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Tropical Analytic Geometry and the Bogomolov Conjecture

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Diophantine geometry

Example

The diophantine equation $x^4 - y^4 = 5$ has only finitely many rational solutions, e.g. $(\frac{3}{2}, \frac{1}{2})$.

For any number field K, we have the **Mordell-conjecture**:

Theorem (Faltings 1983)

An algebraic curve of genus $g > 1$ has only finitely many points with coordinates in K.

- A central tool is the **height** of a point.
- The height measures the arithmetic complexity of the point.
- e.g. $H(\frac{3}{2}, \frac{1}{2}) = 3$, as we have the projective solution $(2:3:1)$.

Heights

Definition

For $P = [x_0 : \cdots : x_n] \in \mathbb{P}^n$ with coordinates $x_0, \ldots, x_n$ in the number field K, we define the **height** by

$$H(P) = \prod_{v \in M_K} \max_j |x_j|_v .$$

- This is independent of the choice of $(x_0, \ldots, x_n) \in K^{n+1}$.
- Using suitable normalizations of the absolute values $v \in M_K$, this is also independent of K and hence well-defined for coordinates in $\overline{K}$.
- We will use $h(P) = \log(H(P)) \geq 0$.
- For a projective variety X, we will choose an embedding into $\mathbb{P}^n$ to get a height by restriction.
- This **Weil height** depends on the choice of the embedding up to bounded functions.

Abelian varieties

- An **abelian variety** A is a projective group variety.
- The elliptic curves are the abelian varieties of dimension 1

Definition

We call

$$\hat{h}(P) = \lim_{m \to \infty} m^{-2} h(mP)$$

the **Néron–Tate-height** of $P \in A$.

- $\hat{h}$ is a positive semi-definite quadratic form.
- The kernel of the associated bilinear form is the torsion group.
- We have a canonical semidistance $d(P, Q) := \hat{h}(P - Q)$ on A.

Bogomolov conjecture

Definition

A **torsion subvariety** of A has the form $B + t$ for an abelian subvariety B and a torsion point t of A.

For a closed subvariety X of A, we have the **Bogomolov conjecture**:

Theorem (Ullmo 1998 for curves, Zhang 1998 in general)

- *There are only finitely many maximal torsion subvarieties in X.*
- *$\hat{h}$ has a positive lower bound on their complement in X.*

- This is a statement for points with coordinates in $\overline{K}$.
- The torsion points are dense in every torsion subvariety.

Corollary (Manin-Mumford conjecture, proved by Raynaud 1983)

If the torsion points are dense in X, then X is a torsion subvariety.

Setup for equidistribution

The proof of the Bogomolov conjecture relies on the following equidistribution result:

- Let X be a closed subvariety of the abelian variety A.
- The absolute Galois group $G := \mathrm{Gal}(\overline{K}/K)$ acts on X.
- Suppose that (P_n) is a small generic sequence in X:
  - *generic* means $\{n \in \mathbb{N} \mid P_n \in Y\}$ is finite for every closed $Y \subsetneq X$.
  - *small* means that $\lim_{n\to\infty} \hat{h}(P_n) = 0$.
- We consider the discrete probability measure μ_n on $X(\mathbb{C})$ which has support GP_n and is equidistributed on this Galois orbit.

Equidistribution theorem

Theorem (Szpiro, Ullmo and Zhang 1997)

The discrete probability measures μ_n converge weakly to a positive volume form μ on $X(\mathbb{C})$.

- The proof uses differential geometric methods from Arakelov geometry.
- If L is any ample symmetric line bundle on A, then

$$\mu = \frac{1}{\deg_L(X)} c_1(L|_X, \|\ \|_{\rm can})^{\wedge \dim(X)}$$

- Chambert-Loir proved a non-archimedean version in case of good reduction.
- Yuan gave a huge generalization to arbitrary projective varieties. An analogue over function fields was proved by Faber and in [Gu3].

