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Mirror instability in space plasmas: solions and cnoidal waves

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Mirror instability in space plasmas: solions and cnoidal waves

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For small p (taking only the linear terms) we have (if $k_z, k_y \neq 0$)

$$\frac{p}{k_z} = \sqrt{\frac{T_{\parallel}}{M}} \; \frac{\frac{8\pi n_o T_{\perp}}{H_o^2}\left(\frac{T_{\perp}}{T_{\parallel}} - 1\right) - \left(1 + \frac{8\pi n_o T_{\perp}}{H_o^2} - \frac{8\pi n_o T_{\parallel}}{H_o^2}\right)\frac{k_z^2}{k_y^2} - 1}{\sqrt{(\frac{1}{2}\pi)}\; 8\pi n_o T_{\perp}^2 / H_o^2 T_{\parallel}} .$$

Accordingly, the criterion of instability ($p > 0$) is

$$\frac{8\pi n_o T_{\perp}}{H_o^2}\;\frac{T_{\perp}}{T_{\parallel}} > 1 + \frac{8\pi n_o T_{\perp}}{H_o^2} .$$

In the limit $H_o \longrightarrow 0$, the condition (8) becomes $T_{\perp} > T_{\parallel}$.

If $k_y = 0$, the terms linear in p disappear from equation (5), and t terms give

$$\frac{p^2}{k_y^2} = - \frac{H_o^2}{4\pi n_o M}\left\{1 + \frac{8\pi n_o T_{\perp}}{H_o^2} - \frac{8\pi n_o T_{\parallel}}{H_o^2}\right\}.$$

Equation (9) corresponds to the Alfvén magnetohydrodynamic branch. For

MI linear growth rate for the arbitrary distribution function

Field line bending

FLR effect

$$\gamma_L = \frac{m p_\perp \left|k_\parallel\right|}{\pi^2 D}\left[L - \frac{k_\parallel^2}{\beta_\perp k_\perp^2}\left(1+\frac{\beta_\perp - \beta_\parallel}{2}\right) - \frac{3}{2\beta_\perp}k_\perp^2\rho_i^2\right]$$

Inverse response

$$\text{L=A-E-}\beta_\perp^{-1} \quad \text{and} \quad \text{D} = -\int\left(W - W_0\right)^2 \frac{\partial F_{res}}{\partial W}\,dW$$

$$W_0 = \frac{e\sum_j q_j\left\langle \mu B_0 \partial F_j / \partial W\right\rangle}{\sum_j \left\langle \partial F_j / \partial W\right\rangle}$$

Instabilty threshold

Notations - arbitrary velocity distribution function

Instability threshold

$$L = A - \frac{1}{\beta_{\perp i}} + \frac{\left(\sum_j q_j \left\langle \mu B_0 \partial F_j / \partial W \right\rangle\right)^2}{2 p_{\perp i} \sum_j q_j \left\langle \partial F_j / \partial W \right\rangle}$$

Anisotropy

$$A = -\frac{1}{2 p_{\perp i}} \sum_j \left\langle \mu^2 B_0^2 \frac{\partial F_j}{\partial W} \right\rangle - \frac{p_\perp}{p_{\perp i}}$$

Notations – Bi-Maxwellean distribution

Anisotropy

$$A = \frac{T_\perp}{T_\parallel} - 1 \quad \text{(bi-Maxwell)}$$

$$E = \frac{\left(T_\perp / T_\parallel - 1\right)^2 T_e}{2 T_\perp \left(1 + T_e / T_\parallel\right)}$$

Threshold

$$L = \frac{T_\perp}{T_\parallel} - 1 - \frac{\left(T_\perp / T_\parallel - 1\right)^2 T_e}{2 T_\perp \left(1 + T_e / T_\parallel\right)} - \frac{1}{\beta_\perp}$$

