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Self-consistent fluid model for a quantum electron gas

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It is shown that, for a large class of statistical mixtures, the Wigner-Poisson (or Hartree) system can be reduced to an effective Schrödinger-Poisson system, in which the Schrödinger equation contains a new nonlinearity. For the case of a zero-temperature one-dimensional electron gas, this additional nonlinearity is of the form $|\Psi|^4$. In the long-wavelength limit, the results obtained from the effective Schrödinger-Poisson system are in agreement with those of the Wigner-Poisson system. The reduced model is further used to describe the stationary states of a quantum electron gas and the two-stream instability.

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I. INTRODUCTION

Understanding the dynamics of a quantum electron gas is an important issue for a variety of physical systems, such as ordinary metals, semiconductors, and even astrophysical systems under extreme conditions (e.g., white dwarfs). Although some level of understanding can be achieved by considering independent electrons, a more accurate description requires the use of self-consistent models, where electron-electron interactions are taken into account. As the treatment of the full N-body problem is clearly out of reach, mean-field models are usually adopted, of which the Hartree and Hartree-Fock models are standard examples.[1] In the Hartree approximation, each electron is described by a one-particle wave function (obeying Schrödinger's equation), and the electrostatic force acting on it results from Poisson's equation. (Fock's correction accounts for the parity of the N-particle wave function for an ensemble of fermions, but this correction will not be considered in this paper.)

The Hartree model can be written in a more compact, but strictly equivalent, form by making use of Wigner functions. The Wigner representation[2] is a useful tool to express quantum mechanics in a phase-space formalism (for reviews, see Ref. 3). In this representation, a quantum state (either pure or mixed) is described by a Wigner function (i.e., a function of the phase-space variables), and the Wigner equation provides an evolution equation that is similar to the Vlasov equation, well known from classical plasma physics. We note that, although the Wigner distribution satisfies most of the standard properties of probability distributions, it cannot be regarded as such, since it may take negative values. The resulting self-consistent model is called the Wigner-Poisson (WP) system, and has been extensively used in the study of quantum transport.[4–6]

Despite its considerable interest, the WP formulation presents some intrinsic drawbacks : (a) it is a nonlocal, integro-differential system, and (b) its numerical treatment requires the discretization of the whole phase space. Moreover, as is often the case with kinetic models, the WP system gives more information than one is really interested in. For these reasons, it would be useful to obtain an accurate reduced model that, though not providing the same detailed information as the kinetic WP system, is still able to reproduce the main features of the physical system under consideration.

After discussing the general validity of the WP model, we will derive an effective Schrödinger-Poisson (SP) system, which, in an appropriate limit, reproduces the results of the kinetic WP formulation. A similar result was recently obtained in the mathematical literature,[7,8] although its physical implications have not been fully analyzed. In this effective SP model, the Schrödinger equation is nonlinear, as it includes an effective potential depending on the modulus of the wave function. The exact form of this effective potential depends on the specific physical system being studied. In order to obtain the effective SP system, we will first derive a system of reduced "fluid" equations by taking moments of the WP system. It will be shown that the pressure term appearing in the fluid equations can be decomposed into a classical and a quantum part. With some reasonable hypotheses on the pressure term, the fluid system can be closed. Finally, the effective SP system will be applied to several physical problems, including linear wave propagation, nonlinear stationary solutions, and the two-stream instability.

II. COUPLING PARAMETER FOR A QUANTUM PLASMA

A classical plasma can be said to be collisionless ("ideal") when long-range self-consistent interactions (described by the Poisson equation) dominate over short-range two-particle interactions (collisions). This happens when the potential energy of two electrons separated by an average interparticle distance is small compared to the average kinetic energy. The potential energy is estimated as $E_{\rm pot} = e^2 n_0^{1/3}/\varepsilon_0$, while the average kinetic energy is simply given by the temperature T (measured in energy units). Here $-e$ is the electron charge, ε_0 the dielectric constant in vacuum, and n_0 the equilibrium particle density. One defines, therefore, a classical coupling parameter,

$$\Gamma_C = \frac{E_{\rm pot}}{E_{\rm kin}} = \frac{e^2 n_0^{1/3}}{\varepsilon_0 T}, \tag{1}$$

such that the collisionless approximation is valid when $\Gamma_C \ll 1$. The classical coupling parameter can be written in a different way, by introducing the plasma frequency, the thermal velocity, and the Debye length,

