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P-adic Nevanlinna Theory and Applications

HA HUY KHOAI

Hanoi Institute of Mathematics
18 Hoang Quoc Viet Road, 10307 Hanoi
Vietnam

P-ADIC NEVANLINNA THEORY AND APPLICATIONS

HA HUY KHOAI

In this talk we give a brief survey on the p-adic Nevanlinna theory and its applications in p-adic function theory and related topics.

1. Nevanlinna theory

One can start by recalling the fundamental theorem of algebra:

Theorem 1.1. *Let*
$$f(z) = a_o + a_1 z + \cdots + a_n z^n$$
be a polynomial of degree $n \geq 1$, with complex coefficients. Then for every complex value a, the equation $f(z) = a$ has n solutions (counting multiplicities).

One can say that a polynomial takes every value a with the same frequency. Thus, the number of solutions of a polynomial depends on its degree, i.e., on the growth.

It is quite naturally to ask: *"what happens if instead of polynomials one considered holomorphic, or more generally, meromorphic functions"*, or , "*how many times does a meromorphic function $f(z)$ take a value $a \in \mathbb{P}^1$?*".

In this case one can not say about "degree", then one hopes to have a relation between the number of zeros of a function and its growth.

The first results in this direction belong to Hadamard.

Theorem 1.2. *Let $f(z)$ be a holomorphic function in the complex plane $\mathbb{C}$. Then,*
$$\text{the number of zeros of } f \text{ in } \{|z| \leq r\} \leq \log \max_{|z|\leq r} |f(z)| + O(1),$$
where O(1) depends on f, but not on r .

This result is not yet "ideal" because of the following two deficiencies.

1. When f is a meromorphic function, we have the infinity in the RHS of the inequality. In this case, the Hadamard theorem does not give an estimation of the number of zeros of f.

2. There are functions, for example, $f(z) = e^z$, which do not have the zeros, but grow very fast. In this case, Hadamard's inequality becomes trivial, because the LHS is zero.

For eliminating these deficiencies, R. Nevanlinna defines the following functions.

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Counting function . Let $a \in \mathbb{C}$. We set

$n(a,r)$ = the number of zeros of $f(z) - a$ in the disk $\{|z| \leq r\}$ (counting multiplicity),

$$N(a,r) = \int_o^r \frac{n(a,t) - n(a,0)}{t} dt + n(a,0) \log r.$$

Characteristic function . Instead of $\log \max_{|z|\leq r} |f(z)|$ we consider the function

$$T(r) = \frac{1}{2\pi} \int_o^{2\pi} \log^+ |f(re^{i\theta})| d\theta + N(\infty, r).$$

By using these two functions one get the following inequality:

$$N(0,r) \leq T(r) + O(1).$$

This inequality is valid and non-trivial for meromorphic functions. For eliminating the second deficiency one notices that, while the function e^z does not have the zeros, it takes many values "approached to zero". Then one can "measure" this set by using

Mean proximity function.

$$m(a,r) = \frac{1}{2\pi} \int_o^{2\pi} \log^+ |\frac{1}{f(re^{i\theta}) - a}| d\theta,$$

where $\log^+ = \max(0, \log)$. It is obviously that $m(a,r)$ becomes large when $f(z)$ approaches to a.

There are two "Main Theorems" and defect relations which occupy a central place in Nevanlinna theory.

First Main Theorem of Nevanlinna. *There is a function $T(r)$ such that for any $a \in \mathbb{P}^1$ we have*

$$m(a,r) + N(a,r) = T(r) + O(1).$$

As $T(r)$ does not depend on a, one can say that a meromorphic function takes every value $a \in \mathbb{P}^1$ (and "approached to a" values) with the same frequency. However, to have the First Main Theorem, as an analog to the fundamental theorem of algebra, one had have to add an "approximity part" (the function $m(a,r)$). Therefore one would hope that this part would be "very small". Indeed, this follows from the following very deep theorem of Nevanlinna.

Second Main Theorem of Nevanlinna *For arbitrary $q \in \mathbb{N}$ and distinct points $a_i \in \mathbb{P}^1, i = 1, \ldots, q,$ we have*

$$\sum_{i=1}^{q} m(a_i, r) < 2T(r) + O(\log(rT(r))),$$

where the inequality is valid beside a set of finite measure.