Function fields

- Let $K = k(B)$ be the function field of a curve B over the algebraically closed field k.
- There is a strong analogy between function fields and number fields.
- The set of places M_K is B and satisfies a product formula, hence we have a theory of heights.
- Many proofs are easier for function fields:
  - Fermat's conjecture:
    Tschebyscheff, Liouville, Korkine, 19. century
  - Mordell conjecture:
    Manin, Grauert, Samuel, 1963-1966

Totally degenerate abelian varieties

Let v be a discrete valuation on K. Then there is a smallest field extension $\mathbb{C}_K$ of K which is algebraically closed and complete with respect to a valuation $w|v$.

Definition

An abelian variety A is called *totally degenerate* wrt v if A is w-analytically isomorphic to T/M for $T = (\mathbb{C}_K^*)^n$ with lattice M.

Example (Tate)

An elliptic curve with multiplicative reduction in v is w-analytically isomorphic to $\mathbb{C}_K^*/M$ for $M = \{q^n \mid n \in \mathbb{Z}\}$ with $q \in \mathbb{C}_K$, $w(q) > 0$.

Bogomolov conjecture for function fields

Theorem (Gu2)

Suppose that X is a closed subvariety of an abelian variety A which is totally degenerate with respect to a place $v \in M_K$:

- *There are only finitely many maximal torsion subvarieties in X.*
- *$\hat{h}$ has a positive lower bound on their complement in X.*

- The Bogomolov conjecture is wrong if X and A are defined over k.
- It is conjectured only if $\mathrm{Tr}_{L/k}(A) = 0$ for all finite L/K.
- Idea of proof: **Tropical analytic geometry replaces the differential geometry** in Zhang's proof.
- Recently, Cinkir proved the Bogomolov conjecture for curves inside their Jacobians using graph theory and Zhang's formula for the arithmetic self-intersection number of the Gross-Schoen cycle.

Tropical algebraic geometry

Let v be a discrete valuation on K with extension w on $\mathbb{C}_K$, $T = (\mathbb{C}_K^*)^n$.

$$\mathrm{val} : T \to \mathbb{R}^n, \quad \mathrm{val}(x_1, \ldots, x_n) = (w(x_1), \ldots, w(x_n))$$

Let X be a closed algebraic subvariety of T, $d := \dim(X)$.

Definition

The closure of $\mathrm{val}(X)$ in $\mathbb{R}^n$ is denoted by $\mathrm{trop}(X)$ and is called the *tropical variety* associated to X.

Theorem (Einsiedler, Kapranov, Lind)

