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Algebraic degeneracy of non-Archimedean Analytic Maps Omitting Divisors with Sufficiently Many Components

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Non-Archimedean Analytic Functions

- Throughout this talk, **F** will denote an algebraically closed field complete with respect to a non-Archimedean absolute value $|\ |$.
- An **analytic function** on **F** (or an entire function) is a formal power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

  in one variable z with coefficens a_n in **F** and infinite radius of convergence.
- For each $r > 0$, one can define a non-Archimedean absolute value $|\ |_r$ on the ring of entire functions by

$$|f|_r = \max |a_n| r^n.$$

  That this is multiplicative and therefore defines an absolute value is essentially Gauss's Lemma.

Maximum Modulus Principle

Note that for each z in $\mathbf{F}$ with $|z| = r$,

$$|f(z)| \leq |f|_r.$$

The non-Archimedean **maximum modulus principle** says in fact that $|f(z)| = |f|_r$ for most z with $|z| = r$.

Location of Zeros of Entire Functions

Recall $|f|_r = \max|a_n|r^n$. If there exists a single non-negative integer k such that

$$|a_k|r^k > |a_n|r^n \qquad \text{for all } n \neq k,$$

then by the non-Archimedean triangle inequality,

$$|f(z)| = |f|_r = |a_k|r^k \neq 0 \qquad \text{for all } z \text{ with } |z| = r.$$

Thus, if $f(z_0) = 0$, then $r = |z_0|$ is such that $\max|a_n|r^n$ is taken on at more than one n. Such values of r are called "critical." Conversely, the theory of Newton or valuation polygons says:

Proposition (Location of Zeros)

For each $r > 0$, let

$$k(f,r) = \min\{k : |a_k|r^k = |f|_r\} \quad \textit{and} \quad K(f,r) = \max\{k : |a_k|r^k = |f|_r\}.$$

Then, f has $K(f,r) - k(f,r)$ zeros in $\mathbf{F}$ with $|z| = r$, counting multiplicity.

Analog of Complex Picard Theorem

Proposition (Location of Zeros)

For each $r > 0$, let

$$k(f, r) = \min\{k : |a_k|r^k = |f|_r\} \quad \textit{and} \quad K(f, r) = \max\{k : |a_k|r^k = |f|_r\}.$$

Then, f has $K(f, r) - k(f, r)$ zeros in $\mathbf{F}$ with $|z| = r$, counting multiplicity.

If f is non-constant, then for r large enough, $K(f, r) > 0$, and hence

Corollary (Picard analog)

A non-constant non-Archimedean entire function always has a zero.

p-adic or Non-Archimedean Value Distribution Theory

Mathematicians have investigated *p*-adic or non-Archimedean analogs of complex value distribution theory beginning at least as far back as the 1971 paper of Adams and Straus:

> W. Adams and E. Straus, Non-Archimedian analytic functions taking the same values at the same points, *Illinois J. Math.* **15** (1971), 418–424,

who proved an analog of Nevanlinna's theorem that a meromorphic function is uniquely determined by the inverse images of five distinct points.

Analog's of Nevanlinna's theory were developed by:

- Hà Huy Khoái
- Capi Corrales-Rodrigàñez
- My Vinh Quang,
- and Abdelbaki Boutabaa.

Why study *p*-adic or non-Archimedean Value Distribution Theory?

- There are many connections between, on the one hand, Nevanlinna theory and hyperbolicity, and, on the other hand, Diophantine approximation theory and scarcity of rational points.

Conjecture (Lang)

Let k be a field finitely generated over the rational numbers $\mathbf{Q}$ and let X be a non-singular projective variety defined over k. Let the "analytic special set" Z_{anal} be the Zariski closure in X of the images of all the non-constant holomorphic curves from $\mathbf{C}$ to X. Let the "arithmetic special set" Z_{arith} be the smallest closed subvariety in X such that for every field F finitely generated over k all but finitely many of the F-rational points of X lie in Z_{arith}. Then, $Z_{\mathrm{anal}} = Z_{\mathrm{arith}}$.