$$\omega_p=\left(\frac{n_0e^2}{m\varepsilon_0}\right)^{1/2},\quad v_T=\left(\frac{T}{m}\right)^{1/2},\quad \lambda_D=\frac{v_T}{\omega_p},\tag{2}$$

which are typical inverse time, velocity, and length scales for a collisionless plasma. With these definitions, the coupling parameter can be expressed as

$$\Gamma_C^{3/2}=\frac{1}{n_0\lambda_D^3},\tag{3}$$

which is the inverse of the number of electrons contained in a Debye volume. When the condition $\Gamma_C\ll 1$ is satisfied, two-body correlations (collisions) can be neglected, and the N-particle Liouville equation can be reduced, via a Bogolyubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy, to the one-particle Vlasov equation. The Vlasov-Poisson system is therefore the standard model to describe classical electrostatic plasmas in the collisionless approximation.

Similarly, it is possible to define a quantum coupling parameter Γ_Q. Let us consider the case of a completely degenerate electron gas. Now the average kinetic energy is given by the Fermi temperature $T_F\sim\hbar^2n_0^{2/3}/m$ (we neglect irrelevant dimensionless constants), so that the quantum coupling parameter becomes

$$\Gamma_Q=\frac{E_{\rm pot}}{T_F}=\frac{e^2m}{\hbar^2\varepsilon_0n_0^{1/3}}.\tag{4}$$

Notice that, according to Eq. (4), a quantum electron gas is more ideal at higher densities. Using the Fermi velocity $v_F=\sqrt{T_F/m}$, one can define a typical length scale for the quantum, plasma

$$\lambda_F=\frac{v_F}{\omega_p}.\tag{5}$$

The quantum coupling parameter can thus be expressed as the inverse of the number of electrons contained in a Fermi volume:

$$\Gamma_Q^{3/2}=\frac{1}{n_0\lambda_F^3}.\tag{6}$$

Finally, another expression for the coupling parameter is the following:

$$\Gamma_Q^{1/2}=\frac{\hbar\omega_p}{T_F},\tag{7}$$

which is valid for any number of dimensions.

The quantum electron gas is collisionless when $\Gamma_Q\ll 1$. In this case, the quantum N-body problem can be reduced to a one-particle Wigner equation. The Wigner-Poisson system is therefore capable of describing a quantum electrostatic plasma in the collisionless approximation.

The previous results were derived in the limiting cases $T\gg T_F$ (classical) and $T\ll T_F$ (quantum degenerate). For intermediate temperatures, simple expressions for the coupling parameters are not available, but one must expect a smooth transition between the two regimes.

For electrons in metal, we have typically

$$n_0\simeq 10^{29}\ \mathrm{m}^{-3},\quad v_F\simeq 10^6\ \mathrm{m\,s}^{-1},\quad \omega_p\simeq 10^{16}\ \mathrm{s}^{-1},$$

$$\lambda_F\simeq 10^{-10}\ \mathrm{m}.\tag{8}$$

These values yield a quantum coupling parameter of order unity. Allowing for the dimensionless constants we have neglected and the different properties of metals, we realize that Γ_Q can be both smaller and larger than unity for typical metallic electrons.

The above values seem to indicate that, as $\Gamma_Q\simeq 1$, electron-electron (e-e) collisions cannot be neglected for metals. If that were the case, one should abandon one-particle models such as the Wigner or Hartree equations, and resort to the full N-body problem. This is hardly a feasible task. Fortunately, however, the exclusion principle comes to the rescue by reducing the collision rate quite dramatically in most cases of interest.[1] This occurs when the electron distribution is close to the Fermi-Dirac equilibrium at relatively low temperatures. The fundamental point is that, when all lower levels are occupied, the exclusion principle disallows a vast number of transitions that would otherwise be possible. In particular, at strictly zero temperature, all electrons have energies below T_F, and no transition is possible, simply because there are no available states for the electrons to occupy. At moderate temperatures, only electrons within a shell of thickness T about the Fermi surface can undergo collisions. The e-e collision rate (inverse of the lifetime τ_{ee}) for such electrons is proportional to $T/\hbar$ (this is a form of the uncertainty principle, energy $\times$ time$=$const). The *average* collision rate is obtained by multiplying the previous expression by the fraction of electrons present in the shell of thickness T about the Fermi surface, which is $\sim T/T_F$. One obtains

