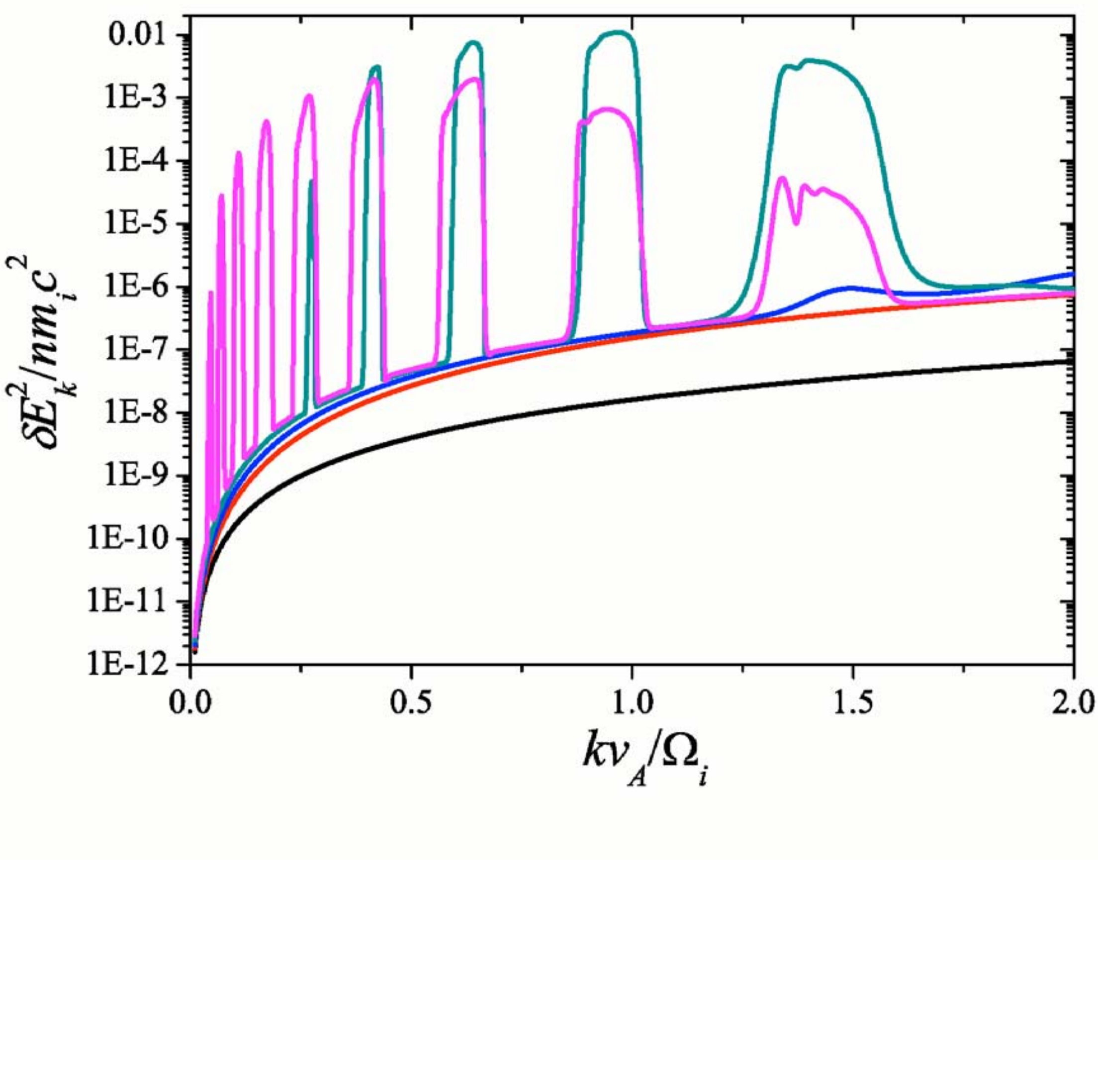

0.030
0.025
0.020
0.015
0.010
0.005
0.000
0
20
40
60
80
100
$\Omega_i\, t$
Quasi-linear
$\delta B^2/nm_i v_A^{\ 2}$
Nonlinear
$\delta E^2/nm_i c^2$



Alfven-ion-cyclotron wave spectrum (Quasi-linear)
0.1
0.01
1E-3
1E-4
1E-5
1E-6
1E-7
$\delta B_k^2/nm_i v_A^2$
50
$\Omega_i t = 100$
20
10
$\Omega_i\ t = 0$
0.0
0.5
1.0
1.5
2.0
kv_A/Ω_i



Alfven-ion-cyclotron wave spectrum (Nonlinear)
$\Omega_i\ t = 0$
$\Omega_i\ t = 10$
$\Omega_i\ t = 20$
$\Omega_i\ t = 50$
$\Omega_i\ t = 100$
$\Omega_i\ t = 50$
$\Omega_i\ t = 20$
10
$\Omega_i\ t = 0$
0.1
0.01
1E-3
1E-4
1E-5
1E-6
1E-7
0.0
0.5
1.0
1.5
2.0
$\delta B_k^2/nm_i v_A^2$
kv_A/Ω_i



0.01
1E-3
1E-4
1E-5
1E-6
1E-7
1E-8
1E-9
1E-10
1E-11
1E-12
$\delta E_k^2/nm_ic^2$
0.0
0.5
1.0
1.5
2.0
kv_A/Ω_i