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p-Adic Spherical Coordinates and their Applications

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Abstract

On the space \mathbb{Q}_p^n , where $p \neq 2$ and p does not divide n , we construct a p -adic counterpart of spherical coordinates. As applications, a description of homogeneous distributions on \mathbb{Q}_p^n and a skew product decomposition of p -adic Lévy processes are given.

Key words: p -adic numbers, spherical coordinates, homogeneous distributions, Lévy processes

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1 INTRODUCTION

Spherical coordinates in \mathbb{R}^n are among the basic tools of real analysis from its very early days. The usefulness of the decomposition $\mathbb{R}^n \setminus \{0\} = S^{n-1} \times \mathbb{R}_+$ is due to the fact that both factors on the right are *smooth* manifolds of dimensions smaller than n . Thus various n -dimensional objects of analysis and geometry are reduced to objects of similar nature in smaller dimensions.

A straightforward generalization to the case of the n -dimensional p -adic space \mathbb{Q}_p^n leads to a different situation. What is usually called a p -adic unit sphere, the set

$$\mathcal{S}_1 = \left\{ x = (x_1, \dots, x_n) \in \mathbb{Q}_p^n : \max_{1 \leq j \leq n} |x_j|_p = 1 \right\},$$

is actually an open-closed subset of \mathbb{Q}_p^n , so that it has the same dimension as the ambient space \mathbb{Q}_p^n . The “radial” component in the decomposition $\mathbb{Q}_p^n \setminus \{0\} = \mathcal{S}_1 \times p^{\mathbb{Z}}$, $p^{\mathbb{Z}} = \{p^N, N \in \mathbb{Z}\}$, given by the equality $x = \{(x_1, \dots, x_n)p^N\} p^{-N}$, where $\max_{1 \leq j \leq n} |x_j|_p = p^N$, is discrete and does not have the same nature as the “spherical” component.

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In this paper, assuming that $p \neq 2$ and p does not divide n , we construct a coordinate system in \mathbb{Q}_p^n resembling the classical spherical coordinates. The idea is to identify \mathbb{Q}_p^n with the unramified extension K of the field \mathbb{Q}_p of degree n and to use such related objects as the Frobenius automorphism and the norm map. The counterpart of the sphere introduced below is a direct product of a finite set by a hypersurface of the group of principal units of K ; the counterpart of \mathbb{R}_+ is a multiplicative subgroup of \mathbb{Q}_p generated by $p^{\mathbb{Z}}$ and (an interesting coincidence of terminology!) the group of positive elements of \mathbb{Q}_p [16]. For $n = 2$, our construction is different from the polar coordinates introduced in [6, 18] though the constructions have some common features.

As applications, we obtain, following [11], a description of all homogeneous distributions on \mathbb{Q}_p^n (earlier such a result was known only for $n = 1$; only an example was considered for an arbitrary n in [18]), and a skew product representation for p -adic Lévy processes. The latter result follows a recent work by Liao [12] who considered a decomposition of a Markov process on a manifold invariant under a Lie group action; for earlier results regarding decompositions of a Brownian motion into a skew product of the radial motion and the spherical Brownian motion with a time change see [5, 14].

2 Preliminaries

Let us recall some notions and results from p -adic analysis and algebraic number theory, which will be used in a sequel. Note that elementary notions and facts regarding p -adic numbers and their properties are used without explanations; see [18]. For further details see [4, 9, 10, 15, 16, 19].

Let p be a prime number, $p \neq 2$. A field K is called a finite extension of degree n of the field \mathbb{Q}_p of p -adic numbers, if \mathbb{Q}_p is a subfield of K , and K is a finite-dimensional vector space over \mathbb{Q}_p , with $\dim K = n$.

For each element $x \in K$, consider a \mathbb{Q}_p -linear operator L_x on K defined as $L_x z = xz$, $z \in K$ (the multiplication in K). Its determinant $N(x) = \det L_x$ is an element of \mathbb{Q}_p . The mapping $x \mapsto N(x)$, $K \rightarrow \mathbb{Q}_p$, is called the norm map. If $x, y \in K$, $\lambda \in \mathbb{Q}_p$, then $N(xy) = N(x)N(y)$, $N(\lambda x) = \lambda^n N(x)$. The norm map is used to define the normalized absolute value on K : $\|x\| = |N(x)|_p$ making K a locally compact totally disconnected topological field.

Denote

$$O = \{x \in K : \|x\| \leq 1\}, \quad P = \{x \in K : \|x\| < 1\}, \quad U = O \setminus P.$$

O is a subring of K called the ring of integers, P is an ideal in O called the prime ideal. The multiplicative subgroup U is called the group of units. The quotient O/P is a finite field of characteristic p consisting of $q = p^v$ elements ($v \in \mathbb{N}$). The field O/P is isomorphic to the standard finite field \mathbb{F}_q consisting of q elements (see [13]). The normalized absolute value $\|\cdot\|$ takes the values q^N , $N \in \mathbb{Z}$, and 0.