Because the right hand side does not depend on q, one can say that the values $m(a_i, r)$ are "very small".

The following "defect relation" gives a "quantitative version" of the above statement. Set

$$\delta(a) = \lim_{r \to \infty} \frac{m(a,r)}{T(r)},$$

Then we have

$$\sum_{a \in \mathbb{P}^1} \delta(a) \leq 2.$$

$\delta(a)$ is called the *defect value* at the point a, and the above inequality is "defect relation". Precisely, $\delta(a) = 0$ for almost all a (except a countable set). Roughly speaking, with two Main Theorems and the defect relation one can work with meromorphic functions as one does with polynomials.

2. Nevanlinna Theory and Number Theory

In the famous paper "De la métaphysique aux mathématiques" ([W]) A. Weil discussed about the role of analogies in mathematics. For illustrating he analysed a "metaphysics" of Diophantine Geometry: the resemblance between *Algebraic Numbers* and *Algebraic Functions*. However, the striking similarity between Weil's theory of heights and Cartan's Second Main Theorem for the case of hyperplanes is pointed out by P. Vojta only after 50 years! P. Vojta observed the resemblance between *Algebraic Numbers* and *Holomorphic Functions*, and gave a "dictionary" for translating the results of Nevanlinna Theory in the one-dimensional case to Diophantine Approximations. Due to this dictionary one can regard Roth's Theorem as an analog of Nevanlinna Second Main Theorem. P. Vojta has also made quantitative conjectures which generalize Roth's theorem to higher dimensions. One can say that P. Vojta proposed a "new metaphysics" of Diophantine Geometry: Arithmetic Nevanlinna Theory in higher dimensions.

Table 6.1: Vojta's Dictionary

Value Distribution	**Diophantine Approximation**
A non-constant function $f:\mathbb{C}\to\mathbb{P}^1$	An infinite sequence $\{x\}$ in a number field F
A radius r	An element x of F
A finite mesure set E of radii	A finite subset of $\{x\}$
An angle θ	An embedding $\sigma: F\to\mathbb{C}$
$\lvert f(re^{i\theta})\rvert$	$\lvert x\rvert_\sigma$
$(\mathrm{ord}_z f)\log\dfrac{r}{\lvert z\rvert}$	$(\mathrm{ord}_\wp x)[F_\wp:\mathbb{Q}_p]\log p$
Characteristic function:	Logarithmic height:
$T(f,r)=\int_0^{2\pi}\log^+\lvert f(re^{i\theta})\rvert\dfrac{d\theta}{2\pi}+N(f,\infty,r)$	$h(x)=\frac{1}{[F:\mathbb{Q}]}\sum\limits_{\sigma:F\hookrightarrow\mathbb{C}}\log^+\lvert x\rvert_\sigma+N(F,\infty,r)$
Mean-proximity function:	Mean-proximity function:
$m(f,a,r)=\int_0^{2\pi}\log^+\left\lvert\dfrac{1}{f(re^{i\theta})-a}\right\rvert\dfrac{d\theta}{2\pi}$	$m(F,a,x)=\sum\limits_{\sigma:F\hookrightarrow\mathbb{C}}\log^+\left\lvert\dfrac{1}{x-a}\right\rvert_\sigma$
Counting function:	Counting function:
$N(f,a,r)=\sum\limits_{\lvert z\rvert<r}(\mathrm{ord}_z^+(f-a))\log\dfrac{r}{\lvert z\rvert}$	$\frac{1}{[F:\mathbb{Q}]}\sum\limits_{\wp\in F}(\mathrm{ord}_\wp^+(x-a))[F_\wp:\mathbb{Q}_p]\log p$
First Main theorem	Height property
$N(f,a,r)+m(f,a,r)=T(f,r)+O(1)$	$N(F,a,x)+m(F,a,x)=h(x)+O(1)$
Second Main theorem :	Roth type conjecture:
$(q-2)T(r,f)-\sum\limits_{j=1}^{q}N^{(1)}(f,a_j,r)$	$(q-2)h(x)-\sum\limits_{j=1}^{q}N^{(1)}(F,a_j,r)$
$\le\log(T(f,r)\psi(T(f,r)))+O(1)$	$\le\log(h(x)\psi(h(x)))+O(1)$
Weaker Second Main theorem:	Roth's Theorem:
$(q-2)T(r,f)-\sum\limits_{j=1}^{q}N(f,a_j,r)\le\varepsilon T(f,r)$	$(q-2)h(x)-\sum\limits_{j=1}^{q}N(F,a_j,r)\le\varepsilon h(x)$
Jensen's Formula:	Artin-Whaples Product Formula:
$\int_0^{2\pi}\log\lvert f(re^{i\theta})\rvert\dfrac{d\theta}{2\pi}$	$\sum\limits_{\sigma\in S_\infty}\log\lvert x\rvert_\sigma$
$=N(f,0,r)-N(f,\infty,r)+O(1)$	$=N(F,0,x)-N(F,\infty,x)$