I first became interested in the *p*-adic analogs of Nevanlinna theory because the theory was more algebraic than the complex analytic counterpart but retained some tools (notably derivatives) available in analysis but not number theory. However, to date, the study of *p*-adic analogs has not resulted in any insight connecting Nevanlinna theory to Diophantine approximation.

Why study *p*-adic or non-Archimedean Value Distribution Theory?

- There are differences between the non-Archimedean (*e.g.* *p*-adic) theory and the complex theory, such as the difference in Picard's theorem already pointed out. This results in some questions about the non-Archimedean case that could be considered interesting in their own right.
- Non-Archimedean analogs of complex analysis more broadly have been used to study Non-Archimedean dynamics, which has been a useful tool in arithmetic dynamics. For instance, in

  R. L. Benedetto, Preperiodic points of polynomials over global fields, *J. Reine Angew. Math.* **608** (2007), 123–153.

  Benedetto bounds the number of preperiodic points for polynomial automorphisms of $\mathbf{P}^1$ over global fields in terms of the number of places of bad reduction. One of the main techniques is an analysis of non-Archimedean Julia sets.

Algebraic Degeneracy

Theorem (Borel; Bloch; Cartan)

A complex holomorphic curve in $\mathbf{P}^n(\mathbf{C})$ omitting $n+2$ hyperplanes in general position is linearly degenerate.

Theorem

A non-Archimedean analytic curve in $\mathbf{P}^n(\mathbf{F})$ omitting 2 hyperplanes in general position is linearly degenerate.

Proof.

By the general position assumption on the hyperplanes, we can choose projective coordinates $X_0, \dots, X_n$ on $\mathbf{P}^n$ such that the hyperplanes are given by $X_0 = 0$ and $X_1 = 0$. Let the map f be given by homogenous coordinate functions $(f_0, \dots, f_n)$. By assumption, f_0 and f_1 are zero free entire functions, hence constant by the non-Archimedean analog of Picard's theorem. □

Algebraic Degeneracy

Theorem (Borel; Bloch; Cartan)

A complex holomorphic curve in $\mathbf{P}^n(\mathbf{C})$ omitting $n+2$ hyperplanes in general position is linearly degenerate.

The canonical divisor K on $\mathbf{P}^n$ has degree $-(n+1)$. Thus one could conjecture that the above theorem of Bloch and Cartan could be generalized to read that a holomorphic curve in a non-singular projective variety X with canonical divisor K and omitting a sufficiently general divisor D such that $K+D$ is positive must be algebraically degenerate.

Theorem

A non-Archimedean analytic curve in $\mathbf{P}^n(\mathbf{F})$ omitting 2 hyperplanes in general position is linearly degenerate.

- Here 2 doesn't depend on n.
- What should the general conjecture be? Something all projective spaces have in common.

How degenerate?

Theorem (Dufresnoy)

A complex holomorphic curve omitting $n+k$ hyperplanes in general position in $\mathbf{P}^n(\mathbf{C})$ must be contained in a linear subspace of dimension at most n/k.

Theorem (non-Archimedean analog)

A non-Archimedean analytic curve ommitting $k+1$ hyperplanes in general position in $\mathbf{P}^n(\mathbf{F})$ must be contained in a linear subspace of dimension at most $n-k$.

Corollary

A non-Archimedean analytic curve omitting $n+1$ hyperplanes in general position in $\mathbf{P}^n(\mathbf{F})$ must be constant.

Theorem (Ta Thi Hoai An)

If f is an analytic map to $X \subset \mathbf{P}^N$ omitting $\dim X + 1$ hypersurfaces in $\mathbf{P}^N$ in general position with X then f is constant.