An extension K of degree n is called unramified, if $\|p\| = q^{-1}$. In this case, $P = pO$, $q = p^n$. It is known that, for any $n \in \mathbb{N}$, there exists an unramified extension of \mathbb{Q}_p of degree n ; it is unique up to an isomorphism. Below we fix n and reserve the letter K for this extension. It is generated over \mathbb{Q}_p by a primitive root of 1 of degree $q - 1$. Thus, K contains the group μ_{q-1} of all the roots of 1 of this degree. On the other hand, if $p \neq 2$, then K does not contain nontrivial roots of 1 of degree p . The Galois group of the extension K , that is the group of automorphisms

of the field K fixing \mathbb{Q}_p , is a cyclic group generated by the Frobenius automorphism \mathfrak{g} . On μ_{q-1} , \mathfrak{g} acts by the rule $\mathfrak{g}(\omega) = \omega^p$, permuting the roots of 1. The norm map is invariant with respect to \mathfrak{g} : $N(\mathfrak{g}(x)) = N(x)$. On the finite field O/P , the Galois group induces the Galois group of \mathbb{F}_q over \mathbb{F}_p ; the automorphism \mathfrak{g} turns into its finite field counterpart given by raising to the power p .

Let $\theta_1, \dots, \theta_n \in O$ be such elements that their images in O/P form a basis in O/P over \mathbb{F}_p . Then $\theta_1, \dots, \theta_n$ form a basis of K over \mathbb{Q}_p (called a canonical basis). The choice of this basis determines an identification of K and \mathbb{Q}_p^n . The isomorphism $\mathbb{Q}_p^n \rightarrow K$ as vector spaces over \mathbb{Q}_p defining this identification has the form

$$(x_1, \dots, x_n) \mapsto \sum_{j=1}^n x_j \theta_j, \quad x_1, \dots, x_n \in \mathbb{Q}_p.$$

The normalized absolute value on K has the following expression: if $x = \sum_{j=1}^n x_j \theta_j$, then

$$\|x\| = \left(\max_{1 \leq j \leq n} |x_j|_p \right)^n \quad (1)$$

(to avoid confusion, note that here and below we consider only the unramified extensions). Below, it will be convenient to assume that $\theta_n = 1$.

The multiplicative group $K^* = K \setminus \{0\}$ can be described as follows (we consider only the case where $p \neq 2$). If $x \in K^*$, then

$$x = p^\nu \omega \left\{ \prod_{j=1}^{n-1} (1 + \theta_j p)^{b_j} \right\} (1 + p)^{b_n} \quad (2)$$

where $\nu \in \mathbb{Z}$, $\omega \in \mu_{q-1}$, $b_j \in \mathbb{Z}_p$ ($j = 1, \dots, n$). The expression $(1 + z)^\beta$, with $z \in K$, $\|z\| < 1$, $\beta \in \mathbb{Z}_p$, is defined as a limit of $(1 + z)^{\beta_m}$ where $\beta_m \in \mathbb{N}$, $\beta_m \rightarrow \beta$, as $m \rightarrow \infty$, in the topology of \mathbb{Z}_p . An equivalent definition is via the Mahler expansion

$$(1 + z)^\beta = 1 + \sum_{i=1}^{\infty} z^i \frac{\beta(\beta-1)\cdots(\beta-i+1)}{i!} \quad (\beta \in \mathbb{Z}_p, \|z\| < 1) \quad (3)$$

convergent in the Banach space of continuous functions on \mathbb{Z}_p with values from K . Obviously, an element (2) has the absolute value $q^{-\nu}$; all the factors in the right-hand side of (2), except the first one, belong to U . The elements $\nu, \omega, b_1, \dots, b_n$ are determined by x in a unique way (thus, for a fixed x , each factor $(1 + \theta_j p)^{b_j}$ contains a fixed p -adic integer b_j , so that $(1 + \theta_j p)^{b_j}$ is just an element from U).

Another canonical representation of an element $x \in K^*$ is

$$x = p^\nu \omega (1 + x_1 p + x_2 p^2 + \cdots) \quad (4)$$

where the first two factors are the same as in (2), $x_1, x_2, \dots \in \mu_{q-1} \cup \{0\}$, the series converges in K , and all the ingredients of (4) are determined in a unique way.

The set U_1 of elements (4) with $\nu = 0$ and $\omega = 1$ is a multiplicative group called the group of principal units. For the unramified extension considered here, the norm map N maps U_1 onto the group $U_1(\mathbb{Q}_p)$ of principal units of the field of p -adic numbers. If $\zeta \in U_1(\mathbb{Q}_p)$, that is $\zeta = 1 + \zeta_1 p + \zeta_2 p^2 + \dots$, $\zeta_j \in \mu_{p-1}$, the powers ζ^β , $\beta \in \mathbb{Z}_p$, are defined in accordance with (3) and belong to $U_1(\mathbb{Q}_p)$.

