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Introduction to Locally Analytic Representation Theory

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Summary of and references to the lectures at the

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Contents

The course's aim is to give an introduction to the theory of locally analytic representations of p-adic locally analytic groups. Systematic foundational work in this area was done by P. Schneider and J. Teitelbaum, whose articles will be our main references. Earlier papers which are related to our topic are those about continuous representations of particular p-adic groups on p-adic Banach spaces, cf. for instance [Di79] and [Tr81]. Inspiring for the work of Schneider and Teitelbaum were the papers of Y. Morita on analytic representations for GL_2, cf. [Mo83], [Mo85]. We have chosen to illustrate the general concepts by discussing in detail parabolically induced locally analytic representations (for general p-adic reductive groups). The article [S06] summarizes a substantial part of our mini course, and is recommended as a secondary reference.

In the following, we fix finite extensions $K \supset L \supset \mathbb{Q}_p$. (For most purposes it suffices that K be a discretely valued complete p-adic field, or even only spherically complete.) By G and H we denote locally L-analytic groups.

1. Locally analytic representations and distribution algebras

In the first lecture we will define the notion of a locally analytic representation of a p-adic analytic group G, and the distribution algebra $D(G,K)$. The aim is to "algebraize" the category of locally analytic representations (or certain subcategories thereof), which means to relates these to equivalent categories of modules over $D(G, K)$. We will treat:

- basic concepts of Non-Archimedean Functional Analysis, e.g., lattices, gauges and locally convex vector spaces, dual spaces and compact inductive limits of Banach spaces; cf. [S02]

- locally analytic manifolds and groups over completely valued fields, and locally analytic functions on these, cf. [Bou]

- locally analytic representations, cf. [ST02a]

- example: locally analytic characters, and principal series representations, cf. [OS07]

- p-adic distribution algebras, cf. [ST02a]

- example: the character variety of a split torus, cf. [ST01b]

2. Fréchet-Stein structures and admissibility

In the second lecture we will study the distribution algebra $D(H, K)$ of a compact p-adic analytic group H in detail. The key notion is that of a Fréchet-Stein algebra. Such an algebra is a projective limit of noetherian Banach algebras with flat transition maps. We will show that $D(H, K)$ is a Fréchet-Stein algebra, using as input some techniques and results from the theory of analytic pro-p groups. Given a Fréchet-Stein K-algebra A, there is a natural class of modules for A, the so-called co-admissible modules. The admissible locally analytic representation of G are those whose topological dual space is a co-admissible module for the Fréchet-Stein algebra $D(H, K)$ of any (equivalently, one) compact open subgroup $H \subset G$. We will discuss:

- basic concepts from the theory of analytic pro-p groups: uniform pro-p groups and p-valuations, cf. [DDMS]

- norms on distribution algebras associated to p-valuations, cf. [ST03]

- distribution algebras of compact p-adic analytic groups are Fréchet-Stein algebras, cf. [ST03]

- co-admissible modules for Fréchet-Stein algebras and admissible representations, equivalences of categories, cf. [ST03]

- example: principal series representations (continued)

3. Jordan-Hölder series for parabolically induced representations

In the third lecture we will use techniques and concepts we have encountered so far to prove some irreducibility results for certain parabolically induced representations. We will actually treat more general representations which are in the image of a (bi-)functor $\mathcal{F}^G_P$ which "globalizes" certain $U(\mathfrak{g})$-modules to locally analytic representations. $U(\mathfrak{g})$ denotes the universal enveloping algebra of the Lie algebra $\mathfrak{g}$ of the reductive p-adic group G. This lecture is based on joint work with S. Orlik, cf. [OS09]. The key points we will treat are:

- the category of $U(\mathfrak{g})$-modules $\mathcal{O}$ of Bernstein, Gelfand and Gelfand, and its parabolic variants $\mathcal{O}^P$ and $\mathcal{O}^P_{\rm alg}$ for standard parabolic subgroups $P \subset G$

- the bi-functor

$$\mathcal{F}^G_P(\cdot,\cdot) : \mathcal{O}^P_{\rm alg} \times {\rm Rep}^{\infty,{\rm f.l.}}_K(L_P) \longrightarrow {\rm Rep}^{\rm loc.an.}_K(G)\,.$$

Here, ${\rm Rep}^{\infty,{\rm f.l.}}_K(L_P)$ denotes the category of smooth admissible finite length representations of a Levi subgroup L_P of P.

- properties of the bi-functor $\mathcal{F}^G_P$, and consequences for Jordan-Hölder series of parabolically induced representations

- example: the case of ${\rm GL}_2$

- sketch of the proof of the irreducibility statement; this part uses results on the completions of $U(\mathfrak{g})$ with respect to the norms defined in terms of uniform pro-p subgroups of G, cf. [Ko07], [Sch08a]

- applications to ${\rm GL}_n$-equivariant vector bundles on Drinfeld symmetric spaces as studied in the papers [ST02b] and [O08]

4. Banach space representations and locally analytic vectors

This last lecture is about continuous representations of G on Banach spaces over K. These are in general not locally analytic representations, but related to those by passage to locally analytic vectors. In the other direction, given a locally analytic representation V of G, and a G-invariant norm on V, one can consider the completion $V^\wedge$ of V with respect to this norm. $V^\wedge$ furnishes then a continuous

Banach space representation of G. The study of G-invariant norms on locally analytic representations and the Banach space representations to which they give rise by completion is a central subject in the p-adic Langlands program. This lecture is based on the papers [ST02c], [Sch08b]. By definition, a K-Banach space representation of G is a linear G-action on a K-Banach space V such that the corresponding map $G \times V \to V$ is continuous. O_K denotes the ring of integers of K. We will discuss

- admissible K-Banach representations: those K-Banach space representations V which have the property that for any compact open subgroup $H \subset G$ and any bounded H-invariant open lattice $M \subset V$, and for any open subgroup $H' \subset H$ the O_K-submodule $(V/M)^{H'}$ of H'-invariant elements in V/M is of cofinite type (i.e. $\mathrm{Hom}_{O_K}((V/M)^{H'}, K/O_K)$ is a finitely generated O_K-module)

- Lazard's theorem: for any compact p-adic Lie group H the completed group ring $O_K[[H]]$ is noetherian

- for a compact p-adic Lie group H we have

$$K[[H]] := O_K[[K]] \otimes_{O_K} K \simeq C(H,K)' = \mathrm{Hom}_K(C(H,K),K)$$

- for a compact p-adic Lie group H there is an anti-equivalence of categories

$$\mathrm{Ban}_H^a(K) \xrightarrow{\simeq} \mathrm{Mod}_{fg}(K[[H]])\,, \quad V \mapsto V'\,,$$

where $\mathrm{Mod}_{fg}(K[[H]])$ is the category of finitely generated $K[[H]]$-modules

- the map $V \mapsto V'$ induces a bijection

$$\begin{array}{c}\text{set of isomorphism classes}\\ \text{of topologically irreducible}\\ \text{admissible } K\text{-Banach space}\\ \text{representations of } G\end{array} \xrightarrow{\sim} \begin{array}{c}\text{set of isomorphism classes}\\ \text{of simple } K[[H]]\text{-modules}\end{array}$$

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