Theorem (Noguchi/Winkelmann)

Let M be a compact Kähler manifold of dimension m. Let $\{D_i\}_{i=1}^{\ell}$ be ℓ irreducible hypersurfaces in general position. Let r be the rank of the group generated by $\{c_1(D_i)\}_{i=1}^{\ell}$. Let W be a closed subvariety of M of dimension n and irregularity q. Suppose there exists an algebraically non-degenerate holomorphic map from the complex plane $\mathbf{C}$ to W that omits each of the D_i that does not contain all of W. Then

(i) $\#\{W \cap D_i \neq W\} + q \leq n + r$;

(ii) *If $\ell > m$ and in addition each of the D_i are ample, then*

$$n \leq \frac{m}{\ell - m} \max\{0, r - q\}.$$

Noguchi and Winkelmann's theorem generalizes Dufresnoy's result and relates the dimension of the image of a holomorphic curve in a projective variety X in terms of two fundamental invariants

- The irreqularity $q =$ the dimension of the space of holomorphic one-forms.
- The rank of the Néron-Severi group, which is the group of divisor classes module algebraic equivalence.

Non-Archimedean analytic maps into semi-Abelian varieties

An immediate corollary of the non-Archimedean Picard theorem is that there are no non-constant analytic maps from $\mathbf{A}^1$ to the multiplicative group $\mathbf{G}_m$, and hence no non-constant analytic maps to multiplicaive tori.

Theorem (Cherry)

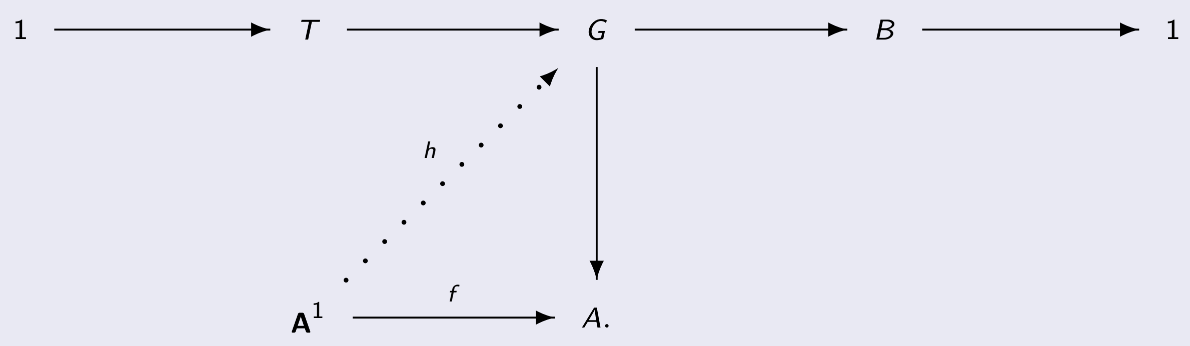
Every non-Archimedean analytic map to an Abelian variety is constant.

Theorem (Cherry)

Every non-Archimedean analytic map to an Abelian variety is constant.

Proof.

Let f be a non-Archimedean analytic map from $\mathbf{A}^1$ to an Abelian variety A. By the semi-Abelian reduction theorem, there is a semi-Abelian variety G that maps onto A and fits into an exact sequence $1 \to T \to G \to B \to 1$, where B is an Abelian variety with good reduction and T is a multiplicative torus. Lifting f to an analytic map to G, for instance by Berkovich theory