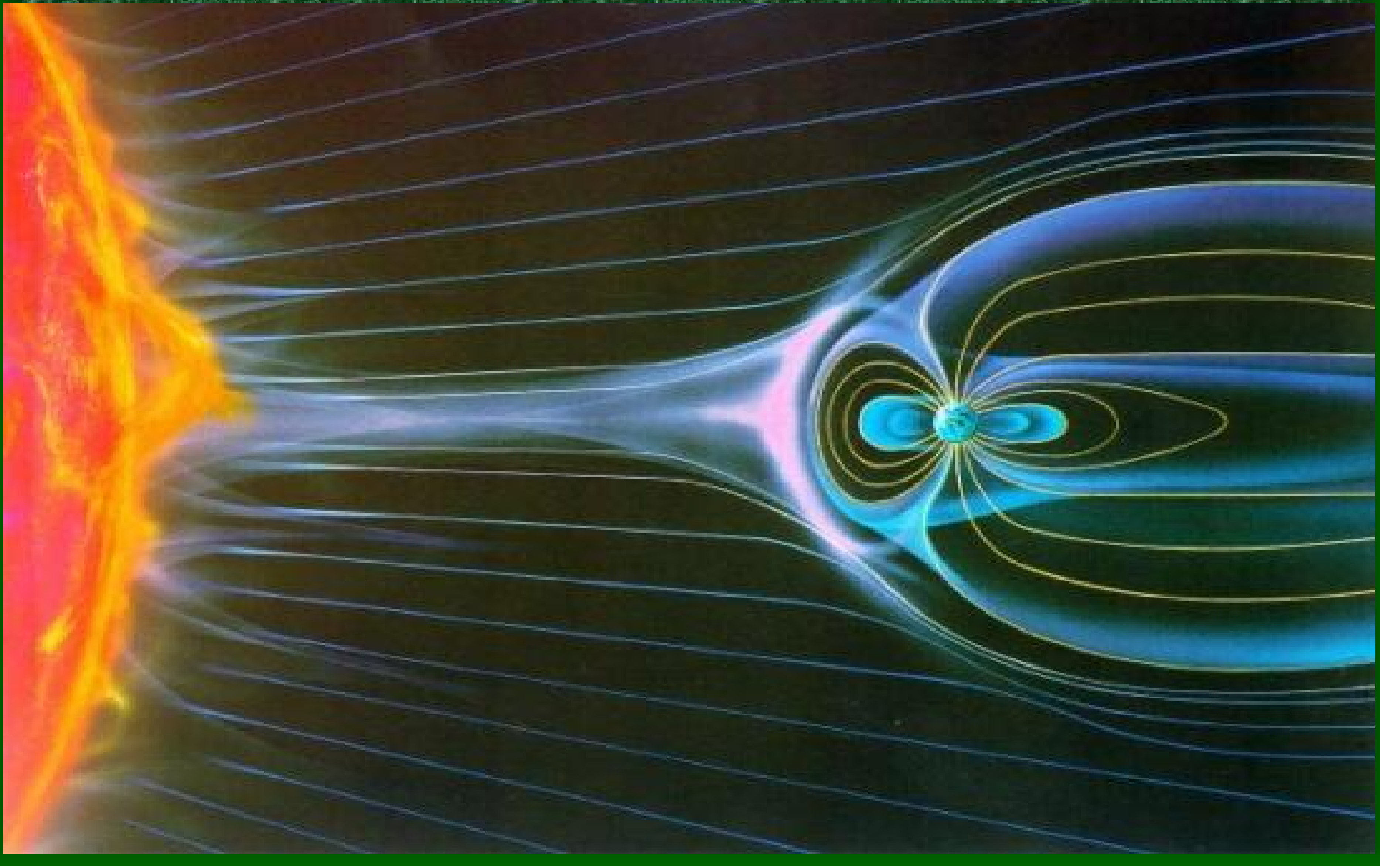
Mirror force

Electrostatic force

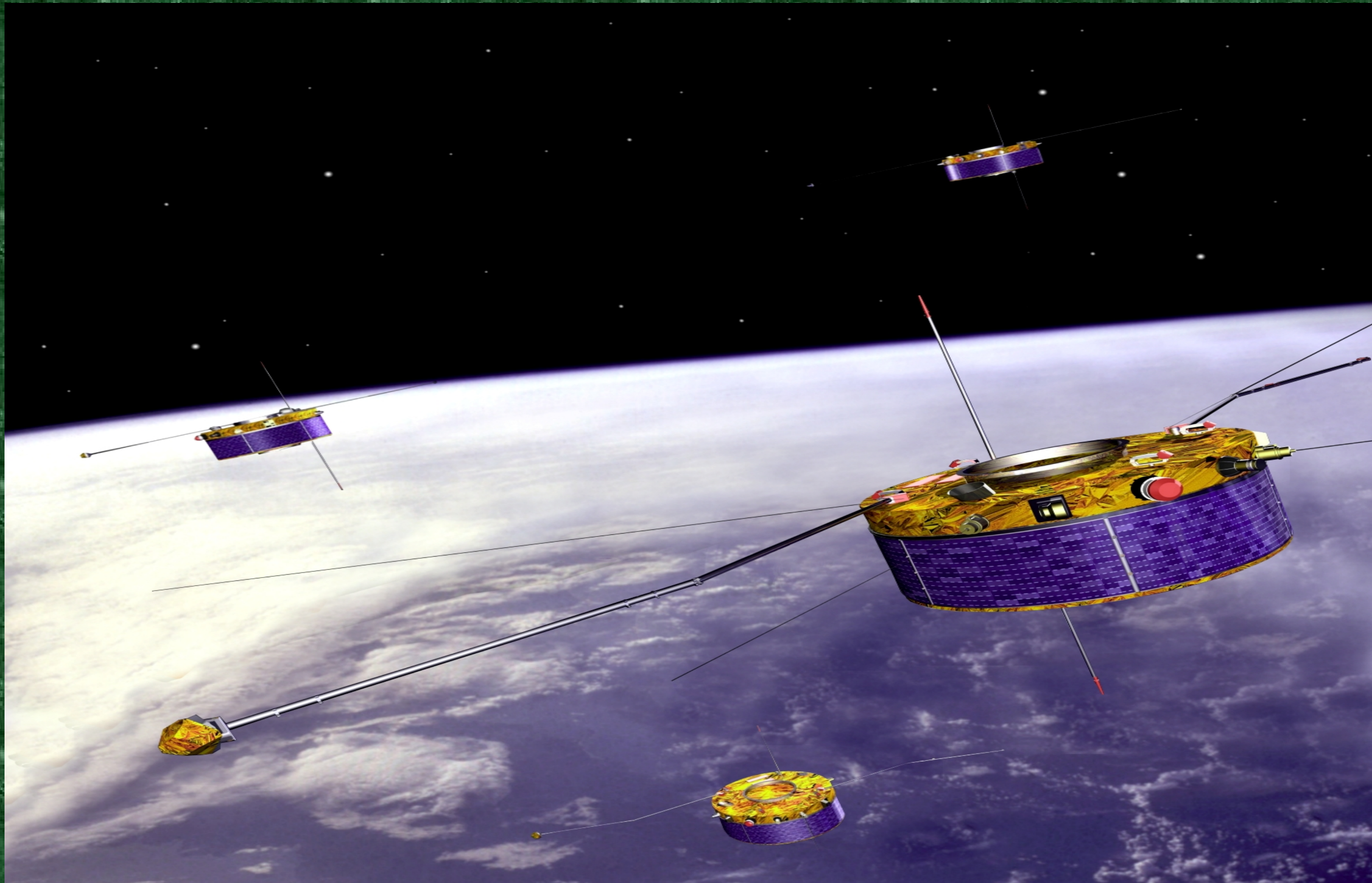
Mirror waves are found in:

- Ring-current plasma
- Earth's magnetosheath
- Planetary magnetosheaths
- Cometary comas
- Wake of Io
- Solar wind
- Laboratory plasmas (teta-pinches)