3. p-adic Nevanlinna Theory.

In the philosophy of Hasse-Minkowski principle one hopes to have an "arithmetic result" if one had have it in p-adic cases for all prime numbers p, and in the real and complex cases. Hence one would naturally have interest to determine how Nevanlinna Theory would look in the p-adic case.

§1. **Two main theorems**

Let p be a prime number, $\mathbb{Q}_p$ the field of p-adic numbers, and $\mathbb{C}_p$ the p-adic completion of the algebraic closure of $\mathbb{Q}_p$. The absolute value in $\mathbb{Q}_p$ is normalized so that $|p|=p^{-1}$. We further use the notion $v(z)$ for the additive valuation on $\mathbb{C}_p$ which extends ord_p.

We define the counting function in exactly the same way as in classical Nevanlinna theory. That is, given a meromorphic function f, we let $n(f,\infty,r)$ denote the number of poles in

$\{|z| \leq r\}$, and we let

$$N(f,\infty,r) = \int_o^r [n(f,\infty,t) - n(f,\infty,0)]\frac{dt}{t} + n(f,\infty,0)\log r =$$
$$\sum_{|z|\leq r, z\neq 0} \max\{0, -ord_z f\}\log\frac{r}{|z|} + \max\{0, -ord_o f\}\log r,$$

where $ord_z f$ denotes the order of vanishing of f at z, negative numbers indicating poles. Counting functions for other values are defined similarly.

For the mean proximity function, note that the norms $|\ |_r$ are multiplicative on entire functions and they extend to meromorphic functions. Thus we define

$$m(f,\infty,r) = \log|f|_r,$$

and for finite a,

$$m(f,a,r) = \log\frac{1}{|f-a|_r}.$$

Note that there is no need to do any sort of "averaging" over $|z| = r$, since by the strong maximum modulus principle, for suitable generic z with $|z| = r$, we know $|f(z)| = |f|_r$. Finally, just as in classical Nevanlinna theory, the characteristic function is given by

$$T(f,a,r) = m(f,a,r) + N(f,a,r).$$

The properties of the valuation polygon imply that

$$\log|f|_r = \sum_{|z|\leq r, z\neq 0}(ord_z f)\log\frac{r}{|z|} + (ord_o f)\log r + 0(1),$$

where the $0(1)$ term depends on the size of the first non-zero coefficient in the Laurent expansion for f at 0. This is of course a non-Archimedean Jensen formula and can be written

$$m(f,\infty,r) + N(f,\infty,r) = m(f,0,r) + N(f,0,r) + 0(1),$$

from which the non-Archimedean analog to *Nevanlinna first Main Theorem* easily follows.

The Second Main Theorem. Let f be a non-constant meromorphic function on $\mathbb{C}_p$, and let $a_1, a_2, \ldots, a_q$ be q distinct points on $\mathbb{C}_p \cup \{\infty\}$. Then, for all $r \geq r_o > 0$,

$$(q-2)T(f,r) - \sum_{j=1}^{q} N(f,a_j,r) - N_{Ram}(f,t) \leq -\log r + 0(1),$$

where

$$N_{Ram}(f,a) = N(f',0,r) + 2N(f,\infty,r) - N(f',\infty,r)$$

measures the growth of the ramification of f, and the $0(1)$ term depends only on the a_j, the function f, and the number r_o.

Corollary. *Let f and $a_1, a_2, \ldots, a_q$ be as in the Theorem. Then for all $r \geq r_o$*

$$(q-2)T(f,r) \leq \sum_{j=1}^{q} N_1(f,a_j,r) - \log r + 0(1),$$

where $N_1(f, a_j, r)$ *denotes a modified counting function in that each point where* $f = a$ *is counted only with multiplicity 1, and again 0(1) term depends on the* a_j, f, r_o.