$$\frac{1}{\tau_{ee}}\sim\frac{1}{\hbar}\frac{T^2}{T_F}.\tag{9}$$

At room temperature, $\tau_{ee}\simeq 10^{-10}$ s, which is much larger than the typical collisionless time scale $\tau_p=\omega_p^{-1}\simeq 10^{-16}$ s. Therefore, for times smaller than τ_{ee}, the effect of e-e collisions can be safely neglected. In addition, it turns out that the typical relaxation time scale is $\tau_r\simeq 10^{-14}$ s, which is again significantly larger than τ_p. In summary, the ordering

$$\tau_p\ll\tau_r\ll\tau_{ee}\tag{10}$$

implies that a collisionless (Wigner) model is appropriate for relatively short time scales.

III. DERIVATION OF THE FLUID MODEL

In one spatial dimension, the Wigner-Poisson system[2,3] reads

$$\frac{\partial f}{\partial t}+v\frac{\partial f}{\partial x}+\frac{iem}{2\pi\hbar}\int\int d\lambda\, dv'\, e^{im(v-v')\lambda} \times\left[\phi\left(x+\frac{\lambda\hbar}{2}\right)-\phi\left(x-\frac{\lambda\hbar}{2}\right)\right]f(x,v',t)=0, \tag{11}$$

$$\frac{\partial^2\phi}{\partial x^2}=\frac{e}{\varepsilon_0}\left(\int f\,dv-n_0\right), \tag{12}$$

where $f(x,v,t)$ is the Wigner distribution function, $\phi(x,t)$ the electrostatic potential, $-e$ and m the electron charge and mass, ε_0 the vacuum dielectric constant, and n_0 a background ionic charge. Notice that the one-particle Wigner function used here actually represents an N-particle system. Indeed, the above Wigner-Poisson system can be derived from the full N-body problem via a BBGKY hierarchy, neglecting two-body correlations and only keeping the mean Coulomb field.[9] Further, it is easy to see that, in the limit $\hbar\to 0$, one recovers the familiar Vlasov-Poisson system for classical collisionless plasmas. For simplicity of notation, only one-dimensional problems will be treated in the rest of this paper, but the results can be readily extended to higher dimensions.

In order to derive a fluid model, we take moments of Eq. (11) by integrating over velocity space. Introducing the standard definitions of density, mean velocity, and pressure,

$$n(x,t)=\int f\,dv,\quad u(x,t)=\frac{1}{n}\int fv\,dv,$$

$$P(x,t)=m\left(\int fv^2dv-nu^2\right), \tag{13}$$

we obtain

$$\frac{\partial n}{\partial t}+\frac{\partial(nu)}{\partial x}=0, \tag{14}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=\frac{e}{m}\frac{\partial\phi}{\partial x}-\frac{1}{mn}\frac{\partial P}{\partial x}. \tag{15}$$

We immediately notice that Eqs. (14) and (15) do not differ from the ordinary evolution equations for a classical fluid. This may seem surprising, but in the following it will appear that the quantum nature of this system is in fact hidden in the pressure term.

The pressure term may be decomposed into a classical and a quantum part. This can be shown as follows. The Wigner distribution for a quantum mixture of states $\psi_\alpha(x,t)$, each characterized by an occupation probability p_α, is written as

$$f(x,v,t)=\frac{m}{2\pi\hbar}\sum_\alpha p_\alpha\int d\lambda\,\psi_\alpha^*\left(x+\frac{\lambda}{2},t\right) \times\psi_\alpha\left(x-\frac{\lambda}{2},t\right)e^{imv\lambda/\hbar}, \tag{16}$$

where the sum extends over all possible states. The numbers p_α, representing probabilities, satisfy the relations $p_\alpha\geqslant 0$, $\Sigma_\alpha p_\alpha=1$. Using the previous expression, one can compute the pressure. After some algebra, one obtains

$$P=\frac{\hbar^2}{4m}\sum_\alpha p_\alpha\left(2\left|\frac{\partial\psi_\alpha}{\partial x}\right|^2-\psi_\alpha^*\frac{\partial^2\psi_\alpha}{\partial x^2}-\psi_\alpha\frac{\partial^2\psi_\alpha^*}{\partial x^2}\right) +\frac{\hbar^2}{4mn}\left[\sum_\alpha p_\alpha\left(\psi_\alpha^*\frac{\partial\psi_\alpha}{\partial x}-\psi_\alpha\frac{\partial\psi_\alpha^*}{\partial x}\right)\right]^2. \tag{17}$$