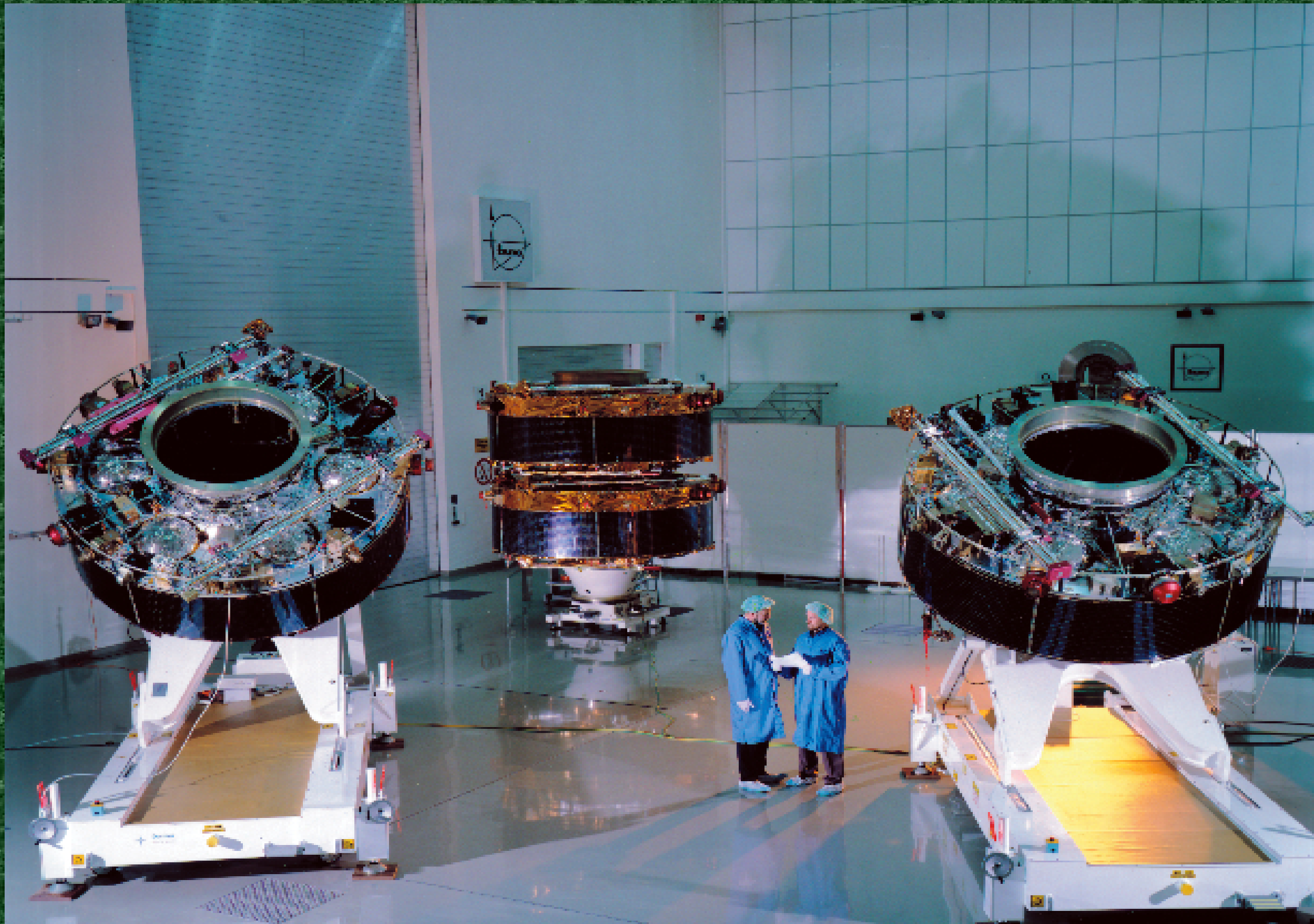
Solar-Terrestrial Interactions



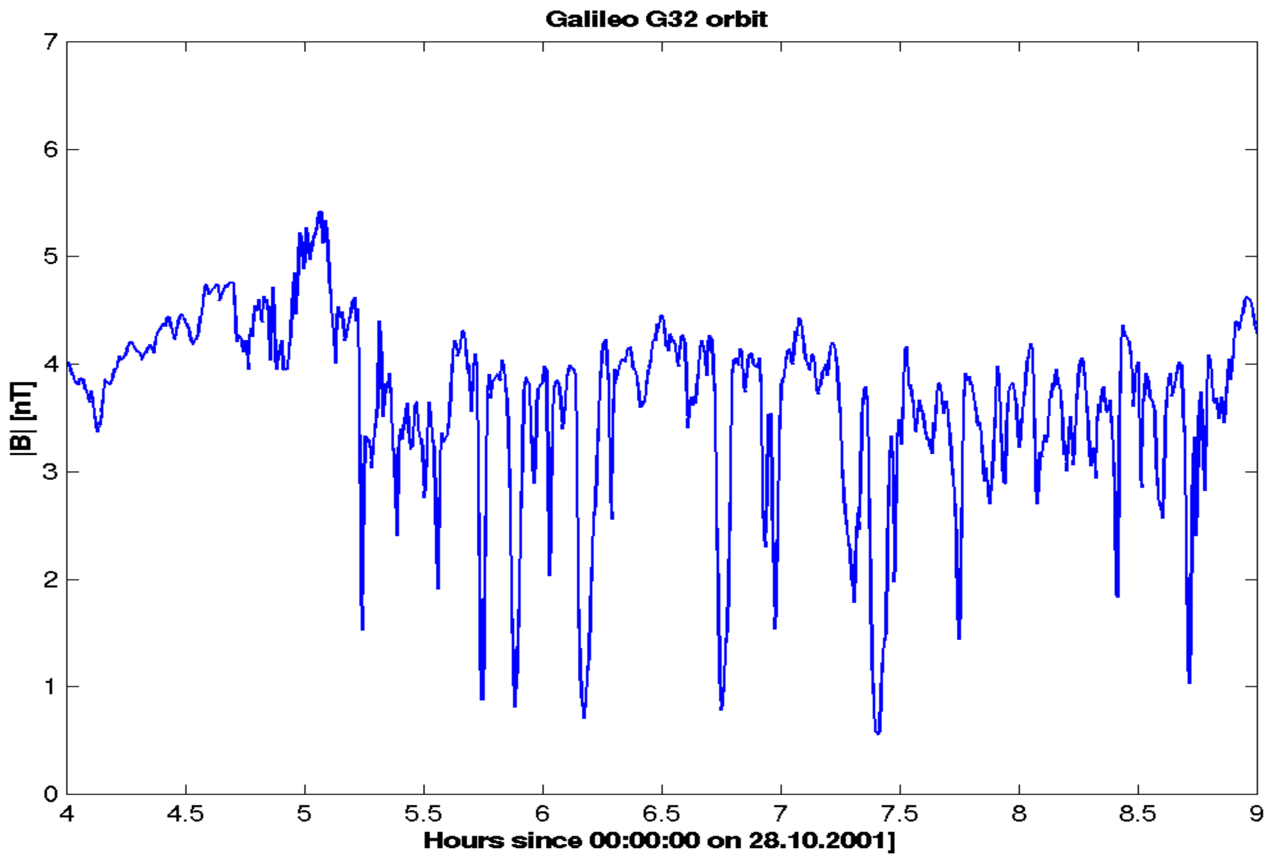
CLUSTER MISSION



Cluster satellites

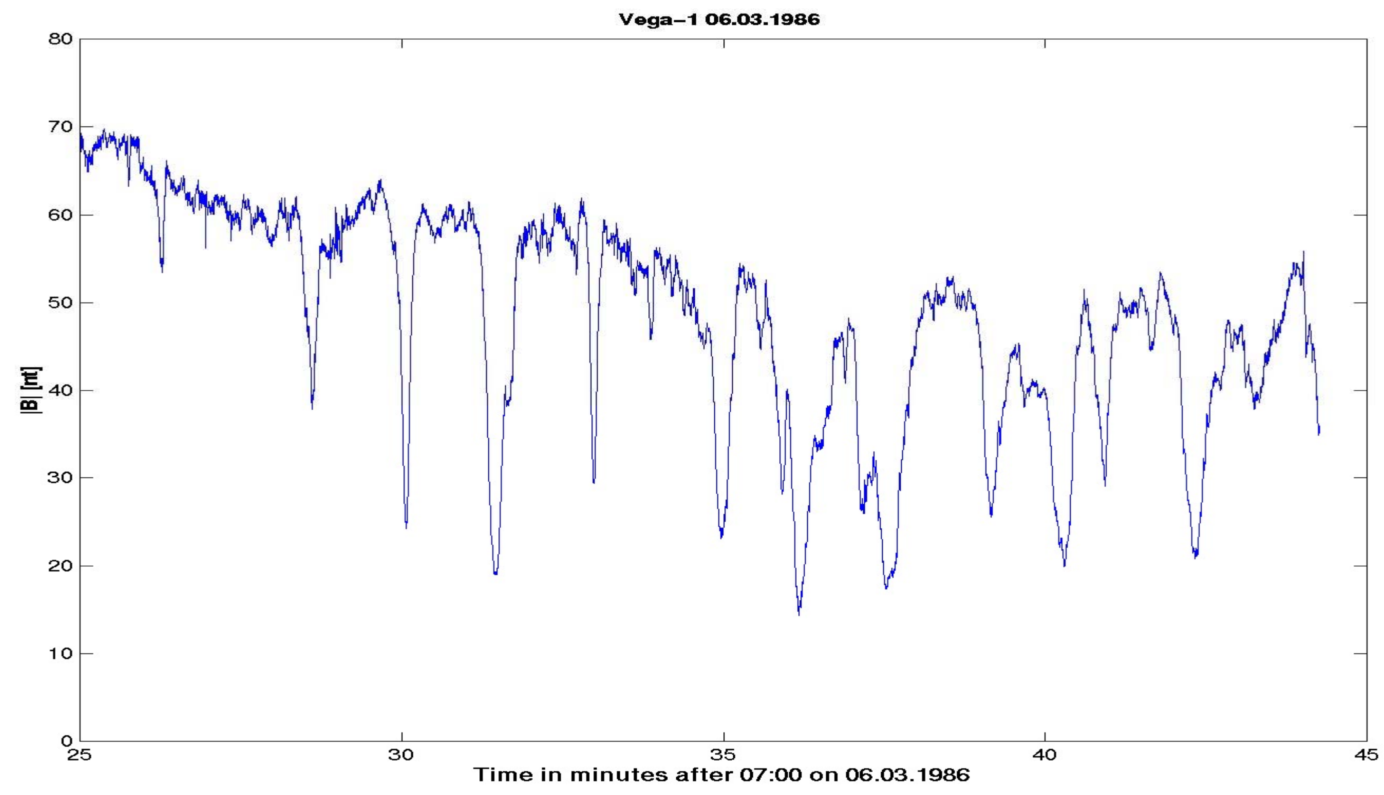


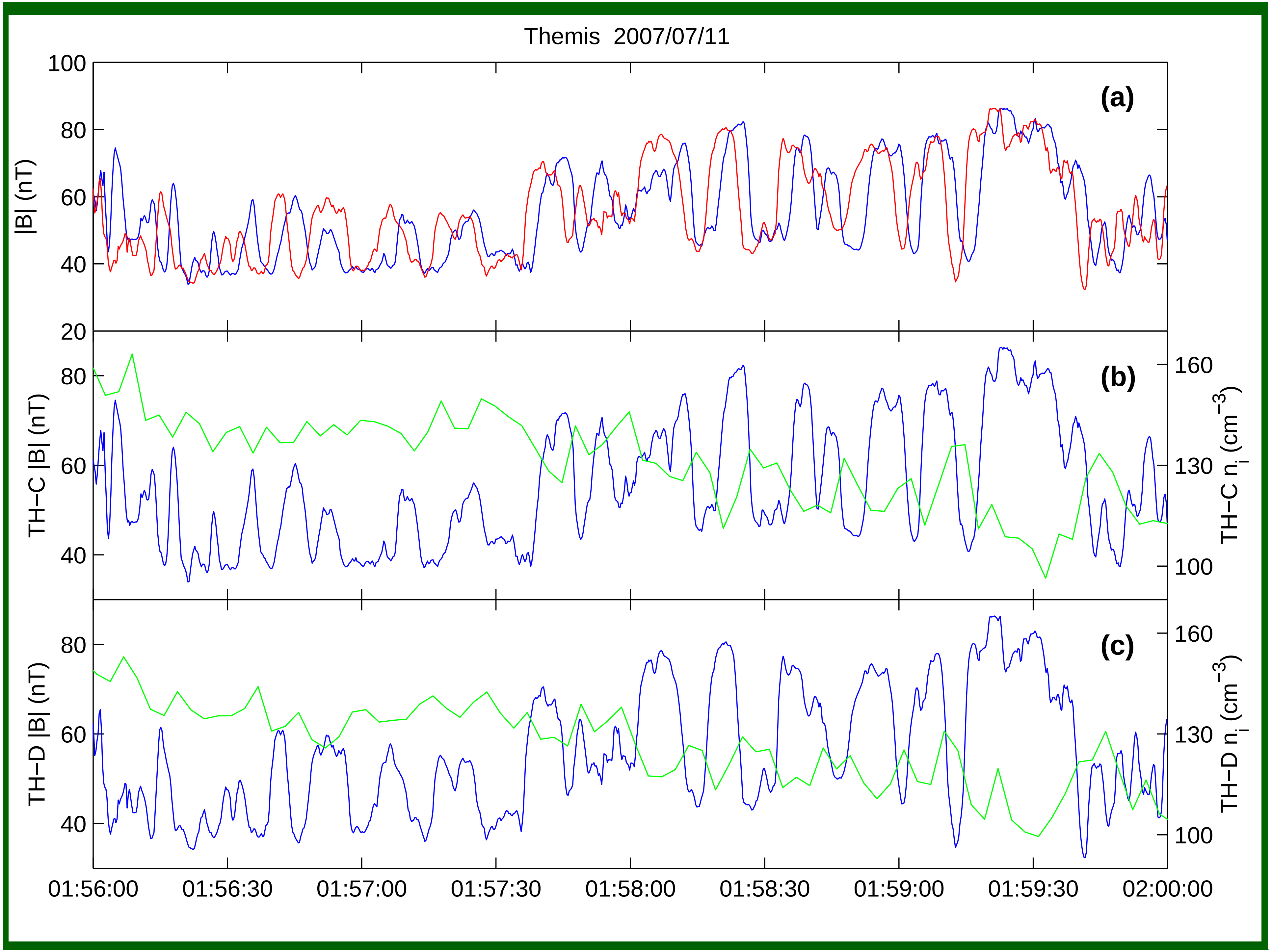
Galileo



Galileo G32 orbit
|B| [nT]
0
1
2
3
4
5
6
7
4
4.5
5
5.5
6
6.5
7
7.5
8
8.5
9
Hours since 00:00:00 on 28.10.2001]