In particular, if p does not divide n , then $\frac{1}{n} \in \mathbb{Z}_p$, and we have a well-defined root $\zeta^{1/n} \in U_1(\mathbb{Q}_p)$. Thus, in this case, for any $x \in K^*$ of the form (4), we may write

$$N(\omega^{-1}x) = p^{n\nu} N(1 + x_1 p + x_2 p^2 + \dots)$$

and define

$$r = (N(\omega^{-1}x))^{1/n} = p^\nu (N(1 + x_1 p + x_2 p^2 + \dots))^{1/n} \quad (5)$$

as an element of

$$\mathbb{Q}_p^{(1)} = \{ \zeta \in \mathbb{Q}_p : \zeta = p^\nu (1 + \zeta_1 p + \zeta_2 p^2 + \dots), \nu \in \mathbb{Z}, \zeta_j \in \mu_{p-1} \cup \{0\} \}, \quad (6)$$

a multiplicative subgroup of \mathbb{Q}_p .

3 Spherical coordinates

For $x \in K^*$, we consider the following elements. Let $\omega = \omega(x) \in \mu_{q-1}$ be the element from (2) or (4). If p does not divide n , set

$$r = r(x) = (N(\omega^{-1}x))^{1/n} \in \mathbb{Q}_p^{(1)}.$$

Finally, let

$$\xi = \xi(x) = \omega^{-1} r^{-1} x,$$

so that

$$x = \omega(x) \xi(x) r(x). \quad (7)$$

We call (ω, ξ, r) the *spherical coordinates* of an element $x \in K^*$.

Denote by Σ_n the compact multiplicative group

$$\Sigma_n = \{ y \in K^* : \omega(y) = 1, N(y) = 1 \}.$$

Theorem 1. *If $p \neq 2$ and p does not divide n , then, for each $x \in K^*$, $\xi(x) \in \Sigma_n$. The representation of an element $x \in K^*$ as a product of elements from μ_{q-1} , Σ_n , and $\mathbb{Q}_p^{(1)}$, is unique. The decomposition (7) defines an isomorphism $K^* \cong \mu_{q-1} \times \Sigma_n \times \mathbb{Q}_p^{(1)}$ of multiplicative topological groups.*

Proof. We will use the representation (2), not with an arbitrary system $\{\theta_j\}$, but with a special one. In order to construct the latter, we begin with an arbitrary canonical basis $\theta_1, \dots, \theta_{n-1}, \theta_n$, where $\theta_n = 1$. Consider the elements

$$\varepsilon_j = \frac{\theta_j - \mathfrak{g}(\theta_j)}{1 + \mathfrak{g}(\theta_j)p}, \quad j = 1, \dots, n-1. \quad (8)$$

Their images $\overline{\varepsilon_j}$ in $O/P \cong \mathbb{F}_q$ have the form $\overline{\varepsilon_j} = \overline{\theta_j} - \overline{\theta_j}^p$ where $\overline{\theta_j}$ is the image of θ_j .

Let us show that the elements $\overline{\varepsilon_1}, \dots, \overline{\varepsilon_{n-1}}, 1$ form a basis in \mathbb{F}_q over \mathbb{F}_p . It is sufficient to prove their linear independence.

First we prove the linear independence of $\overline{\varepsilon_1}, \dots, \overline{\varepsilon_{n-1}}$. Let $c_j \in \mathbb{F}_p$, $\sum_{j=1}^{n-1} c_j \overline{\varepsilon_j} = 0$. Since $c_j^p = c_j$, we have

$$0 = \sum_{j=1}^{n-1} c_j \overline{\theta_j} - \sum_{j=1}^{n-1} c_j \overline{\theta_j}^p = \sum_{j=1}^{n-1} c_j \overline{\theta_j} - \left(\sum_{j=1}^{n-1} c_j \overline{\theta_j} \right)^p,$$

so that $\sum_{j=1}^{n-1} c_j \overline{\theta_j} \stackrel{\text{def}}{=} \lambda \in \mathbb{F}_p$ and

$$\sum_{j=1}^{n-1} c_j \overline{\varepsilon_j} - \lambda \overline{\theta_n} = 0 \quad (\overline{\theta_n} = 1).$$

Since $\overline{\theta_1}, \dots, \overline{\theta_{n-1}}, \overline{\theta_n}$ are linearly independent, we find that $c_1 = \dots = c_{n-1} = \lambda = 0$, which proves the linear independence of $\overline{\varepsilon_1}, \dots, \overline{\varepsilon_{n-1}}$.

Now, let $d_1, \dots, d_n \in \mathbb{F}_p$

$$d_1 \left(\overline{\theta_1} - \overline{\theta_1}^p \right) + d_2 \left(\overline{\theta_2} - \overline{\theta_2}^p \right) + \dots + d_{n-1} \left(\overline{\theta_{n-1}} - \overline{\theta_{n-1}}^p \right) + d_n = 0. \quad (9)$$

Raising to the power p we obtain successively that

$$d_1 \left(\overline{\theta_1}^p - \overline{\theta_1}^{p^2} \right) + d_2 \left(\overline{\theta_2}^p - \overline{\theta_2}^{p^2} \right) + \dots + d_{n-1} \left(\overline{\theta_{n-1}}^p - \overline{\theta_{n-1}}^{p^2} \right) + d_n = 0,$$

$$\dots \dots \dots$$

$$d_1 \left(\overline{\theta_1}^{p^{n-1}} - \overline{\theta_1}^{p^n} \right) + d_2 \left(\overline{\theta_2}^{p^{n-1}} - \overline{\theta_2}^{p^n} \right) + \dots + d_{n-1} \left(\overline{\theta_{n-1}}^{p^{n-1}} - \overline{\theta_{n-1}}^{p^n} \right) + d_n = 0,$$

$$d_1 \left(\overline{\theta_1}^{p^n} - \overline{\theta_1}^{p^{n+1}} \right) + d_2 \left(\overline{\theta_2}^{p^n} - \overline{\theta_2}^{p^{n+1}} \right) + \dots + d_{n-1} \left(\overline{\theta_{n-1}}^{p^n} - \overline{\theta_{n-1}}^{p^{n+1}} \right) + d_n = 0.$$