If we represent each state as

$$\psi_\alpha(x,t)=A_\alpha(x,t)\exp(iS_\alpha(x,t)/\hbar), \tag{18}$$

where A_α (amplitude) and S_α (phase) are real functions, we obtain $P=P^C+P^Q$, where the classical P^C and quantum P^Q parts of the pressure are

$$P^C=\frac{1}{2mn}\sum_{\alpha,\beta}p_\alpha p_\beta A_\alpha^2A_\beta^2\left(\frac{\partial S_\alpha}{\partial x}-\frac{\partial S_\beta}{\partial x}\right)^2, \tag{19}$$

$$P^Q=\frac{\hbar^2}{2m}\sum_\alpha p_\alpha\left[\left(\frac{\partial A_\alpha}{\partial x}\right)^2-A_\alpha\frac{\partial^2A_\alpha}{\partial x^2}\right]. \tag{20}$$

Notice that P^Q only depends on the amplitudes A_α, and that for a pure state only P^Q survives. It can be easily shown that P^C represents the standard pressure, resulting from the dispersion of velocities. To prove this, one has to remember that the phases S_α are related to the mean velocity u_α of each wave function through the relation $mu_\alpha=\partial S_\alpha/\partial x$ [the u_α's should not be confused with the *global* mean velocity u defined in Eq. (13)]. Thus, by expanding Eq. (19), one obtains after some algebra

$$P^C=mn\left[\frac{\Sigma_\alpha p_\alpha A_\alpha^2u_\alpha^2}{n}-\left(\frac{\Sigma_\alpha p_\alpha A_\alpha^2u_\alpha}{n}\right)^2\right]. \tag{21}$$

With an appropriate definition of averages, we can rewrite the above equation as $P^C=mn(\langle u_\alpha^2\rangle-\langle u_\alpha\rangle^2)$, which is the standard expression for the pressure. The contribution P^Q, on the other hand, is a purely quantum pressure, with no classical counterpart.

In order to close the fluid system, some equation of state, relating P^C and P^Q to the density n, must be used. In this paper, we consider a statistical mixture in which all the amplitudes are equal (but not constant), $A_\alpha(x)=A(x)$. This gives, using Eqs. (13) and (16), $n=A^2$. With the help of Eq. (20), the quantum pressure becomes

$$P^Q=\frac{\hbar^2}{2m}\left[\left(\frac{\partial}{\partial x}\sqrt{n}\right)^2-\sqrt{n}\frac{\partial^2}{\partial x^2}\sqrt{n}\right]. \tag{22}$$

For the classical part of the pressure, we make the standard assumption that it is some function of the density, $P^C=P^C(n)$. With these hypotheses, the force equation (15) can be written as

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=\frac{e}{m}\frac{\partial \phi}{\partial x}-\frac{1}{mn}\frac{\partial P^C}{\partial x}-\frac{1}{mn}\frac{\partial P^Q}{\partial x}. \tag{23}$$

Defining the effective potential

$$W(n)=\int^n \frac{dn'}{n'}\frac{dP^C(n')}{dn'}, \tag{24}$$

and using the identity

$$\frac{1}{mn}\frac{\partial P^Q}{\partial x}=-\frac{\hbar^2}{2m^2}\frac{\partial}{\partial x}\left(\frac{\partial^2(\sqrt{n})/\partial x^2}{\sqrt{n}}\right), \tag{25}$$

the force equation (23) reduces to

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=\frac{e}{m}\frac{\partial \phi}{\partial x}-\frac{1}{m}\frac{\partial W}{\partial x}+\frac{\hbar^2}{2m^2}\frac{\partial}{\partial x}\left(\frac{\partial^2(\sqrt{n})/\partial x^2}{\sqrt{n}}\right). \tag{26}$$

Now comes the crucial point: it is possible to combine Eqs. (14) and (26) into an effective Schrödinger equation. Indeed, let us define the effective wave function,

$$\Psi=\sqrt{n(x,t)}\exp(iS(x,t)/\hbar), \tag{27}$$

with $S(x,t)$ defined according to $mu(x,t)=\partial S(x,t)/\partial x$. We obtain that $\Psi(x,t)$ satisfies the equation

$$i\hbar\frac{\partial \Psi}{\partial t}=-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2}-e\phi\Psi+W\Psi. \tag{28}$$