$\mathrm{trop}(X)$ *is a finite connected union of d-dimensional polyhedrons.*

Examples

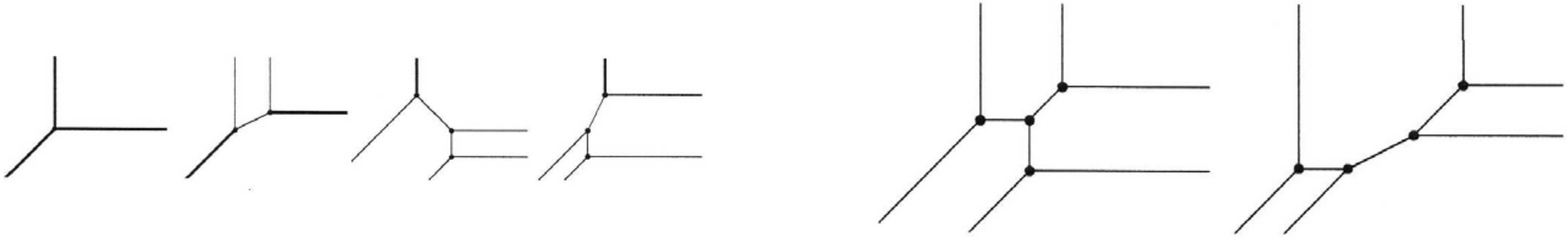

Figure: Plane conics

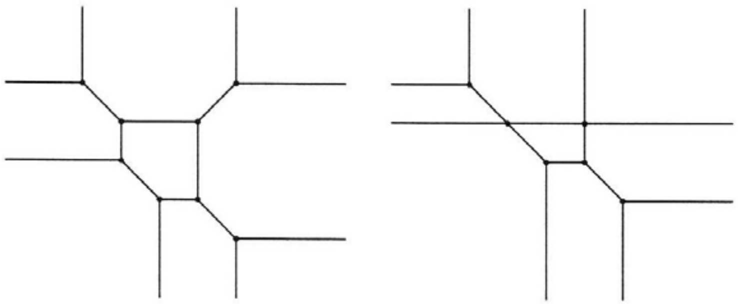

Figure: Plane biquadratic curves

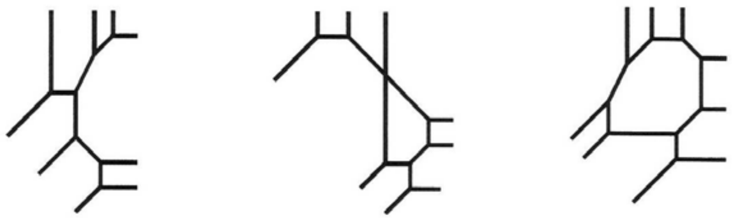

Figure: Plane cubics

Tropical analytic geometry

Let X be a closed analytic subvariety of T of dimension d.

Definition

The closure of $\mathrm{val}(X)$ in $\mathbb{R}^n$ is denoted by $\mathrm{trop}(X)$ and is called the *tropical variety* associated to X.

Theorem (Gu1)

$\mathrm{trop}(X)$ *is a connected locally finite union of d-dimensional polytopes.*

- methods of proof:
  - Analytic spaces of Berkovich
  - Formal geometry over the valuation ring

Periodical tropical geometry

Let A be a totally degenerate abelian variety with respect to v, i.e. $A^{\mathrm{an}} = T/M$ and $\Lambda = \mathrm{val}(M)$ is a lattice in $\mathbb{R}^n$:

$$\begin{array}{ccc} T & \xrightarrow{\mathrm{val}} & \mathbb{R}^n \\ \downarrow & & \downarrow \\ A & \xrightarrow{\mathrm{Val}} & \mathbb{R}^n/\Lambda \end{array}$$

Let X be a d-dimensional closed algebraic subvariety of A.

Definition

The closure of $\mathrm{Val}(X)$ in $\mathbb{R}^n/\Lambda$ is called the *tropical variety* and is denoted by $\mathrm{Trop}(X)$.

Theorem (Gu1)

$\mathrm{Trop}(X)$ *is a finite union of d-dimensional polytopes in* $\mathbb{R}^n/\Lambda$.

Tropical equidistribution theorem [Gu2,§5]

- Let $(P_n)_{n\in\mathbb{N}}$ be a small generic sequence in X as before.
- Let us consider the following discrete probability measure on $\mathrm{Trop}(X)$:

$$\mu_n = \frac{1}{|GP_n|} \sum_{Q \in GP_n} \delta_{\mathrm{Val}(Q)}.$$

Theorem (Gu2)

Then μ_n converges weakly to a strictly positive volume form μ on $\mathrm{Trop}(X)$, i.e. $\mathrm{Trop}(X)$ is a finite union of d-dimensional polytopes Δ such that $\mu|_\Delta$ is a positive multiple of the Lebesgue measure.

Methods of proof

- Analytic spaces of Berkovich
- Formal models over the valuation ring (Bosch, Lütkebohmert, Raynaud)
- Corresponding intersection theory of divisors
- Mumford models of A associated to polytopal decompositions $\mathscr{C}$ of $\mathbb{R}^n/\Lambda$
- If we choose $\mathscr{C}$ generically, then the irreducible components of X are toric varieties.
- Measures $c_1(\overline{L}|_X)^{\wedge \dim(X)}$ of Chambert-Loir on X^{an}
- Their explicit computation in terms of convex geometry
- A Riemann-Roch argument to prove a fundamental inequality of heights.
- A variational principle for metrics on L is used to deduce the equidistribution theorem from the fundamental inequality.