Note that $\overline{\theta_j}^{p^{n+1}} = \overline{\theta_j}^{qp} = \overline{\theta_j}^p$ and add up all the equalities. We find that $nd_n = 0$ in \mathbb{F}_p , and since p does not divide n , $d_n = 0$. Then (9) implies the equalities $d_1 = \dots = d_{n-1} = 0$.

Thus, we know that the collection $\overline{\varepsilon_1}, \dots, \overline{\varepsilon_{n-1}}, 1$ is a \mathbb{F}_p -basis in O/P . Therefore we may write a representation like (2) based on the canonical basis $\varepsilon_1, \dots, \varepsilon_{n-1}, 1$: for every $x \in K^*$,

$$x = p^\nu \omega \left\{ \prod_{j=1}^{n-1} (1 + \varepsilon_j p)^{b'_j} \right\} (1 + p)^{b'_n} \quad (10)$$

where $\nu \in \mathbb{Z}$, $\omega = \omega(x) \in \mu_{q-1}$, $b'_1, \dots, b'_n \in \mathbb{Z}_p$.

By (8), we can write

$$1 + \varepsilon_j p = \frac{1 + \theta_j p}{1 + \mathfrak{g}(\theta_j) p},$$

whence

$$\prod_{j=1}^{n-1} (1 + \varepsilon_j p)^{b'_j} = \frac{y}{\mathfrak{g}(y)}, \quad y = \prod_{j=1}^{n-1} (1 + \theta_j p)^{b'_j}. \quad (11)$$

It follows from (11) that

$$\prod_{j=1}^{n-1} (1 + \varepsilon_j p)^{b'_j} \in \Sigma_n.$$

On the other hand, $p^\nu(1+p)^{b'_n} \in \mathbb{Q}_p^{(1)}$, and $N(\omega^{-1}x) = p^{n\nu}(1+p)^{nb'_n}$. Thus

$$r(x) = p^\nu(1+p)^{b'_n},$$

and we have got the representation (7) with $\xi(x) = \frac{y}{\mathfrak{g}(y)}$.

If we have another representation $x = \omega_1 \xi_1 r_1$, $\omega_1 \in \mu_{q-1}$, $\xi_1 \in \Sigma_n$, and $r_1 \in \mathbb{Q}_p^{(1)}$, then it follows directly from the definitions of Σ_n and $\mathbb{Q}_p^{(1)}$ that $\omega_1 = \omega$. Then, applying N we get $N(\omega^{-1}x) = r_1^n$, so that $r_1 = r$ and $\xi_1 = \xi$.

The fact that the representation (7) is compatible with the algebraic operations and the topologies on the corresponding topological groups follows immediately from the properties of the norm map and the group of principal units as a topological \mathbb{Z}_p -module. \blacksquare

Below we will often denote our spherical coordinates by $x = (\eta, r)$ where $\eta = (\omega, \xi) \in Z_n = \mu_{q-1} \times \Sigma_n$. As in the classical situation, sometimes it is convenient to extend the spherical coordinates to the whole of K – for $x = 0$, we set $r = 0$ while η is not defined.

In order to derive a formula for integration in spherical coordinates, denote by $d\xi$ the Haar measure on Σ_n normalized by the relation $\int_{\Sigma_n} d\xi = 1$. A Haar measure on $\mathbb{Q}_p^{(1)}$ is induced by the multiplicative Haar measure on \mathbb{Q}_p^* having the form $\frac{dr}{|r|_p}$ where dr is the additive Haar measure. Similarly (see [2]), $\frac{dx}{\|x\|}$ is the Haar measure on K^* . As usual, we assume that $\int_O dx = \int_{\mathbb{Z}_p} dr = 1$.

For the direct product $K^* = \mu_{q-1} \times \Sigma_n \times \mathbb{Q}_p^{(1)}$, we have the integration formula

$$\int_{K^*} f(x) \frac{dx}{\|x\|} = c \sum_{\omega \in \mu_{q-1} \Sigma_n} \int_{\mathbb{Q}_p^{(1)}} d\xi \int f(\omega \xi r) \frac{dr}{|r|_p} \quad (12)$$

valid, for example, for any continuous function on K^* with a compact support. In order to find the normalization constant c , we take for f the indicator function of the group of units U .

It is known [10] that

$$\int_U dx = 1 - q^{-1},$$

$$\int_{r \in \mathbb{Q}_p^{(1)}, |r|_p=1} dr = \frac{1}{p-1} \left(1 - \frac{1}{p}\right) = p^{-1}.$$

Therefore $1 - q^{-1} = c(q-1)p^{-1}$, whence $c = \frac{1}{p^{n-1}}$.