This is a nonlinear Schrödinger equation, as the effective potential W depends on the wave function through Eq. (24), where $n=|\Psi|^2$. Separating Eq. (28) into its real and imaginary parts, we indeed find the continuity (14) and force (26) equations. Finally, the complete effective SP system is composed of Eq. (28) and the Poisson equation

$$\frac{\partial^2\phi}{\partial x^2}=\frac{e}{\varepsilon_0}(|\Psi|^2-n_0). \tag{29}$$

To summarize what we have achieved so far, we notice that, in general, the dynamics of a statistical mixture must be treated with the full Wigner-Poisson system, or, equivalently, with a set of Schrödinger equations, coupled by Poisson's equation (Hartree's model). In the present section, we have shown that one can reduce the problem of quantum transport to a single nonlinear Schrödinger equation plus Poisson's equation. Also, notice that the nature of the interaction (electrostatic in our case) is not of essential importance. The main result is that we can reduce the (phase-space) Wigner equation to a (real-space) nonlinear Schrödinger equation.

The two hypotheses used for this reduction are as follows: (a) all states composing the mixture have the same amplitude [which leads to Eq. (22) for the quantum pressure], and (b) the equation of state for the classical pressure is $P^C=P^C(n)$. Hypothesis (b) is the standard fluid closure, and needs no further comment. Hypothesis (a) means that all electrons are distributed in space according to the same probability distribution $n(x)=A^2(x)$. What distinguishes the electrons from one another is their phase S_α, and therefore their velocity u_α. This approximation can be viewed as a first step beyond the standard homogeneous equilibrium of a fermion gas, for which each state is represented by a plane wave,

$$\psi_\alpha(x,t)=A\exp(imu_\alpha x/\hbar),$$

with the amplitude A and the velocity u_α spatially constant. In our approximation, both the amplitude and the velocity can be spatially modulated, although we still restrict ourselves to the case in which the amplitude is the same for all states. This appears to be a reasonable closure assumption for systems that are not too far from the Fermi-Dirac equilibrium.

IV. APPLICATIONS

As a relevant example of the above theory, we consider a zero-temperature one-dimensional electron gas, with Fermi velocity v_F and equilibrium density n_0. In this case, the classical pressure is

$$P^C=\frac{mv_F^2}{3n_0^2}n^3. \tag{30}$$

(Notice that the term "classical" is somewhat inappropriate here, as P^C will contain Planck's constant through the Fermi velocity.) We also note that the Fermi velocity in one spatial dimension

$$v_F=\frac{\pi}{2}\frac{\hbar n_0}{m} \tag{31}$$

is proportional to n_0, whereas in three dimensions $v_F\propto n_0^{1/3}$.

Using Eq. (30), the effective potential defined in Eq. (24) turns out to be

$$W=\frac{mv_F^2}{2n_0^2}|\Psi|^4. \tag{32}$$

Notice that the effective potential is repulsive, and tends to flatten the electron density. This is quite natural, as W derives from the pressure P^C, which in turn is a manifestation of the dispersion of velocities in a fermion gas. When the gas is at equilibrium, $W\sim n^2\sim$ const, and this term has no effect.

We also point out that a similar nonlinear Schrödinger equation with a $|\Psi|^4$-dependent potential has recently been derived in the study of low-dimensional Bose condensates.[10] We stress, however, that such a boson-fermion duality only applies to one-dimensional systems. For a D-dimensional fermion system, the classical part of the pressure has the form $P^C\sim n^{(D+2)/D}$, so that the effective potential becomes $W\sim n^{2/D}$.[11]

A. Linear wave propagation

As a first application, let us study wave propagation for the effective SP system (28) and (29) with W given by Eq.

(32). Linearizing around the homogeneous equilibrium $\Psi = \sqrt{n_0}$, $e\phi = mv_F^2/2$, we obtain the following dispersion relation (for waves with frequency ω and wave number k):

$$\omega^2 = \omega_p^2 + k^2 v_F^2 + \frac{\hbar^2 k^4}{4m^2}. \tag{33}$$

For $v_F = 0$, we recover the dispersion relation of the standard SP system.[6] Equation (33) can be written in dimensionless units by using Eqs. (5) and (7), which are valid both in one and three spatial dimensions,

$$\frac{\omega^2}{\omega_p^2} = 1 + k^2 \lambda_F^2 + \frac{k^4 \lambda_F^4}{4} \Gamma_Q. \tag{34}$$

Note that quantum-*mechanical* effects (dispersion of the wave packet) are first order in the coupling parameter Γ_Q, whereas quantum-*statistical* effects (Fermi-Dirac distribution) appear at leading order.