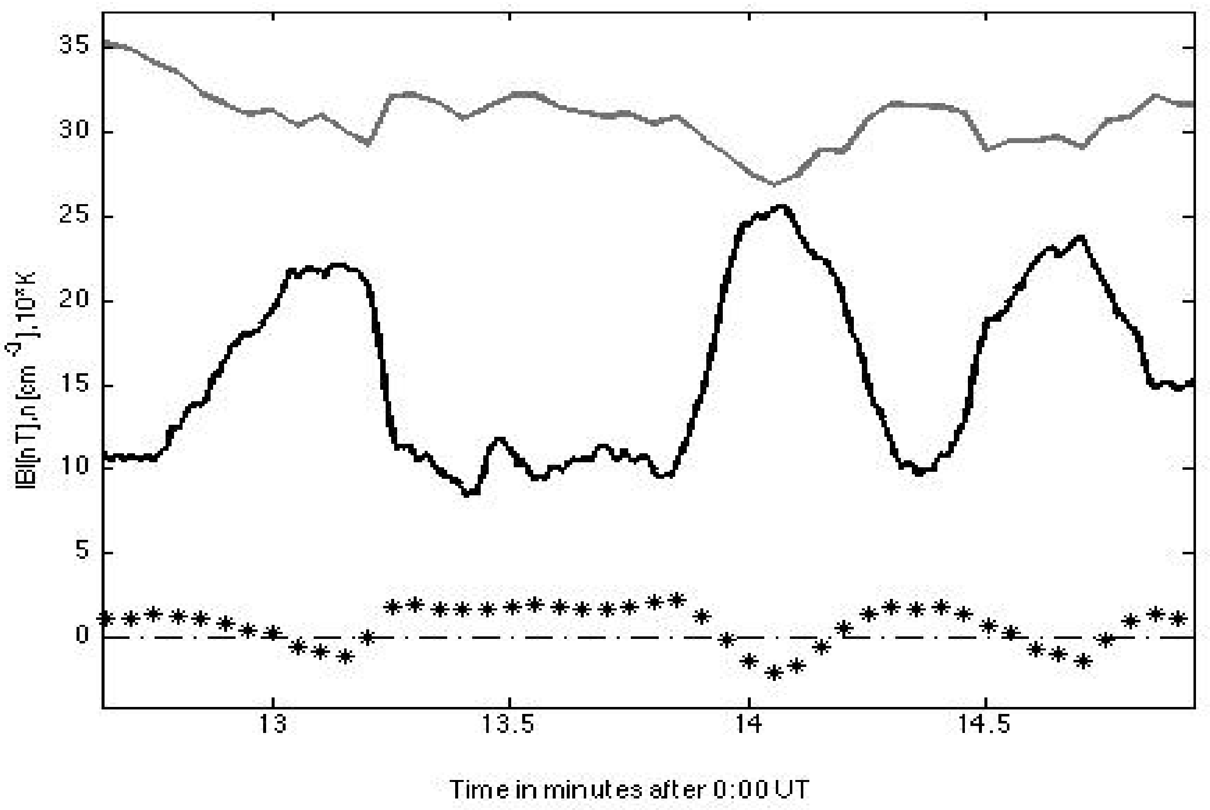
Vega


Vega-1 06.03.1986
|B| [nT]
Time in minutes after 07:00 on 06.03.1986
80
70
60
50
40
30
20
10
0
25
30
35
40
45



Themis 2007/07/11
(a)
(b)
(c)
|B| (nT)
TH−C |B| (nT)
TH−D |B| (nT)
TH−C n_i (cm^{-3})
TH−D n_i (cm^{-3})
100
80
60
40
20
160
130
100
01:56:00
01:56:30
01:57:00
01:57:30
01:58:00
01:58:30
01:59:00
01:59:30
02:00:00

THEMIS observations

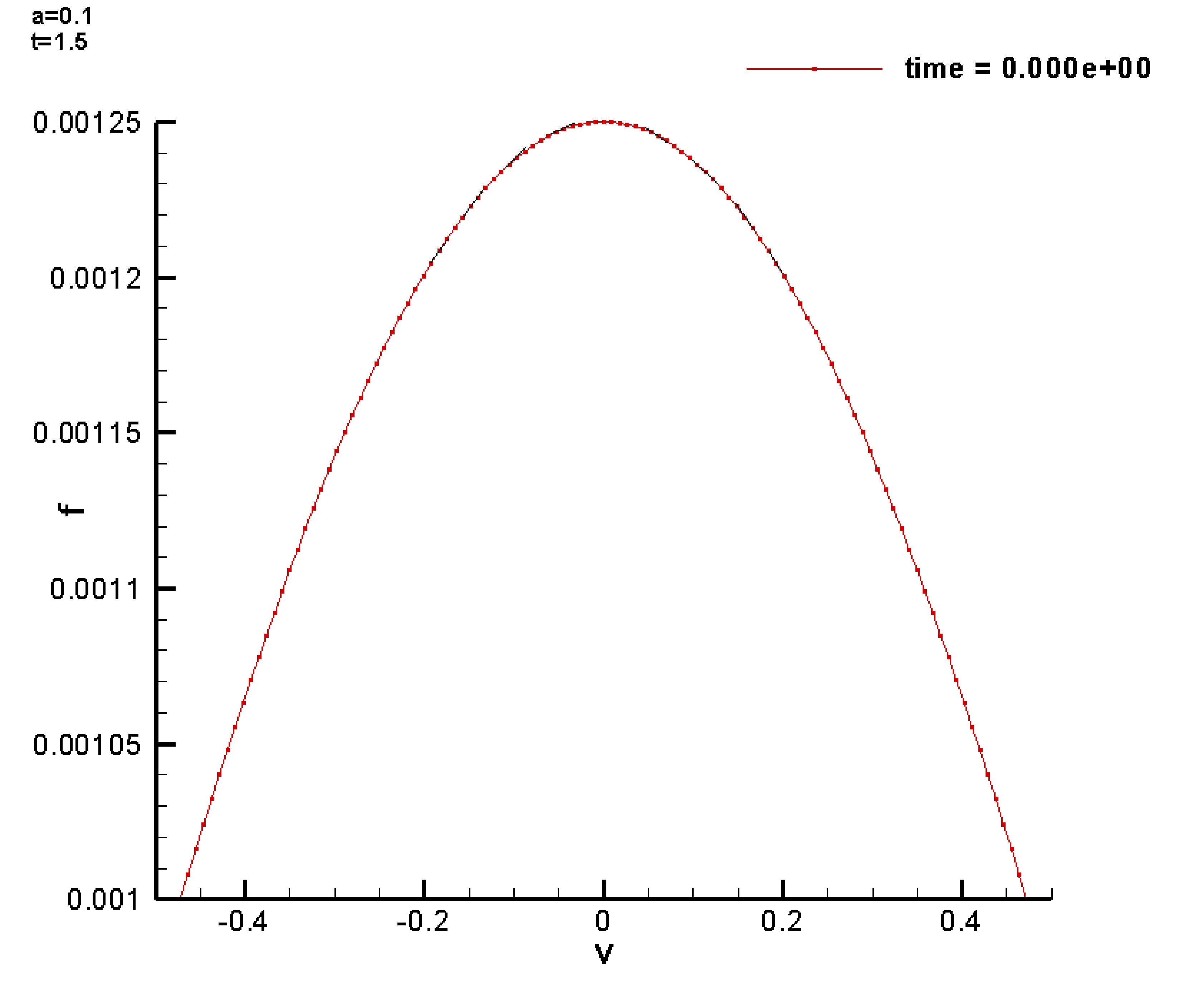


Perpendicular plasma pressure balance

$$\delta p_\perp + \frac{B_0 \delta B_z}{\mu_0} - \frac{3}{2}\rho_i^2 \nabla_\perp^2 \frac{B_0 \delta B_z}{\mu_0}$$

$$= -\left(1 + \frac{\beta_\perp - \beta_\parallel}{2}\right) \nabla_\perp^{-2} \frac{\partial^2}{\partial z^2} \frac{B_0 \delta B_z}{\mu_0}$$

$$\delta p_\perp = \delta p_\perp^{res} + \delta p_\perp^{ad} + \delta p_\perp^{hd}$$

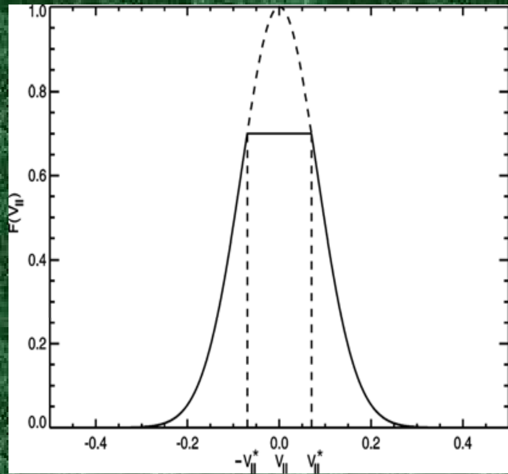

a=0.1
t=1.5
time = 0.000e+00
0.00125
0.0012
0.00115
0.0011
0.00105
0.001
f
-0.4
-0.2
0
0.2
0.4
V

NL effects associated with trapped particles

$$\delta p_{\perp}^{res} = -\pi m \frac{\partial b}{\partial t}\int_0^{\infty} v_{\perp}^5 dv_{\perp} \lim_{\nu \to 0} \int_{v_{\parallel}^*}^{\infty} d v_{\parallel} \frac{\nu}{\nu^2 + k_{\parallel}^2 v_{\parallel}^2} \frac{\partial F}{v_{\parallel} \partial v_{\parallel}}$$

where $v_{\parallel}^* = v_{\perp} \, |b|^{1/2}$ and F is

$$F = \frac{n}{\pi^{3/2} v_{T_\perp}^2 v_{T_\parallel}} e^{-\frac{v_\perp^2}{v_{T_\perp}^2} - \frac{v_\parallel^2}{v_{T_\perp}^2}}$$