§2. **The height function.**

In the p-adic case we can use so-called "the height function". Notice that, the Newton polygon gives expression to one of the most basic deffrences between p-adic analytic functions and complex analytic functions. Namely, the modulus of a p-adic analytic function depends only on the modulus of the argument, except at a discrete set of values of the modulus of argument. This fact often makes it easier to prove the p-adic analogs of classical results. Now we give the definition of the height function.

Let $f(z)$ be an analytic function on $\mathbb{C}_p$, it is represented by a convergent power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

For each n we draw the graph Γ_n which depicts $v(a_n z^n)$ as a function of $v(z) = t$. This graph is a straight line with slope n. Since we have $\lim_{n\to\infty}\{v(a_n)+nt\} = \infty$ for all t it follows that for every t there exists an n for which $v(a_n) + nt$ is minimal. Let $h(f, t)$ denote the boundary of the intersection of all of the half-planes lying under the lines Γ_n. Then in any finite segment $[r, s]$, there are only finitely many Γ_n which appear in $h(f, t)$. Thus, $h(f, t)$ is a polygonal line. This line is what we call *the height* of the function $f(z)$. The points t at which $h(f, t)$ has vertices are called the critical points of $f(z)$. A finite segment contains only finitely many critical points. If t is a critical point, then $v(a_n)+nt$ attains its minimum at least at two values of n. If $v(z) = t$ is not a critical point, then $|f(z)| = p^{-h(f,t)}$.

The height of a function $f(z)$ gives complete information about the number of zeros of $f(z)$. Namely, f has zeros when $v(z) = t_i$ (a critical point) and the number of zeros of f such that $v(z) = t_i$ is equal to the difference $n_{i+1} - n_i$ between slopes of $h(f, t)$ at t_{i-0} and t_{i+0}

For a meromorphic function $f = \dfrac{\phi}{\psi}$, the height of f is defined by $h(f, t) = h(\phi, t)-h(\psi, t)$. We also use the notation

$$h^+(f, t) = -h(f, t).$$

Then we have:

Theorem 3.1. *Let* f *be a meromorphic function and let* $a_1, a_2, \ldots, a_q$ *be* q *distinct points in* $\mathbb{C}_p \cup \{\infty\}$. *Then for* t *sufficiently small,*

$$(q - 2)h^+(f, t) \leq \sum_{j=1}^{q} N_1(f, a_j, t) + t + 0(1),$$

where $N_1(f, a, t)$ *denotes a modified function in that each point where* $f = a$ *is counted only with multiplicity 1, and the 0(1) is a bounded value as* $t \longrightarrow -\infty$.

The height function is applied to the interpolation problem (see [K1]). Let $u = \{u_1, u_2, \dots\}$ be a sequence of points in $\mathbb{C}_p$. In that follows we shall only consider sequences u for which the numbers of points u_i satisfying $v(u_i) \geq t$ is finite for every t. We shall always assume that $v(u_i) \geq v(u_{i+1}),\ (i = 1, 2, \dots)$.

Definition 1. *The sequence $u = \{u_i\}$ is called an interpolating sequence of f if the sequence of interpolating polynomials for f on u converges to f.*

For every sequence u we define a holomorphic function Φ_u as follows. We set

$$N_u(t) = \#\{u_i | v(u_i) \geq t\}.$$

We can write the sequence u in the form:

$$u = \{u_1, u_2, \dots, u_{n_1}, u_{n_1+1}, \dots, u_{n_2}, \dots\},$$

where

$$v(u_i) = t_k \ \text{ for } n_{k-1} < i \leq n_k$$

(we take $u_o = 0$), and

$$\lim_{k\to\infty} t_k = -\infty.$$

We choose a sequence a_k with the property

$$v(a_o) = -n_1 t_1,\ v(a_{k+1}) = v(a_k) + (n_k - n_{k+1})t_{k+1},\ (k = 1, 2, \dots).$$