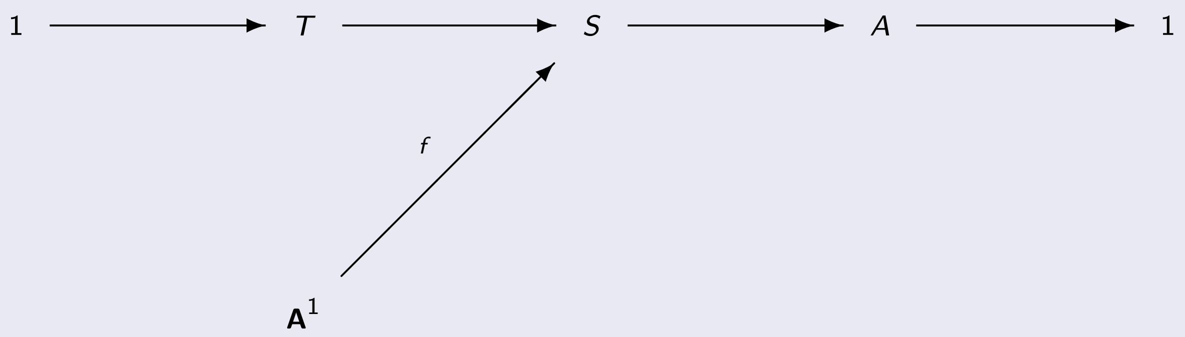


we get a map to B. The map to B must be constant because, again by Berkovich theory, it lies above a single closed point in the reduction, which is isomorphic to an open ball. Hence the image of h lies in a translate of T in G and is thus constant. □

Corollary (An/Cherry/Wang)

Every non-Archimedean analytic map to a semi-Abelian variety is constant.

Proof.



□

Algebraic degeneracy

Theorem (An/Cherry/Wang)

Let Y be a possibly singular projective variety and let $\iota : Y \to X$ be a morphism to a smooth projective variety X. Let $\{D_i\}_{i=1}^{\ell}$ be ℓ irreducible effective divisors on X such that $\{\iota^* D_i\}_{i=1}^{\ell}$ form ℓ distinct effective Cartier divisors on Y. Assume the number of irreducible components ℓ is larger than the rank of the subgroup generated by the $c_1(D_i)$ in $\mathrm{NS}(X)$. Then, any analytic map from $\mathbf{A}^1$ to Y is either algebraically degenerate or intersects the support of at least one of the $\iota^* D_i$.

Example

Let f be an algebraically non-degenerate analytic map from $\mathbf{A}^1$ to $\mathbf{A}^2$. Let X be obtained by blowing up $r-1$ general points in $\mathbf{P}^2$, none of which are contained in a fixed hyperplane H and which are also not contained in the image of f. Let $\{D_i\}_1^r$ consist of the $r-1$ exceptional divisors and the strict transform of H. Then, lifting f to X results in an algebraically non-degenerate map omitting r effective divisors.

Theorem (An/Cherry/Wang)

Let $\iota : Y \to X$ and $f : \mathbf{A}^1 \to Y$ be algebraically non-degenerate. Suppose $\{\iota^* D_i\}_{i=1}^{\ell}$ form $\ell > \mathrm{rk}\langle c_1(D_i)\rangle$ distinct effective Cartier divisors on Y. Then, f intersects some D_i.

Proof.

Let $f : \mathbf{A}^1 \to Y$ and lift to the normalization $\widetilde{Y}$.
We can find integers a_i not all zero so that $\sum a_i c_1(D_i) = 0$. Thus, $\sum a_i \tilde{\iota}^* D_i$ is a non-zero divisor algebraically equivalent to zero on $\widetilde{Y}$.
If there is a non-constant rational map from $\widetilde{Y}$ to an Abelian variety, then f is already algebraically degenerate. Hence, assume $\mathrm{Pic}^0(\widetilde{Y})$ is trivial.
Find a non-constant rational function h on $\widetilde{Y}$ such that

$$\mathrm{div}(h) = \sum a_i \tilde{\iota}^* D_i.$$

If f omits the supports of all the $\iota^* D_i$, then $h \circ f$ is an entire function without zeros, and hence constant. □

Adding the assumption that the D_i are ample to the previous result and using the same argument in Noguchi and Winkelmann's paper then gives:

Corollary

Let Y be a closed positive dimensional subvariety of a non-singular projective variety X. Let $\{D_i\}_{i=1}^{\ell}$ be ℓ irreducible, effective, ample divisors in general position on X. Let r be the rank of the subgroup of $\mathrm{NS}(X)$ generated by $\{c_1(D_i)\}_{i=1}^{\ell}$. If there exists an algebraically non-degenerate analytic map from $\mathbf{A}^1$ to Y omitting each of the D_i that does not contain all of Y, then

$$\ell \le \max\left\{r + \mathrm{codim}\ Y, r \cdot \frac{\dim X}{\dim Y}\right\} \quad \text{and} \quad \dim Y \le \max\left\{r + \dim X - \ell, \frac{r}{\ell}\dim X\right\}.$$

Remarks

- *When $X = P^n$, the above inequality was proven by An, Wang, and Wong.*
- *With the assumption that the components D_i are ample, I suspect a bound can be independent of r.*

Work of Lin and Wang

Theorem (Lin/Wang)

Let $f : \mathbf{A}^1 \to X \setminus (D_1 \cup \cdots \cup D_\ell)$. be a non-Archimedean analytic curve.

1. *If each D_i is pseudo-ample (a.k.a big) and $\cap_{i=1}^{\ell} D_i = \emptyset$ then f is algebraically degenerate.*
2. *If each D_i is ample and $\ell > \dim X$, then f is constant.*

Conjecture (Lin/Wang)

A non-Archimedean analytic curve omitting ℓ ample divisors in general position in a non-singular projective variety X is contained in a proper algebraic subvariety of codimension at least $\ell - 1$.

Defect Relations

In complex analysis, the maximal deficiency sum that is a consequence of the Second Main Theorem is the same as the maximum number of "things" that can be omitted. This is not true in the non-Archimedean case.

Example

Choose an algebraically non-degenerate map $f = (1, f_0, \ldots, f_n)$ *to* $\mathbf{P}^n$ *such that* f_n *grows much faster than any of the other coordinate functions. Then, the first* n *coordinate hyperplanes have deficiency* 1 *and so the deficiency sum for* f *is* n*.*

Big Picard Type Extension Theorems

Theorem

A non-Archimedean analytic map from the punctured disc to an elliptic curve with good reduction ($|j| \leq 1$) extends to an analytic map from the disc.

Proof.

By Berkovich theory, the image of the map in the elliptic curve E lies above a single closed point in the reduction, which is isomorphic to an open disc, and hence the result follows by the non-Archimedean Riemann extension theorem. □

Example

Restricting the non-Archimedean covering map from $\mathbf{G}_m$ to an elliptic curve with bad reduction ($|j| > 1$) to the disc does not extend to an analytic map from the disc.

Conclusion! Extension theorems in the non-Archimedean case do not depend only on coarse geometric invariants of the target.

K3 surfaces

A variety is said to be **pseudo-canonical** (a.k.a. of general type) if its canonical divisor is pseudo-ample.

Conjectures

Green-Griffiths *A holomorphic curve in a pseudo-canonical variety is algebraically degenerate.*

Lang *If X is a non-singular pseudo-canonical projective variety, then there is a proper algebraic subvariety Z of X which contains the images of all the non-constant holomorphic curves in X (and also all but finitely many rational points over fields finitely generated over $\mathbf{Q}$.)*

Cherry *The image of a non-Archimedean analytic curve in a K3 surface (simply connected, trivial canonical divisor) must be contained in a rational curve.*

- The non-Archimedean conjecture has some features in common with the Green-Griffiths Conjecture in the complex case. However, K3 surfaces have more structure than arbitrary pseudo-canonical varieties.
- Some (presumably all) K3 surfaces contain infinitely many rational curves. Thus, there is not a fixed proper subvariety of a K3 surface that contain all the images of non-constant non-Archimedean analytic maps. Thus, a solution to the non-Archimedean problem from K3 surfaces might give some insight into whether the strong Lang conjecture is true.

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