When $v_{\parallel}^* << v_{T_\parallel}$, i.e. $|b| << 1$

$$\delta p_{\perp}^{res} = \frac{2nm}{\pi^{1/2} |k_{\parallel}| v_{T_\perp}^2 v_{T_\parallel}^3} \frac{\partial b}{\partial t} \int_0^{\infty} v_{\perp}^5 e^{-\frac{v_\perp^2}{v_{T_\parallel}^2}} dv_{\perp} \lim_{\nu \to 0} \left(\frac{\pi}{2} - \arctan \frac{v_{\perp} |k_{\parallel}| |b|^{1/2}}{\nu} \right)$$

when amplitude is small one has the standard linear (Landau) response

$$\delta p_{\perp}^{res} = p_{\perp} \frac{2\pi^{1/2}}{|k_{\parallel}| v_{T_\parallel}} \frac{T_{\perp}}{T_{\parallel}} \frac{\partial b}{\partial t}$$

when $|k_{\parallel}| v_{\parallel}^* >> \nu \qquad \delta p_{\perp}^{res} \to 0$

Adiabatic part of the pressure peturbation

$$\delta p_{\perp}^{ad} = \frac{3p_{\perp}}{k_{\parallel}^{2} v_{T_{\parallel}}^{2}} \left(\frac{T_{\perp}}{T_{\parallel}} \right)^{1/2} \frac{1}{|b|^{1/2}} \frac{\partial^2 b}{\partial t^2}$$

Contrary to the linear limit in the NL regime the expansion parameter $\omega / k_{\parallel} v_{T_{\parallel}}$ is now replaced by $\omega / k_{\parallel} v_{\parallel}^{*} = \omega / k_{\parallel} v_{T_{\perp}} |b|^{1/2}$

Quasi-Hydrodynamic part

$$\delta p_{\perp}^{hd} = -2Ap_{\perp}b + \frac{15}{8}Ap_{\perp}\left(\frac{T_{\perp}}{T_{\parallel}}\right)^{1/2}|b|^{1/2}b$$

Mirror force

Effect of flattening of the ion distribution function

NL MI dispersion relation near the saturated state

Near saturate state $\delta p_{\perp}^{res} \to 0$ and

$$[\beta_{\perp} A - 1 - \frac{k_{\parallel}^2}{k_{\perp}^2}(1 + \frac{\beta_{\perp} - \beta_{\parallel}}{2}) - \frac{3}{2} k_{\perp}^2 \rho_i^2$$

$$- \frac{15}{8}\left(\frac{T_{\perp}}{T_{\parallel}}\right)^{1/2} |b|^{1/2}]b - \frac{(3/2)}{k_{\parallel}^2 v_{T_{\parallel}}^2}\left(\frac{T_{\perp}}{T_{\parallel}}\right)^{1/2} \frac{\beta_{\perp}}{|b|^{1/2}} \frac{\partial^2 b}{\partial t^2} = 0$$

The model equation (in the dimensionless form)

$$\frac{\partial^2}{\partial \xi^2}\left(1+\frac{\partial^2}{\partial \xi^2}-|h|^{1/2}\right)h+\frac{\partial^2 h}{\partial \tau^2}=0$$

NL growth rate

$$\gamma_{NL} = \frac{2^{1/2}\left|b\right|^{1/4}\left|k_{\|}\right| v_{T_{\|}}}{3^{1/2}\beta_{\perp}^{1/2}}\left(\frac{T_{\|}}{T_{\perp}}\right)^{1/4} [\beta_{\perp}A - 1 - \frac{3}{2}k_{\perp}^{2}\rho_{i}^{2}$$

$$-(1+\frac{\beta_{\perp}-\beta_{\|}}{2}) - \frac{15}{8}\left(\frac{T_{\perp}}{T_{\|}}\right)^{1/2}\left|b\right|^{1/2}]^{1/2}$$

Maximum growth rate

The maximum growth is attained at

$$k_\perp^2 \rho_i^2 = \frac{2}{9}\left(\beta_\perp A - 1 - \frac{15}{8}\left(\frac{T_\perp}{T_\parallel}\right)^{1/2} |b|^{1/2}\right)$$

$$k_\parallel^2 \rho_i^2 = \frac{2}{27}\frac{\beta_\perp}{(1+\frac{\beta_\perp - \beta_\parallel}{2})}\left(\beta_\perp A - 1 - \frac{15}{8}\left(\frac{T_\perp}{T_\parallel}\right)^{1/2} |b|^{1/2}\right)^2$$

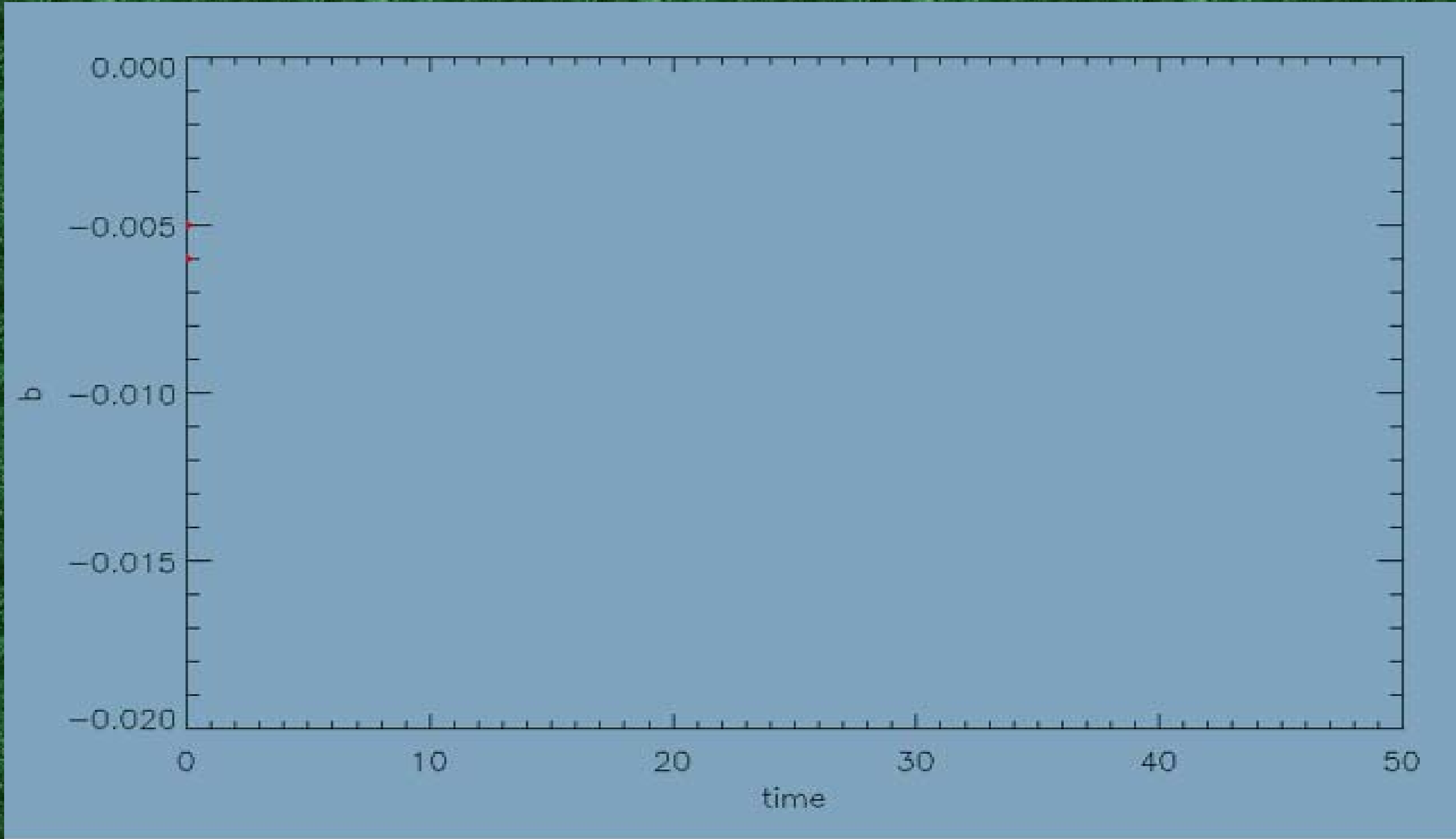
$$\frac{\gamma_{NL}}{\omega_{ci}} = \frac{|b|^{1/4}}{9\sqrt{3}\beta_\perp^{1/2}(1+\frac{\beta_\perp - \beta_\parallel}{2})^{1/2}}\left(\frac{T_\parallel}{T_\perp}\right)^{3/4}\left(\beta_\perp A - 1 - \frac{15}{8}\left(\frac{T_\perp}{T_\parallel}\right)^{1/2} |b|^{1/2}\right)^{3/2}$$