It is easy to rewrite (12) in terms of additive Haar measures. Substituting $f(x)\|x\|$ for $f(x)$ in (12) we find that

$$\int_K f(x) dx = \frac{1}{p^{n-1}} \sum_{\omega \in \mu_{q-1}\Sigma_n} \int_{\mathbb{Q}_p^{(1)}} d\xi \int_{\mathbb{Q}_p^{(1)}} f(\omega\xi r) |r|_p^{n-1} dr. \quad (13)$$

We will not study exact conditions on f , under which (13) is valid. It is sufficient if f is continuous on K and has a compact support.

4 Homogeneous Distributions

Let $\pi : \mathbb{Q}_p^* \rightarrow \mathbb{C}$ be a multiplicative quasicharacter, that is $\pi(z) = |z|_p^s \theta(z)$ where $s \in \mathbb{R}$, $\theta(z)$ is a multiplicative character, $|\theta(z)| = 1$, such that $\theta(p) = 1$.

A continuous function $f : \mathbb{Q}_p^n \rightarrow \mathbb{C}$ is called a homogeneous function of degree π , if

$$f(\lambda x_1, \dots, \lambda x_n) = \pi(\lambda) f(x_1, \dots, x_n) \quad (14)$$

for any $\lambda \in \mathbb{Q}_p^{(1)}$, $x = (x_1, \dots, x_n) \in \mathbb{Q}_p^n$.

A Bruhat-Schwartz distribution $f \in \mathcal{D}'(\mathbb{Q}_p^n)$ is called a homogeneous distribution of degree π , if

$$\langle f, \varphi_\lambda \rangle = \pi(\lambda) |\lambda|_p^n \langle f, \varphi \rangle, \quad \varphi_\lambda(x) = \varphi(\lambda^{-1}x) \quad (x \in \mathbb{Q}_p^n), \quad (15)$$

for any $\varphi \in \mathcal{D}(\mathbb{Q}_p^n)$, $\lambda \in \mathbb{Q}_p^{(1)}$. The definitions (14) and (15) are slightly more general than the usual ones [6, 18, 1] – we take only “positive” $\lambda \in \mathbb{Q}_p^{(1)}$.

As before, studying the structure of homogeneous distributions we identify \mathbb{Q}_p^n with the unramified extension K . The definition (15) makes sense in this case too. **Below we assume that $p \neq 2$ and p does not divide n .** Then we may use the spherical coordinates $x = \omega\xi r$ ($x \in K^*$, $\omega \in \mu_{q-1}$, $\xi \in \Sigma_n$, and $r \in \mathbb{Q}_p^{(1)}$).

It follows from the representations (10) and (11) that the group Σ_n is isomorphic to the direct product of $n - 1$ copies of \mathbb{Z}_p . Therefore we have natural spaces of test functions $\mathcal{D}(\Sigma_n)$ (consisting of locally constant functions on $\Sigma_n \cong \mathbb{Z}_p^{n-1}$) and $\mathcal{D}(Z_n)$, $Z_n = \mu_{q-1} \times \Sigma_n$, as well as the spaces of distributions $\mathcal{D}'(\Sigma_n)$ and $\mathcal{D}'(Z_n)$. If $F \in \mathcal{D}'(Z_n)$ is generated by an ordinary function $F(\omega, \xi)$, that means that

$$\langle F, \psi \rangle = \frac{1}{q-1} \sum_{\omega \in \mu_{q-1}\Sigma_n} \int F(\omega, \xi) \psi(\omega, \xi) d\xi, \quad \psi \in \mathcal{D}(Z_n).$$

If f is a continuous homogeneous function, then it follows from (14) that

$$f(\omega\xi r) = \pi(r) f(\omega\xi). \quad (16)$$

A function f of the form (16) with $\pi(r) = |r|_p^s \theta(r)$, $\text{Re } s > -n$, determines a distribution from $\mathcal{D}'(K)$ in a straightforward way. Using the integration formula (13) we get, for any $\varphi \in \mathcal{D}(K)$, that

$$\langle f, \varphi \rangle = \int_K f(x) \varphi(x) dx = p^{1-n} \sum_{\omega \in \mu_{q-1}\mathbb{Q}_p^{(1)}} \int_{\Sigma_n} |r|_p^{s+n-1} \theta(r) dr \int f(\omega\xi) \varphi(r\omega\xi) d\xi.$$

More generally, if $\text{Re } s > -n$, $F \in \mathcal{D}'(Z_n)$, then the distribution $f = \pi(r)F$ is given by the relation

$$\langle f, \varphi \rangle = p^{1-n}(p^n - 1) \int_{\mathbb{Q}_p^{(1)}} \langle F, \varphi(r \cdot) \rangle |r|_p^{s+n-1} \theta(r) dr,$$

$\varphi \in \mathcal{D}(K)$.