We want to compare this dispersion relation to the one obtained from the complete WP system (11) and (12), which, in the most general case, reads[6,12]

$$1 - \frac{\omega_p^2}{n_0} \int \frac{f_0(v)dv}{(\omega - kv)^2 - \hbar^2 k^4/4m^2} = 0. \tag{35}$$

In our case, $f_0(v)$ is given by the Fermi-Dirac distribution for a zero-temperature one-dimensional electron gas at equilibrium, i.e., $f_0(v) = n_0/2v_F$ if $|v| < v_F$ and $f_0(v) = 0$ if $|v| > v_F$. Substituting into Eq. (35), one obtains (without any further approximation)

$$\frac{\omega^2}{\omega_p^2} = \frac{\Omega^2}{\omega_p^2} \coth\left(\frac{\Omega^2}{\omega_p^2}\right) + k^2 \lambda_F^2 + \frac{k^4 \lambda_F^4}{4} \Gamma_Q, \tag{36}$$

where

$$\frac{\Omega^2}{\omega_p^2} = \frac{\hbar k^3 v_F}{m \omega_p^2} = k^3 \lambda_F^3 \Gamma_Q^{1/2}. \tag{37}$$

Now we expand the first term on the right-hand side of Eq. (36) in the long-wavelength (fluid) limit $\Omega \ll \omega_p$. Using the expansion $x \coth(x) = 1 + x^2/3 - x^4/45 + \cdots$, one obtains

$$\frac{\omega^2}{\omega_p^2} = 1 + k^2 \lambda_F^2 + \left(\frac{k^4 \lambda_F^4}{4} + \frac{k^6 \lambda_F^6}{3}\right) \Gamma_Q - \frac{1}{45} k^{12} \lambda_F^{12} \Gamma_Q^2 + \cdots. \tag{38}$$

This is a double expansion in powers of the parameters Γ_Q and $k\lambda_F$. The collisionless regime is in principle characterized by $\Gamma_Q \ll 1$, although, as was seen in Sec. II, electron-electron interactions can be neglected even when $\Gamma_Q \simeq 1$, as is the case for metals. On the other hand, the fluid regime is characterized by small wave numbers ($\Omega \ll \omega_p$). Indeed, keeping terms to fourth order in $k\lambda_F$, Eq. (38) reduces to the dispersion relation for the effective SP system, Eq. (34). This is a further indication that the effective SP system is a good approximation to the complete WP system for long wavelengths.

We also note that for $\Gamma_Q \to 0$, the dispersion relation reduces to

$$\omega^2 = \omega_p^2 + k^2 v_F^2. \tag{39}$$

This is exactly the dispersion relation obtained from the classical Vlasov-Poisson system with a zero-temperature Fermi-Dirac equilibrium. In other words, when the quantum coupling parameter is vanishingly small, a classical dynamical equation can be used, as the only quantum effects come from the Fermi-Dirac statistics. This situation may apply to extremely dense astrophysical systems such as white dwarfs.

B. Stationary solutions

As a second illustration, we use the present formalism to describe the stationary states of the electron gas.[13] This result is more easily obtained by using the fluid version of our model. In the time-independent case, the continuity equation (14) and the force equation (26) possess the following first integrals:

$$J = A^2 u, \quad E = \frac{mu^2}{2} - e\phi + W - \frac{\hbar^2}{2mA} \frac{d^2 A}{dx^2}, \tag{40}$$

where $A = \sqrt{n}$. The first integrals in Eq. (40) corresponds to current (J) and energy (E) conservation. We can always choose $E = 0$ by a shift in the electrostatic potential. In this way, we can reduce the description of the stationary states to a set of second-order nonlinear ordinary differential equations for the amplitude A and the electrostatic potential ϕ. For a zero-temperature one-dimensional electron gas, the effective potential W is given by Eq. (32); thus from Eqs. (40) and (29) we get

$$\hbar^2 \frac{d^2 A}{dx^2} = m\left(\frac{mJ^2}{A^3} - 2eA\phi + \frac{mv_F^2}{n_0^2} A^5\right), \tag{41}$$

$$\frac{d^2 \phi}{dx^2} = \frac{e}{\varepsilon_0}(A^2 - n_0). \tag{42}$$

Notice that, if the amplitude $A(x)$ is a slowly varying function of x, the second derivative on the left-hand side of Eq. (41) can be neglected. With this assumption, Eq. (41) reduces to an algebraic equation, which can be solved for A, and the result plugged into Eq. (42). This becomes a nonlinear differential equation for the electrostatic potential, which is merely the Thomas-Fermi approximation to our model.