General case

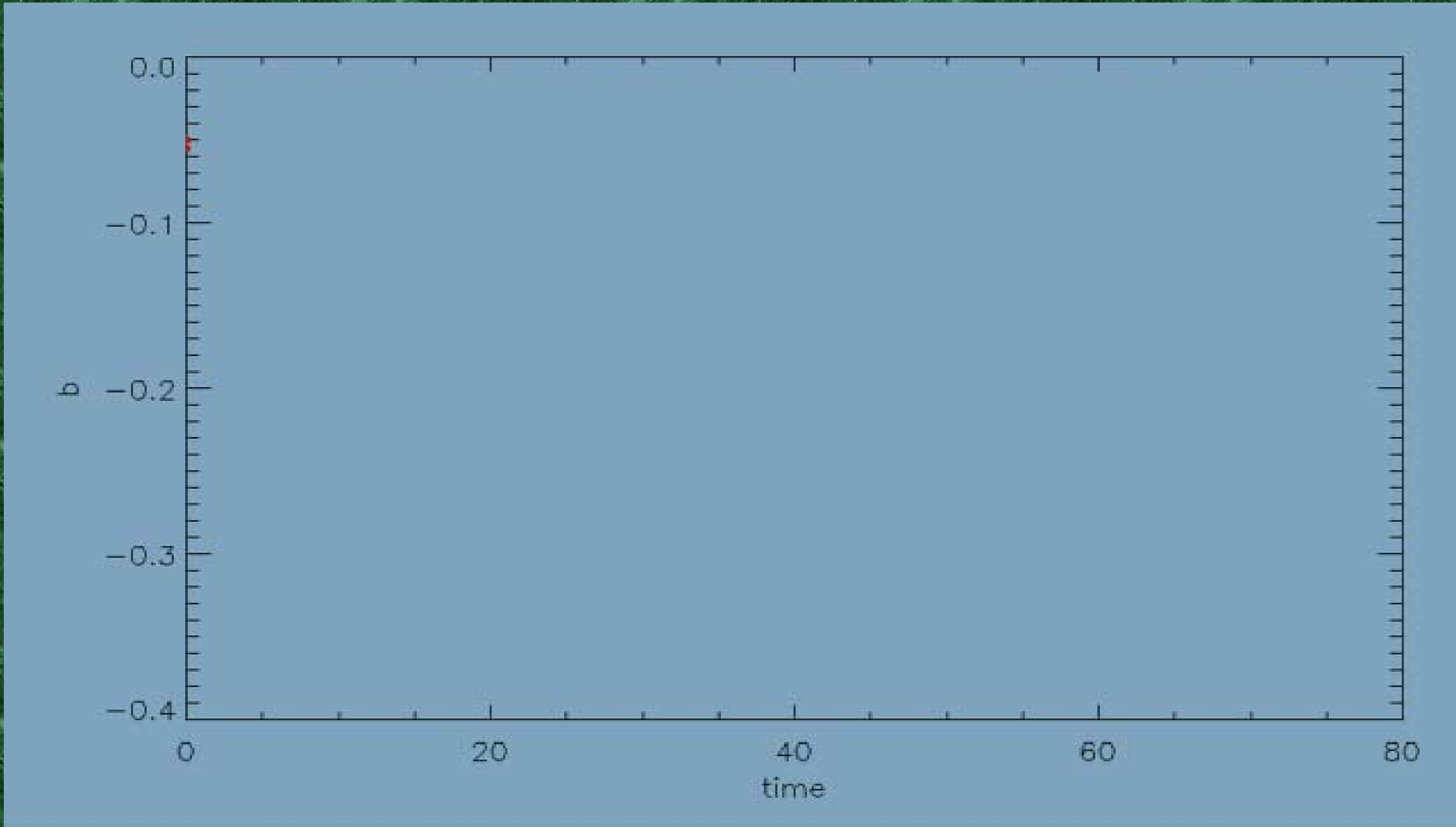
$$\frac{1}{|b|^{1/2}}\frac{\partial^2 b}{\partial t^2}+\left(1-\frac{2}{\pi}\arctan\frac{|b|^{1/2}}{\lambda}\right)\frac{\partial b}{\partial t}=\left(L-|b|^{1/2}\right)b$$

MI growth rate

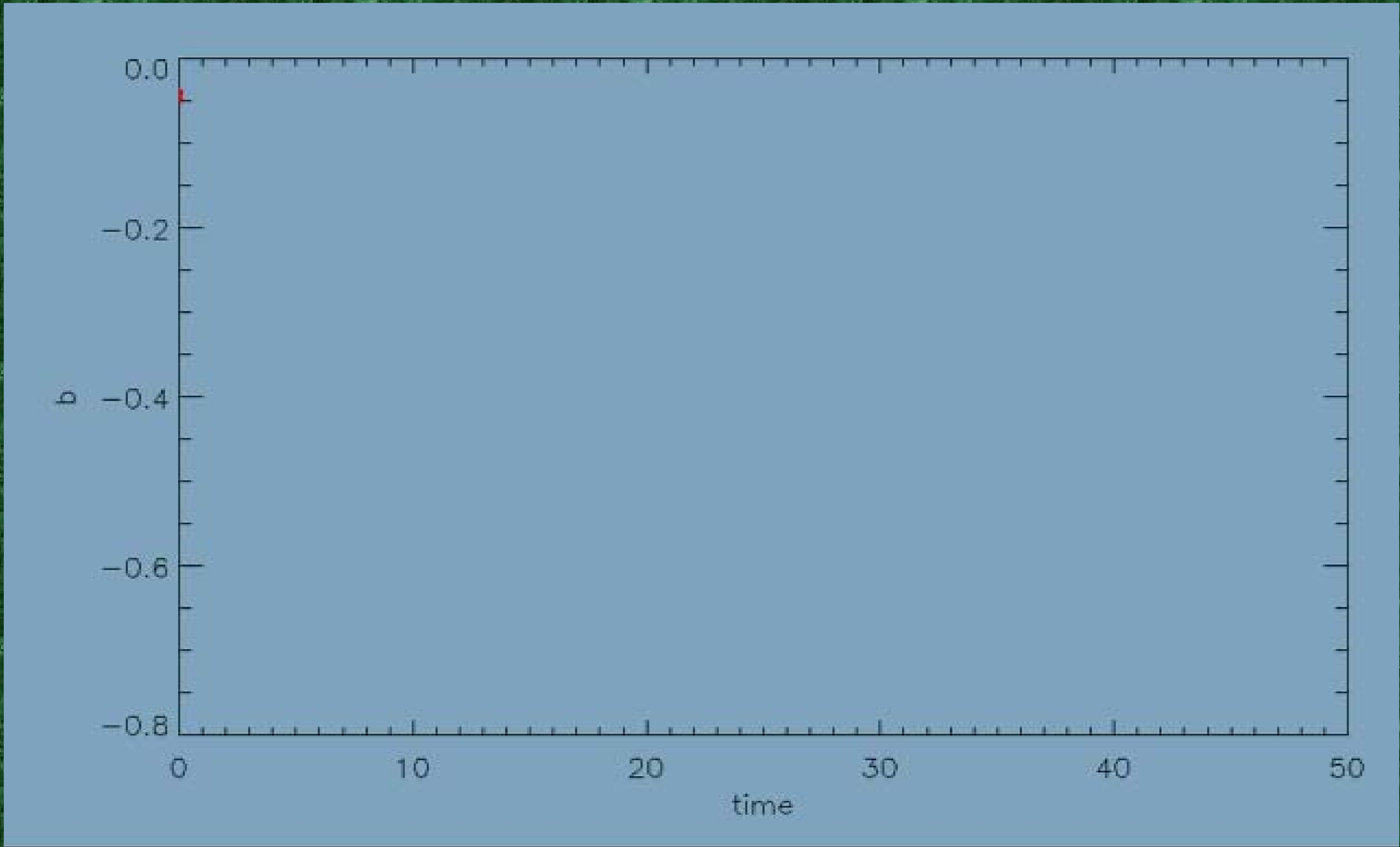
(K=0.1, h=-0.005)



MI growth rate (K=0.5, h=0.05)



MI growth rate
(K=0.8, h=-0.05)



Basic nonlinear equations

$$\frac{\delta p_{\perp}}{2 p_{\perp 0}} + \frac{1}{\beta_{\perp}}\left(1 - \frac{3}{4}\frac{\rho_i^2}{(1+b)^2}\nabla_{\perp}^2\right) b + \frac{b^2}{2\beta_{\perp}}$$

$$= -\left(1 + \frac{\beta_{\perp} - \beta_{\parallel}}{2}\right)\nabla_{\perp}^{-2}\frac{\partial^2}{\partial s^2} b$$

$$p_{\perp} = B^2 \int \mu f d\mu \, d v_{\parallel}$$

$$\frac{\partial f}{\partial t} + v_{\parallel}\frac{\partial f}{\partial s} - \frac{\mu}{m}\frac{\partial B}{\partial s}\frac{\partial f}{\partial v_{\parallel}} = 0$$

Reduction of the kinetic equation

New variables

$$2\xi = k_{\parallel} s - \omega t$$

$$\dot{\xi} = k_{\parallel} v_{\parallel} / 2$$

$$\frac{\partial f}{\partial t} + \dot{\xi}\frac{\partial f}{\partial \xi} - \frac{\sin 2\xi}{2\tau^2}\frac{\partial f}{\partial \dot{\xi}} = 0$$

where $\tau = (m/k_{\parallel}^2 \mu B_0 |b|)^{1/2}$

If b=const then

$$\kappa^{-2} = \tau^2 \dot{\xi}^2 + \sin^2 \xi = const$$

Trapped particles - $|\kappa| > 1$

Untrapped particles - $|\kappa| < 1$

Particles with different magnetic moments can be decomposed into distinct populations

i. $\mu > \mu_1$, where $4m\gamma^2 / B_0 k_{\parallel}^2 b$. This is the adiabatic case.

ii. $\mu < \mu_1$, Linear approximation is applied.