The function $r \mapsto \langle F, \varphi(r \cdot) \rangle$ is locally constant and has a compact support by virtue of the compactness of Z_n . In particular, suppose that $\varphi(r\omega\xi) = 0$ if $|r|_p > p^\nu$. Then

$$\langle f, \varphi \rangle = p^{1-n} \sum_{\omega \in \mu_{q-1}} \left\{ \int_{r \in \mathbb{Q}_p^{(1)}: |r|_p \leq p^\nu} \langle F, \varphi(r \cdot) - \varphi(0) \rangle |r|_p^{s+n-1} \theta(r) dr \right. \\ \left. - \varphi(0) \langle F, 1 \rangle \int_{r \in \mathbb{Q}_p^{(1)}: |r|_p \leq p^\nu} |r|_p^{s+n-1} \theta(r) dr \right\} \quad (17)$$

The first integral in (17) is an entire function of s . Next,

$$\int_{r \in \mathbb{Q}_p^{(1)}: |r|_p \leq p^\nu} |r|_p^{s+n-1} \theta(r) dr = \sum_{j=-\infty}^{\nu} p^{j(s+n-1)} \int_{r \in \mathbb{Q}_p^{(1)}: |r|_p = p^j} \theta(r) dr \quad (18)$$

If the character θ is nontrivial on the group of principal units of \mathbb{Q}_p , then (making the change of variables $r = a\rho$ where a is a principal unit with $\theta(a) \neq 1$) we find that all the integrals in the right-hand side of (18) equal zero. If $\theta(r) \equiv 1$, then [17]

$$\int_{r \in \mathbb{Q}_p^{(1)}: |r|_p = p^j} dr = p^{j-1},$$

so that

$$\int_{\mathbb{Q}_p^{(1)}} |r|_p^{s+n-1} dr = p^{-1} \sum_{j=-\infty}^{\nu} p^{j(s+n)} = \frac{p^{\nu(s+n)-1}}{1 - p^{-s-n}}.$$

It follows that the distribution $f = \pi(r)F$ from $\mathcal{D}'(K)$ is defined by the analytic continuation procedure for any quasicharacter π and any distribution $F \in \mathcal{D}'(Z_n)$, with a single exception, the case where $\pi(r) = |r|_p^{-n}$ (this quasicharacter will be called exceptional) and $\langle F, 1 \rangle \neq 0$. The residue of $\langle f, \varphi \rangle$ at $s = -n + \frac{2\pi ki}{\log p}$ equals

$$p^{1-n} \varphi(0) \langle F, 1 \rangle \text{Res}_{s=-n} \frac{p^{\nu(s+n)-1}}{1 - p^{-s-n}} = \frac{1}{p^n \log p} \varphi(0) \langle F, 1 \rangle,$$

so that

$$\text{Res}_{s=0} f = \frac{\langle F, 1 \rangle}{p^n \log p} \delta.$$

For a non-exceptional quasicharacter π , the distribution $f = \pi(r)F$ obviously satisfies (15). Below we show that the above construction covers the whole class of homogeneous distributions. However we need some auxiliary results.

Denote $\mathbb{Q}_p^{(+)} = \mathbb{Q}_p^{(1)} \cup \{0\}$. With the metric, topology, and (additive) Haar measure induced on $\mathbb{Q}_p^{(+)}$ from \mathbb{Q}_p , we denote by $\mathcal{D}(\mathbb{Q}_p^{(+)})$ the space of all locally constant functions on $\mathbb{Q}_p^{(+)}$ with compact supports, and by $\mathcal{D}'(\mathbb{Q}_p^{(+)})$ the dual space containing in a standard manner all the “ordinary” functions. The definition (15) (with $n = 1$) of a homogeneous distribution makes sense for distributions on $\mathbb{Q}_p^{(+)}$.

Lemma 1. *A homogeneous distribution on $\mathbb{Q}_p^{(+)}$ of degree π , where $\pi(r) \neq |r|_p^{-1}$, has the form $C\pi$, $C = \text{const}$.*

The *proof* is identical to the one known for distributions on \mathbb{Q}_p (see [18]).

Consider an arbitrary test function (a locally constant function with a compact support) $\varphi \in \mathcal{D}(K)$.

Lemma 2. *The function φ admits a decomposition into a finite sum*

$$\varphi(\omega\xi r) = \varphi(0)\Delta_l(r) + \sum_{m=1}^M \varphi(\omega\xi r_m)\Delta_l(r - r_m), \quad \omega \in \mu_{q-1}, \xi \in \Sigma_n, r \in \mathbb{Q}_p^{(+)}, \quad (19)$$

where r_m are some points of $\mathbb{Q}_p^{(1)}$ depending only on the function φ , $\Delta_l(z)$ is the indicator function of some ball $\{z \in \mathbb{Q}_p : |z|_p \leq p^l\}$ ($l \in \mathbb{Z}$).

Proof. Suppose first that $\varphi(0) = 0$, that is $\text{supp } \varphi \subset C$ where

$$C = \{x \in K : q^\nu \leq \|x\| \leq q^N\}, \quad \nu \leq N \ (\nu, N \in \mathbb{Z}),$$

and $\varphi(x+y) = \varphi(x)$ for any $x \in K$, if $\|y\| \leq q^l$, and we may assume that $l < \nu$.