It can be easily verified that the $J = 0$ case cannot sustain small-amplitude, periodic solutions. Hence, we assume $J = n_0 u_0$ with $u_0 \neq 0$ and introduce the following rescaling:

$$\hat{x}=\frac{\omega_p x}{u_0}, \qquad \hat{A}=\frac{A}{\sqrt{n_0}}, \qquad \hat{\phi}=\frac{e\phi}{mu_0^2}, \tag{43}$$

$$H=\frac{\hbar\omega_p}{mu_0^2}, \qquad V_F=\frac{v_F}{u_0}.$$

We obtain, in the transformed variables (omitting the caret for simplicity of notation),

$$H^2\frac{d^2A}{dx^2}=-2\phi A+\frac{1}{A^3}+V_F^2A^5, \tag{44}$$

$$\frac{d^2\phi}{dx^2}=A^2-1, \tag{45}$$

a system that only depends on the rescaled parameters H and V_F. Note that the quantum coupling parameter can be written as $\Gamma_Q=H/V_F^2$.

It is interesting to perform a linear stability analysis in order to see in what conditions the system supports small-amplitude spatially periodic solutions. Writing

$$A=1+A'\exp(ikx), \quad \phi=(1+V_F^2)/2+\phi'\exp(ikx), \tag{46}$$

and retaining in Eqs. (44) and (45) only terms up to first order in the primed variables, we obtain the relation

$$H^2k^4-4(1-V_F^2)k^2+4=0. \tag{47}$$

This second degree equation has solutions

$$k^2=\frac{2(1-V_F^2)\pm 2\sqrt{(1-V_F^2)^2-H^2}}{H^2}. \tag{48}$$

Clearly, spatially oscillating solutions only exist when k^2 is real and positive, which yields the condition

$$V_F^2<1-H, \tag{49}$$

or equivalently

$$mu_0^2>mv_F^2+\hbar\omega_p. \tag{50}$$

This expression sets a lower bound on the speed u_0, below which no oscillating stationary solution can exist.

C. Two-stream instability

A classical plasma composed of two counterstreaming electronic populations with velocities $\pm u_0$ can give rise, for certain wave numbers, to an instability. In a previous paper,[13] we have shown that quantum effects modify the dispersion relation, and give rise to a new instability branch. These results were obtained by neglecting the effects of quantum statistics, and are therefore valid in the limit $v_F\ll u_0$. Here, we perform the same calculations for finite values of v_F.

We consider two electronic populations, which are both distributed according to a zero-temperature Fermi-Dirac equilibrium, but with average velocities $\pm u_0$. The motionless ions provide a neutralizing background. The dispersion relation for such a two-stream plasma can be found in the following way. For a single stream propagating at velocity $\pm u_0$, our fluid model yields the following dielectric constant (thus valid for long wavelengths):

$$\epsilon_\pm(k,\omega)=1-\frac{\omega_p^2}{(\omega\mp ku_0)^2-k^2v_F^2-\hbar^2k^4/4m^2}. \tag{51}$$

Setting $\epsilon_\pm(k,\omega)=0$ leads to the dispersion relation found previously, Eq. (33), with the appropriate Doppler shift. The dielectric constant for the two-stream case is found by averaging the contributions from each stream $\epsilon(k,\omega)=(\epsilon_+ +\epsilon_-)/2$. Using the normalization of Eqs. (43), we obtain

$$\epsilon(k,\omega)=1-\frac{1/2}{(\omega+k)^2-k^2V_F^2-H^2k^4/4} -\frac{1/2}{(\omega-k)^2-k^2V_F^2-H^2k^4/4}. \tag{52}$$

Setting $\epsilon(k,\omega)=0$, we obtain the dispersion relation for the two-stream plasma,

$$\omega^4-\left(1+2k^2(1+V_F^2)+\frac{H^2k^4}{2}\right)\omega^2-k^2\left(1-V_F^2-\frac{H^2k^2}{4}\right) \times\left(1-(1-V_F^2)k^2+\frac{H^2k^4}{4}\right)=0. \tag{53}$$