We set

$$\Phi_u(z) = 1 + \sum_{k=1}^{\infty} a_k z^{n_k}.$$

Then the series converges for $z \in \mathbb{C}_p$ and determines an analytic function $\Phi_u(z)$ on $\mathbb{C}_p$, for which the number of zeros in each region $\{z | v(z) > t\}$ is equal to $N_u(t)$, and

$$h(\Phi_u, t) = \int_{\infty}^{t} N_u(t) dt.$$

Theorem 3.2. *The sequence $u = \{u_i\}$ is an interpolating sequence of the function $f(z)$ if and only if*

$$\lim_{t\to\infty} \{h(f, t) - h(\Phi_u, t)\} = \infty.$$

Remark 3.3. *This is the first interpolation theorem for p-adic analytic functions not necessarily bounded. A similar theorem for analytic functions in the unit disc implies that the p-adic L-functions associated to modular forms are uniquelly defined by the values on Dirichlet characters (see [K2]).*

Remark 3.4. *We can use the interpolation theorem to recover a p-adic meromorphic function if we know the preimages (with multiplicity) of to points (see [K3]).*

For *high dimensions*, as well as in the complex case, instead of the study the preimage of a point, we should consider the preimage of a divisor of codimension one. The reason is that in the p-adic case there exist also Fatou-Bieberbach domains (see [K]):

Theorem 3.5. *There exists an injective holomorphic map* $F : \mathbb{C}_p^2 \longrightarrow \mathbb{C}_p^2$ *with the Jacobian* $J(F) \equiv 1$ *and the complement of the image of* F *contains a non-empty open set in* $\mathbb{C}_p^2$

Now let $f = (f_1, \ldots, f_{n+1}) : \mathbb{C}_p \to \mathbb{P}^n(\mathbb{C}_p)$ be a p-adic holomorphic curve, where the functions f_j have no common zeros.

Definition 2. *The height of the holomorphic curve* f *is defined by*

$$h(f,t) = \min_{1 \le j \le n+1} h(f_j, t),$$

where $h(f_j,t)$ *is the height of* p*-adic holomorphic function on* $\mathbb{C}_p$.

Notices that the height of a curve is well defined up to a bounded value.

The following theorem is a p-adic version of the Second Main Theorem in the case of holomorphic curves.

Theorem 3.6. *([KT) Let* $H_1, \ldots, H_q$ *be* q *hyperplanes in general position, and let* f *be a non-degenerate holomorphic curve in* $\mathbb{P}^n(\mathbb{C}_p)$*. Then we have*

$$(q-n-1)h^+(f,t) \le \sum_{j=1}^{q} N_n(f.H_j,t) + \frac{n(n+1)}{2}.t + 0(1),$$

where 0(1) is bounded when $t \to -\infty$ *and* $h^+(h,t) = -h(f,t)$.

Cherry and Ye [CY] extend the theorem to several variables. Moreover, they considered the case of degenerate curves by using so called Nochka's weights ([N]), Hu and Yang [HY] obtain similar results for moving targets.

For the case of hypersurfaces we have the following

Theorem 3.7. *([KA1]) Let* $H_1, \ldots, H_q$ *be hypersurfaces of degree* d *in* $\mathbb{P}^n(\mathbb{C}_p)$ *in general position. Let* f *be a non-degenerate holomorphic curve . Then*

$$(q-n)h^+(f,t) \le \sum_{j=1}^{q} \frac{N(f \circ H_j, t}{d} + 0(1),$$

where 0(1) is bounded when $t \to -\infty$.

This is a p-adic version of Eremenko-Sodin's theorem ([ES]).

We conclude this section by a conjecture.

Recall that a holomorphic curve f is said to be *k-non-degenerate* if the image of f is contained in a linear subspace of dimension k and is not contained in any linear subspace of dimension $k-1$.

Conjecture 1. *Let $H_1, \ldots, H_q$ be hypersurfaces of degree d_j, $j = 1, \ldots, q$ in $\mathbb{P}^n(\mathbb{C}_p)$ in general position, let f be a k- non-degenerate holomorphic curve, and s be an integer $\geq k$, or $s = \infty$. Then*

$$(q - 2n + k - 1)h^+(f, t) \leq \sum_{j=1}^{q} \frac{N_s(H_j \circ f, t)}{d_j} + 0(1).$$

Remark 3.8. *In the complex case the above conjecture corresponds to the following cases:*