In spherical coordinates, we have $C = \mu_{q-1} \times \Sigma_n \times \tilde{C}$ where

$$\tilde{C} = \{r \in \mathbb{Q}_p^{(1)} : p^\nu \leq |r|_p \leq p^N\},$$

and the above local constancy of φ is equivalent to the local constancy of the function $r \mapsto \varphi(\omega\xi r)$:

$$\varphi(\omega\xi(r+r')) = \varphi(\omega\xi r), \quad \text{if } |r'|_p \leq p^l.$$

Let us take a finite covering of \tilde{C} by non-intersecting balls of radius p^l with the centers r_m , $m = 1, \dots, M$. We get the representation

$$\varphi(\omega\xi r) = \sum_{m=1}^M \varphi(\omega\xi r_m)\Delta_l(r - r_m). \quad (20)$$

If $\varphi(0) \neq 0$, consider the function $\varphi_1(\omega\xi r) = \varphi(\omega\xi r) - \varphi(0)\Delta_l(r)$. Clearly, $\varphi_1 \in \mathcal{D}(K)$,

$$\varphi_1(x) = \begin{cases} 0, & \text{if } \|x\| \leq q^l; \\ \varphi(x), & \text{if } \|x\| > q^l. \end{cases}$$

Applying (20) to the function φ_1 and noticing that $\Delta_l(r) \sum_{m=1}^M \Delta_l(r - r_m) \equiv 0$, we obtain the equality (19). ■

Now we can give a description of homogeneous distributions in the sense of (15).

Theorem 2. *Suppose that $p \neq 2$, p does not divide n , and π is a non-exceptional quasicharacter. Then any homogeneous distribution f of degree π has the form $f = \pi(r)F$, $F \in \mathcal{D}'(Z_n)$.*

Proof. Let $\varphi \in \mathcal{D}(\mathbb{Q}_p^{(+)})$, $\varphi(0) = 0$, $\psi \in \mathcal{D}(Z_n)$. Then $(\varphi \otimes \psi)(\omega\xi r) = \psi(\omega\xi)\varphi(r)$ is a test function from $\mathcal{D}(K)$. The linear mapping

$$\varphi \mapsto \langle f, \varphi \otimes \psi \rangle$$

is a homogeneous distribution from $\mathcal{D}'(\mathbb{Q}_p^{(+)})$ of degree π_1 , $\pi_1(\lambda) = \pi(\lambda)|\lambda|_p^{n-1}$. By Lemma 1, for each φ ,

$$\langle f, \varphi \otimes \psi \rangle = C_\psi \langle \pi_1, \varphi \rangle$$

where C_ψ is some constant.

Let $\varphi \in \mathcal{D}(\mathbb{Q}_p^{(+)})$ be such a test function that $\varphi(0) = 0$ and $\langle \pi_1, \varphi \rangle = \frac{p^{n-1}}{p^n - 1}$. Define a distribution $F \in \mathcal{D}'(Z_n)$ setting

$$\langle F, \psi \rangle = \langle f, \varphi \otimes \psi \rangle, \quad \psi \in \mathcal{D}(Z_n).$$

For any $\Phi \in \mathcal{D}(\mathbb{Q}_p^{(+)})$, such that $\Phi(0) = 0$, we have

$$\begin{aligned} \langle \pi F - f, \Phi \otimes \psi \rangle &= p^{1-n}(p^n - 1) \int_{\mathbb{Q}_p^{(+)}} \pi_1(r) \langle F, \psi \rangle \Phi(r) dr - \langle f, \Phi \otimes \psi \rangle \\ &= p^{1-n}(p^n - 1) \langle F, \psi \rangle \langle \pi_1, \Phi \rangle = C_\psi p^{1-n}(p^n - 1) \langle \pi_1, \varphi \rangle \langle \pi_1, \Phi \rangle - C_\psi \langle \pi_1, \Phi \rangle = 0. \end{aligned}$$

Taking into account Lemma 2, we see that the distribution $\pi F - f \in \mathcal{D}'(K)$ is concentrated at the origin. Therefore [6, 18] there exists such a constant $c \in \mathbb{C}$ that $\pi F - f = c\delta$. Since $\pi F - f$ is a homogeneous distribution of degree π , and π is non-exceptional, we find that $c = 0$, as desired. ■

5 Skew Product Decompositions of p -Adic Lévy Processes

As before, we assume that $p \neq 2$ and p does not divide n , and identify \mathbb{Q}_p^n with the unramified extension K .

Let X_t be a rotation-invariant temporally and spatially homogeneous process on K with independent increments. Its transition probability has the form

$$P_t(x, B) = \Phi_t(B - x), \quad x \in K, \quad B \in \mathcal{B}(K),$$

where Φ_t is a semigroup of measures, $\mathcal{B}(K)$ denotes the Borel σ -algebra of K . Below we assume that Φ_t is absolutely continuous with respect to the Haar measure (see [10, 20] for a complete description of such processes). In this case

$$P_t(x, B) = \int_B \Gamma(t, \|x - y\|) dy, \quad x \in K, \quad B \in \mathcal{B}(K).$$

See [10, 20] for an explicit expression of the density Γ .

Consider the process $R_t = r(X_t)$.