Adiabatic invariants

If b ≠ const

Trapped particles

$$\nu = \frac{[E(\kappa^{-1}) - (1-\kappa^{-2})K(\kappa^{-1})]}{\tau} = const$$

Untrapped particles

$$\sigma = \frac{E(\kappa)}{\kappa\tau} = const$$

Distribution function

Trapped particles

$$f(v_{\parallel}) = \frac{1}{2}\left[f_0(4\sigma / \pi k_{\parallel}) + f_0(-4\sigma / \pi k_{\parallel})\right]$$

Untrapped particles

$$f(v_{\parallel}) = f_0(v_{\parallel 0}) = f_0(4\nu / \pi k_{\parallel})$$

Plasma pressure

$$\mathrm{p}_{\perp} = \frac{B^2}{B_0^2}\mathrm{p}_{\perp 0} + B^2 \int_0^{\mu_1} \mu \, d\mu \int_{-\infty}^{\infty} \delta f(\mu, v_{\parallel})\, dv_{\parallel}$$

Adiabatic contribution

Nonadiabatic contribution

NL pressure variation

$$\frac{\delta p_\perp}{2p_{\perp 0}} = b + \frac{b^2}{2} - \frac{b}{2}\frac{T_\perp}{T_\parallel}\left(1 - \frac{\pi^{1/2}\gamma}{|k_\parallel| v_{T_\parallel}}\right)\Phi(\alpha)$$

where $\alpha = \mu_1 B_0 / T_\perp = 8\gamma^2 / k_\parallel^2 v_{T_\perp}^2 |b|$

and

$$\Phi = 2 - 2\alpha e^{-\alpha} - \alpha^2 e^{-\alpha} - 2e^{-\alpha}$$

Nonlinear evolution

The transition from the linear to

NL regime arises when $\mu_1 \simeq T_\perp / B$.

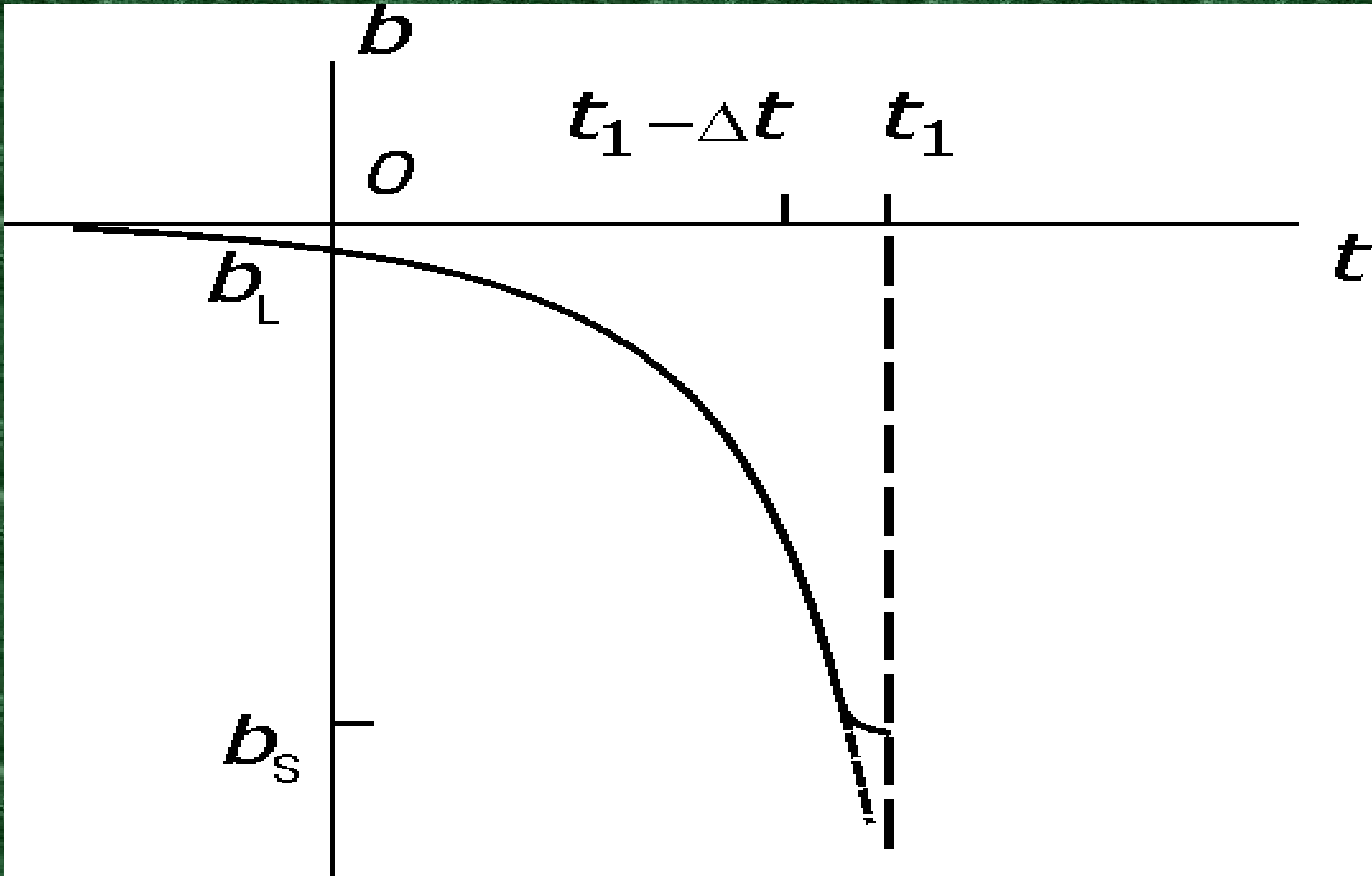
Then NL evolution has explosive behavior.

The explosive dependence is arrested and

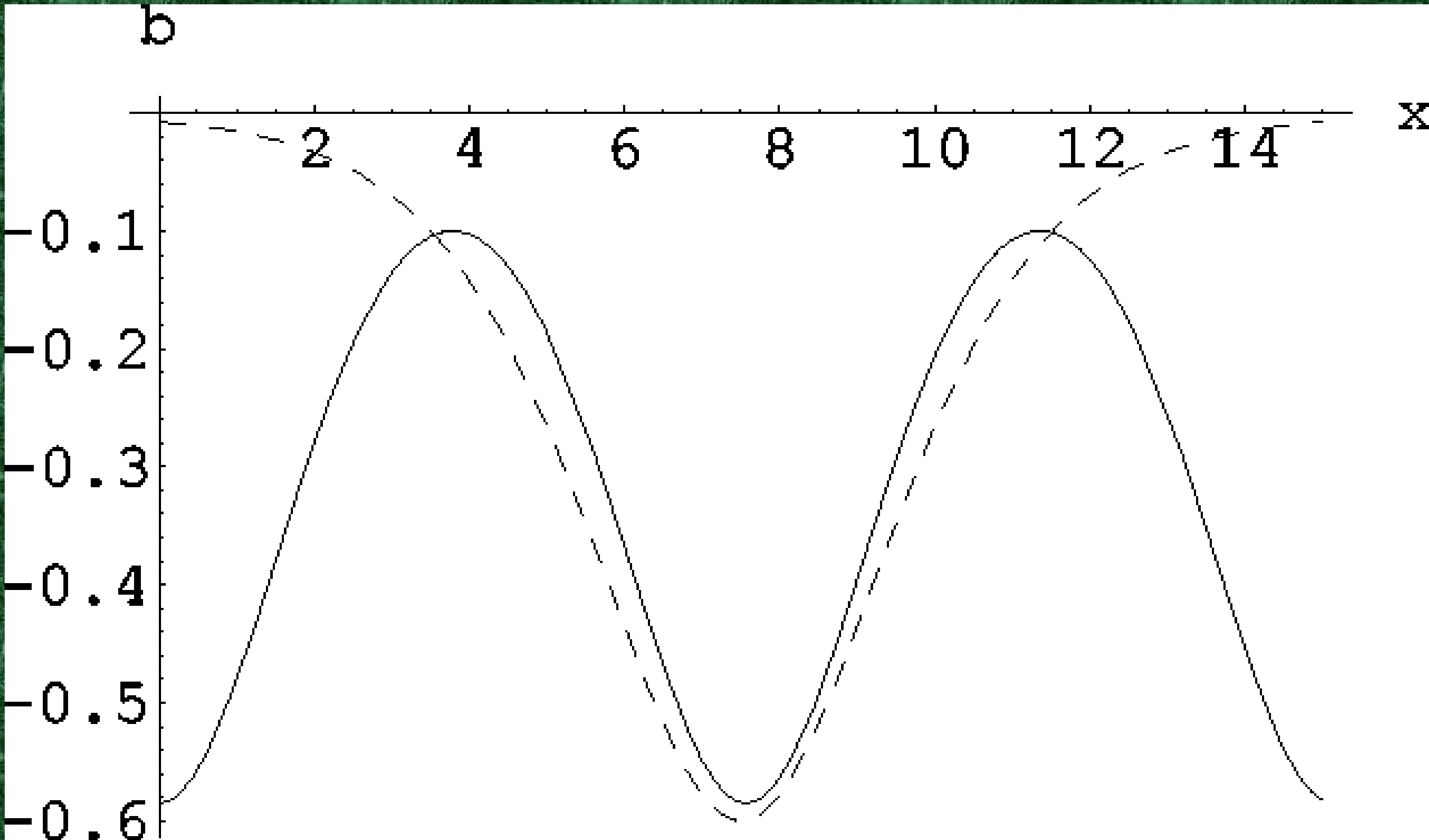
then the system gradually approaches the saturated