1. *Nevanlinna's Second Main Theorem: $n = 1$, $k = 1$, $d_j = 1$, $s = \infty$.*
2. *Cartan Theorem: $\forall n$, $k = n$, $d_j = 1$, $s = n$.*
3. *Nochka Theorem (Cartan's conjecture): $\forall n, \forall k \leq n, s = k$, $d = 1$.*
4. *Eremenko-Sodin's theorem: $\forall n, k = n, \forall d_j, s = \infty$.*

Remark 3.9. *In recent years the Nevanlinna theory is developed also for the case of positive characteristic ([ ], [ ]).*

4. Defect relation and Borel's Lemmas.

Let H be a hyperplane of $\mathbb{P}^n(\mathbb{C}_p)$ such that the image of f is not contained in H. We say that f ramifies at least d ($d > 0$) over H if for all $z \in f^{-1}H$ the degree of the pull-back divisor f^*H, $deg_z f^*H \geq d$. In case $f^{-1}H = \emptyset$ we set $d = \infty$.

Theorem 4.1. *Let $H_1, \ldots, H_q$ be q hyperplanes in general position. Assume f is linearly non-degenerate and ramifies at least d_j over H_j. Then*

$$\sum_{j=1}^{q}(1 - \frac{n}{d_j}) < n + 1.$$

Remark 4.2. *In the complex case we have a similar inequality, but with the sign $\leq$. The reason is that in the p-adic case, the error term in Second Main Theorem is simpler that the complex one. This is important for applications.*

From Theorem 4.1 one can prove the following p-adic version of Borel's Lemma.

Theorem 4.3. *(p-adic Borel's Lemma [Q]). Let $f_1, f_2, \ldots, f_n$ $(n \geq 3)$ be p-adic holomorphic functions without common zeros on $\mathbb{C}_p$ such that $f_1 + f_2 + ... + f_n = 0$.*

Then the functions $f_1, \ldots, f_{n-1}$ are linearly dependent if for $j = 1, \ldots, n$ every zero of f_j is of multiplicity at least d_j and the following condition holds:

$$\sum_{j=1}^{n} \frac{1}{d_j} \leq \frac{1}{n-2}.$$

By using the defect relation one can prove some generalizations of Borel's lemma.

Theorem 4.4. *(p-adic analogue of Masuda-Noguchi's Theorem [M-N]).*

Let

$$M_j = z_1^{\alpha_{j,1}} \dots z_{n+1}^{\alpha_{j,n+1}}, \qquad 1 \le j \le s,$$

be distinct monomials of degree l with non-negative exponents. Let X be a hypersurface of degree dl of $\mathbb{P}^n(\mathbb{C}_p)$ defined by

$$X: \quad c_1 M_1^d + \dots c_s M_s^d = 0,$$

where $c_j \in \mathbb{C}_p^*$ are non-zero constants.

Let $f = (f_1, .., f_{n+1}) : \mathbb{C}_p \longrightarrow X$ be a non-constant holomorphic curve such that any $f_j \not\equiv 0$. Assume that

$$d \ge s(s-2).$$

Then there is a decomposition of indices,$\{1, 2, ..., s\} = \cup I_\gamma$, such that

i) Every I_γ contains at least 2 indices.

ii) The ratio of $M_j^d \circ f(z)$ and $M_j^k \circ f(z)$ is constant for $j, k \in I_\gamma$.

iii) $\sum_{j \in I_\gamma} c_j M_j^d \circ f(z) \equiv 0$ for all γ .

Corollary 4.5. *For $d \ge 3$ there is no solutions of the following equation in the set of p-adic non-constant holomorphic functions having no common zeros:*

$$x^d + y^d = z^d$$

5. P-ADIC HYPERBOLIC SPACES

Recall that a complex space is said to be hyperbolic if every holomorphic curve in it is a constant curve. In the complex case, the Borel Lemma is often used to establish the hyperbolicity of a complex space. In what follows we show some applications of p-adic Borel's Lemma in the study of p-adic hyperbolic hypersurfaces.

Although the set of hyperbolic hypersurfaces of degree d large enough with respect to n is conjectured to be Zariski dense [Ko]), it is not easy to construct explicit examples of hyperbolic hypersurfaces.

The first example of smooth hyperbolic surfaces of even degree $d \ge 50$ was given by R. Brody and M. Green ([BG]). Now we show how to use p-adic Borel's Lemmas to construct explicit examples of p-adic hyperbolic hypersurfaces.