Theorem 3. *The process R_t is a Markov process in $\mathbb{Q}_p^{(+)}$ with the transition probability*

$$\tilde{P}_t(r(x), \tilde{B}) = P_t(x, r^{-1}(\tilde{B})), \quad x \in K, \quad \tilde{B} \in \mathcal{B}(\mathbb{Q}_p^{(+)}).$$

Proof. By the general theorem on transformations of the phase space of a Markov process ([3], Theorem 10.13), it is sufficient to check that if $x' \in K$, $r(x') = r(x)$, then

$$P_t(x, r^{-1}(\tilde{B})) = P_t(x', r^{-1}(\tilde{B})). \quad (21)$$

For the above x and x' , we have $\|x\| = |N(x)|_p$, so that $\|x\| = \|\omega^{-1}(x)x\| = |N(\omega^{-1}x)|_p = |r(x)|_p^n$, and $\|x'\| = \|x\|$. On the other hand,

$$P_t(x, r^{-1}(\tilde{B})) = \int_{r(y) \in \tilde{B}} \Gamma(t, \|x\| \cdot \|1 - x^{-1}y\|) dy.$$

The change of variables $x^{-1}y = z$ yields

$$\begin{aligned} P_t(x, r^{-1}(\tilde{B})) &= \|x\| \int_{r(x)r(z) \in \tilde{B}} \Gamma(t, \|x\| \cdot \|1 - z\|) dz \\ &= \|x'\| \int_{r(x')r(z) \in \tilde{B}} \Gamma(t, \|x'\| \cdot \|1 - z\|) dz = P_t(x', r^{-1}(\tilde{B})), \end{aligned}$$

and we have proved the equality (21). \blacksquare

Let us turn to the ‘‘angular’’ process $z_t = \eta(X_t)$ on $Z_n = \mu_{q-1} \times \Sigma_n$ generated by the process X_t . Let $D_T(K)$ be the space of càdlàg functions $[0, T] \rightarrow K$ endowed with the Skorokhod topology (see [7] regarding this notion for functions with values from a metric space). The mappings r and η induce the corresponding mappings $D_T(K) \rightarrow D_T(\mathbb{Q}_p^{(1)})$ and $D_T(K) \rightarrow D_T(Z_n)$. Let \mathbf{P}_x be the probability measure on $D_T(K)$ corresponding to the process X_t with the starting point x . Denote by ζ the first hitting time for the point 0, that is

$$\zeta = \inf\{t > 0 : X_t = 0 \text{ or } X_{t-} = 0\}$$

where we assume that $\inf \emptyset = \infty$.

Let $\mathcal{F}_{0,T}^{\mathbb{Q}_p^{(+)}}$ be the σ -algebra on $D_T(\mathbb{Q}_p^{(+)})$ generated by the process R_t . It induces the σ -algebra $r^{-1}\left(\mathcal{F}_{0,T}^{\mathbb{Q}_p^{(+)}}\right)$ on $D_T(K)$. By the existence of regular conditional distributions (see Theorem 5.3 in [8]), there exists a probability kernel $W_z^{R(\cdot)}$ from $Z_n \times D_T(\mathbb{Q}_p^{(1)})$ to $D_T(Z_n)$, such that for any $x \in K$, $x \neq 0$, and any measurable $F \subset D_T(Z_n)$,

$$W_{\eta(x)}^{r(X(\cdot))} = \mathbf{P}_x \left[X(\cdot) \in \eta^{-1}(F) \middle| r^{-1}\left(\mathcal{F}_{0,T}^{\mathbb{Q}_p^{(+)}}\right) \right],$$

for \mathbf{P}_x -almost all $X(\cdot)$ in $[\zeta > T] \subset D_T(K)$.

The probability measure $W_z^{R(\cdot)}$ is considered as the conditional distribution of the process z_t , given $z_0 = z$ and a radial path $R(\cdot)$ in $D_T(\mathbb{Q}_p^{(1)})$.

It can be proved in exactly the same way as in [12] that for $x \neq 0$, $r\mathbf{P}_x$ -almost all $R(\cdot)$ in $[\zeta > T] \subset D_T(\mathbb{Q}_p^{(1)})$, the process z_t is a non-homogeneous Lévy process, that is there exists a two-parameter semigroup of random measures $\nu_{s,t}$ on Z_n , such that for any natural number m , any continuous function f on Z_n^m , any points $t_1 < t_2 < \dots < t_m$ from $[0, T]$, and for $x = \rho z$, $\rho \in \mathbb{Q}_p^{(1)}$, $z \in Z_n$,

$$\begin{aligned} \mathbf{E}_{\rho z} \left[f(z_{t_1}, \dots, z_{t_m}) \middle| \mathcal{F}_{0,T}^{\mathbb{Q}_p^{(+)}} \right] \\ = \int_{Z_n^m} f(z z_1, z z_1 z_2, \dots, z z_1 \cdots z_m) \nu_{0,t_1}(dz_1) \nu_{t_2,t_1}(dz_2) \cdots \nu_{t_m,t_{m-1}}(dz_m). \end{aligned}$$

As in the classical situation, the pair (R_t, z_t) forms a “skew product” representation of the process X_t .

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