state

Temporal evolution



Stationary state



Saturated state

$$\Phi \simeq \frac{1}{3}\alpha^{3} - \frac{1}{4}\alpha^{4}$$

and

$$\frac{\delta p_{\perp}}{2 p_{\perp 0}} = b + \frac{b^{2}}{2}$$

$$\frac{d^{2} b}{d x^{2}} = \frac{3}{5} b + \frac{3}{2} b^{2}$$

General solution - cnoidal wave

$$b = b_2 + (b_3 - b_2) cn^2[\frac{(b_1 - b_3)^{1/2}}{2} x, \kappa]$$

or KdV soliton

$$b = -\frac{3/5}{\cosh^2(x/2)}$$

Total plasma pressure

$$\delta p_\perp^{\text{total}} = \delta p_\perp + \delta p_\perp^{\text{disp}}$$

$$\delta p_\perp^{\text{total}} = \left(1 + \frac{3}{4} \frac{\rho_i^2}{(1+b)^2} \nabla_\perp^2 \right) \delta p_\perp$$

$$\simeq \delta p_\perp - \frac{3}{4} \frac{B_0^2}{\mu_0} \frac{\rho_i^2}{(1+b)^2} \nabla_\perp^2 b$$

$$\frac{\delta p_\perp^{\text{total}}}{2 p_{\perp 0}} + \frac{1}{\beta_\perp} \left(b + \frac{b^2}{2} \right) \simeq 0$$

Nonzero electron temperature effects

$$\frac{\delta p_{\perp i}+\delta p_e}{2p_{\perp 0}}+\frac{1}{\beta_\perp}\left(1-\frac{3}{4}\frac{\rho_i^2}{(1+b)^2}\nabla_\perp^2\right)b+\frac{b^2}{2\beta_\perp}$$

$$=-\left(1+\frac{\beta_\perp-\beta_\parallel}{2}\right)\nabla_\perp^{-2}\frac{\partial^2}{\partial s^2}b$$

$$p_{\perp i}=B^2\int \mu\, f d\mu\, dv_\parallel$$

$$\frac{\partial f}{\partial t}+v_\parallel\frac{\partial f}{\partial s}-\left(\frac{\mu B_0}{m}\frac{\partial b}{\partial s}+\frac{e}{m}\frac{\partial \Psi}{\partial s}\right)\frac{\partial f}{\partial v_\parallel}=0$$

$$\delta p_e=n_0 T_e \exp\left(\frac{e\Psi}{T_e}\right)$$

Nonlinear pendulum

$$\frac{\partial f}{\partial t} + \dot{\xi}\frac{\partial f}{\partial \xi} - \frac{\sin(2\xi)}{2\tau^2}\frac{\partial f}{\partial \dot{\xi}} = 0$$

$$\tau = \left[\frac{m}{k_\parallel^2 \mu B_0 |b| \left(1 + \frac{e\Psi}{b\mu B_0}\right)}\right]^{1/2}$$

Nonlinear functions

$$\Phi_1(\alpha) = 1 - (\alpha + 1)e^{-\alpha}$$

$$\Phi_2(\alpha) = 1 - e^{-\alpha}$$

Quasi – neutrality condition

$$a = -\frac{T_e}{T_\perp}\left[\frac{(T_\perp / T_\parallel)\Phi_1(\alpha) - 1}{1 + (T_e / T_\parallel)\Phi_2(\alpha)} - \frac{(T_\perp\Phi_1(\alpha) + T_e\Phi_2(\alpha))}{(T_\parallel + T_e\Phi_2(\alpha))^2}\left(1 - \frac{\pi^{1/2}\gamma(v_\parallel)}{|k_\parallel| v_{T_\parallel}}\right)\right]$$

$$n = \frac{B}{B_0}n_0 + B\int_0^{\mu_1} d\mu\, dv_\parallel \delta f$$

where

$$\delta f = (\mu B_0 b + e\Psi)\left(1 - \frac{\pi\gamma(v_\parallel)}{|k_\parallel|}\frac{\partial F}{m v_\parallel \partial v_\parallel}\right)$$

$$n = \frac{B}{B_0}n_0\left[1 - \frac{T_\perp}{T_\parallel}b[\Phi_1(\alpha) + a\Phi_2(\alpha)]\right]\left(1 - \frac{\pi^{1/2}\gamma(v_\parallel)}{|k_\parallel| v_{T_\parallel}}\right)$$

where

$$\alpha = \frac{4\gamma^2 m}{k_\parallel^2 T_\perp |b|} - a \quad \text{and} \quad \frac{e\Psi}{T_\perp} = ab$$

$$p_{\perp i} = \frac{B^2}{B_0^2}p_{\perp i}^0 + B^2\int_0^{\mu_1}\mu d\mu \int_{-\infty}^{\infty}\delta f(\mu, v_\parallel)dv_\parallel$$

bi-Maxwellian distribution

$$p_{\perp i} = \frac{B^2}{B_0^2}p_{\perp i}^0 - bp_{\perp i}^0\frac{T_\perp}{T_\parallel}[\Phi_3(\alpha) + a\Phi_1(\alpha)]\left(1 - \frac{\pi^{1/2}\gamma}{|k_\parallel| v_{T_\parallel}}\right)$$

Here

$$\Phi_3(\alpha) = 2 - \alpha^2 e^{-\alpha} - 2\alpha e^{-\alpha} - 2e^{-\alpha}$$

Linear approximation

$$\frac{e\Psi}{T_\perp b} = a = -\frac{T_e}{T_\perp}\left[\frac{T_\perp - T_\parallel}{T_e + T_\parallel} - \frac{T_\parallel (T_\perp + T_e)}{(T_e + T_\parallel)^2}\frac{\pi^{1/2}\gamma}{|k_\parallel| v_{T_\parallel}}\right]$$

$$\frac{\delta p_\perp}{2p_{\perp i}} = -b\left(\frac{T_\perp}{T_\parallel} - 1 - \frac{(T_\perp / T_\parallel - 1)^2 T_e}{2T_\perp (1 + T_e / T_\parallel)}\right)$$

$$+b\frac{(1+T_e/T_\parallel)^2 + (1+T_e/T_\perp)^2}{2(1+T_e/T_\parallel)^2}\frac{T_\perp}{T_\parallel}\frac{\pi^{1/2}\gamma}{|k_\parallel| v_{T_\parallel}}$$

Saturated state

$$\frac{\delta p_\perp}{2p_{\perp i}^0} = \left(1 + \frac{T_e}{2T_\perp}\right) b + \frac{b^2}{2}$$

$$b = -\frac{3}{5}\left(\frac{1 + \beta_\perp + \beta_e / 2}{1 + \beta_\perp + 2\beta_e / 5}\right) \cosh^{-2}\left[\left(\frac{3}{20}\right)^{1/2} x\right]$$

where

$$x = r_\perp / \rho_i [9 / 20(1 + \beta_\perp + 2\beta_e / 5)]^{1/2}$$

With the growth of the electron temperature the depth of the magnetic hole varies from 60 to 75 percent of the original magnetic field

Variation of total pressure in the presence of NET

$$\frac{\delta p_{\perp}}{2p_{\perp i}^{0}} = (1+\frac{a}{2})b + \frac{b^2}{2} - \frac{b}{2}\frac{T_{\perp}}{T_{\|}}[\Phi_3(\alpha) + a\Phi_1(\alpha)]$$

Conclusions

- The main nonlinear mechanism responsible for mirror instability saturation is associated with modification of the shape of the background ion distribution function in the region of small parallel particle velocities.
- The MI initially produces a depression in the magnetic field that has an explosive behavior as it was predicted before.
- In the course of NL saturation the mirror mode evolution ends with formation of cnoidal waves or solitons.

Nonzero electron temperature (NET) effects

- The MI threshold is enhanced in the presence of NET effect
- The transition from the linear to nonlinear regime occurs for smaller amplitudes
- NET effect results in a weak decrease of the spatial dimensions of the holes and in the decrease of hole depth

Thank you!