Notice that for $V_F=0$, we recover the dispersion relation obtained in Ref. 13. Solving for ω^2, one obtains

$$\omega^2=\frac{1}{2}+k^2\left(1+V_F^2+\frac{H^2k^2}{4}\right) \pm\frac{1}{2}\left[1+8k^2\left(1+2k^2V_F^2+\frac{H^2k^4}{2}\right)\right]^{1/2}. \tag{54}$$

The solution for ω^2 has two branches, one of which is always positive and gives stable oscillations. The other solution is negative ($\omega^2<0$) provided that

$$[H^2k^2-4(1-V_F^2)][H^2k^4-4(1-V_F^2)k^2+4]<0. \tag{55}$$

We immediately notice that, if $V_F\geqslant 1$, Eq. (55) is never verified, and therefore there is no instability. This is a quite natural result. Indeed, mathematically, the instability is due to the fact that the two-stream velocity distribution has a "hole" around $v=0$. When $V_F\geqslant 1$, the hole is filled up, and no instability can occur. To put it differently, there can be instability only when the equilibrium distribution is a nonmonotonic function of the energy, which ceases to be true when $V_F\geqslant 1$.

When $V_F<1$, Eq. (55) bifurcates for $H=1-V_F^2$. If $H\geqslant 1-V_F^2$, the second factor is always positive, and instability occurs for $H^2k^2<4(1-V_F^2)$. If $H<1-V_F^2$, there is instability if either

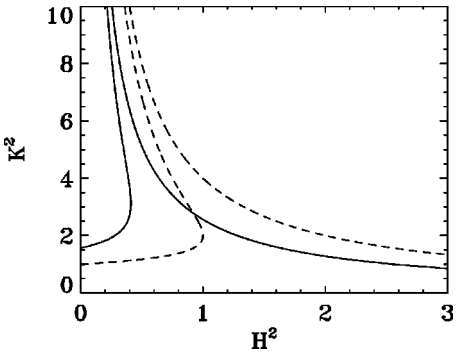

FIG. 1. Stability diagram for the two-stream plasma, with $V_F = 0.7$ (solid lines) and $V_F = 0$ (dashed lines). The curves correspond to Eqs. (56) and (57). For both cases, the region of the plane containing the H^2 axis is unstable.

$$0 < H^2 k^2 < 2(1 - V_F^2) - 2\sqrt{(1 - V_F^2)^2 - H^2} \tag{56}$$

or

$$2(1 - V_F^2) + 2\sqrt{(1 - V_F^2)^2 - H^2} < H^2 k^2 < 4(1 - V_F^2). \tag{57}$$

This yields the stability diagram plotted in Fig. 1, which is similar to the one obtained in the limiting case $V_F = 0$. The presence of a finite Fermi velocity has the effect of reducing the region of instability. Numerical simulations yield similar results to those observed in the $V_F = 0$ case, which are reported in Ref. 13.

V. CONCLUSION

In this paper, we have first established the conditions of validity of the Wigner-Poisson system. Subsequently, by taking moments of the Wigner equation, we have derived an effective Schrödinger-Poisson system that captures the essential features of a quantum electron gas. In the long-wavelength limit, this model correctly reproduces the results of the linear analysis of the Wigner-Poisson system. The advantages of the effective SP model are manifold: it is local in space (compared to the nonlocal WP system); it is cast into the ordinary space, rather than the phase space; and it has a straightforward interpretation in terms of fluid quantities. Furthermore, it is easily amenable to numerical studies, given the abundance of accurate numerical techniques for the Schrödinger equation (in comparison, numerical methods for the Wigner equation[12] are much scarcer and more cumbersome to implement). The crucial points in the derivation of the model are (a) the decomposition of the pressure into a classical and a quantum contribution, and (b) the restriction to an appropriate class of statistical mixtures (composed of states with the same amplitude but different phases). We believe that this class is wide enough to describe a significant range of relevant physical systems.

For the case of a completely degenerate electron gas, the effective SP model can be put in a particularly simple form, in which the Schrödinger equation exhibits a $|\Psi|^4$ nonlinearity. This model has been applied to the study of linear wave propagation, nonlinear stationary solutions, and the two-stream instability. The simplicity of the resulting system of equations makes it a useful tool for the study of quantum transport in solid-state plasmas.

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