Let X be a hypersurface defined as above, and let $d \ge s(s-2)$. Suppose that X is not hyperbolic, and let

$$f = (f_1, ..., f_{n+1}) : \mathbb{C}_p \longrightarrow X$$

be a nonconstant holomorphic curve in X. We are going to show that $\{c_j\}$ belongs to an algebraic subset of $(\mathbb{C}_p^*)^s$. We may assume that any $f_j \not\equiv 0$.

By Theorem 4.4, there is a decomposition of indices $\{1, \dots, s\} = \cup I_\xi$ such that

i) every I_ξ contains at least 2 indices,

ii) the ratio of $M_j^d \circ f(z)$ and $M_k^d \circ f(z)$ is constant for $j,\, k \in I_\xi$,

iii) $\sum_{j\in I_\xi} c_j\, M_j^d \circ f(z) \equiv 0$ for all ξ.

Now for a decomposition of $\{1,\dots,s\}$ as above, we set $b_{jk} = M_j^d \circ f(z)\,/\,M_k^d \circ f(z)$. Then the linear system of equations

$$AY = B$$

where A is the matrix $\{\alpha_{j\ell} - \alpha_{k\ell}\}$, $Y = \begin{pmatrix} y_0 \\ \vdots \\ y_n \end{pmatrix}$, $B = \log b_{jk}$, has the solution $\log f_0, ..., \log f_n$.

Thus, the matrix A satisfies certain conditions on the rank. On the other hand, by the condition iii) there exist $(A_0, ..., A_n) \in \mathbb{P}^n$ such that $(c_i) \in (\mathbb{C}_p^*)^s$ satisfies the following equations

$$\sum_{i\in I_\xi} c_i\, A_o^{\alpha_{io}} \cdots A_n^{\alpha_{in}} = 0$$

Hence, $(c_i) \in (\mathbb{C}_p^*)^s$ belongs to the projection $\Sigma \subset (\mathbb{C}_p^*)^s$ of an algebraic subset in $(\mathbb{C}_p^*)^s \times \mathbb{P}^n$. If we take $(c_i) \notin \Sigma$, we have a hyperbolic hypersurface.

5.1.**Examples**.

Let $N = 4n - 3, k = N(N-2) = 16(n-1)^2$. Then for generic linear functions $H_j(z_0,\dots,z_n) \in \mathbb{C}_p^{n+1}$ $(1 \le j \le n)$ the hypersurface

$$X : \sum_{j=1}^{N} H_j^k = 0$$

is hyperbolic. This is p-adic version of a result of Siu and Yeung ([SY], 1997).

Proof. By p-adic Borel's lemma, if $f : \mathbb{C}_p \longrightarrow X$ is a non-constant holomorphic curve, then $\mathrm{Im} f \subset \cap_\xi X_\xi$, where

$$X_\xi : \sum_{j\in I_\xi} H_j^k = 0.$$

The genericity of $\{H_j\}$ implies $\cap X_\xi = \emptyset$.

For the case of surfaces in $\mathbb{P}^3$ we can use the following method. Take at first a surface $X \subset \mathbb{P}^3$ such that every holomorphic curve in X is degenerate. This means that the image of a holomorphic map from $\mathbb{C}_p$ in to X, $f : \mathbb{C}_p \longrightarrow X$, is contained in a proper algebraic subset of X. If one could prove that the image $f(\mathbb{C}_p)$ is contained in a curve of genus at least 1, then f is a constant map (Berkovich's theorem).

5.2. **Example** Let X be a surface in $\mathbb{P}^3(\mathbb{C}_p)$ defined by the equation

$$X :\; z_1^d + z_2^d + z_3^d + z_4^d + c z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3} z_4^{\alpha_4} = 0,$$

where $c \neq 0$, $\sum_{i=1}^{4} \alpha_i = d$, and if there is an exponent $\alpha_i = 0$, the others must be $\neq 1$. Then X is hyperbolic if $d \geq 24$.

5.3. **Example** Let X be a curve in $\mathbb{P}^2(\mathbb{C}_p)$ defined by the following equation:

$$X: \; z_1^d + z_2^d + z_3^d + cz_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3} = 0,$$

where $d \geq 24$, $d > \alpha_i \geq 0$, $\sum \alpha_i = d$. Then the complement of X is p-adic hyperbolic in $\mathbb{P}^2(\mathbb{C}_p)$.

6. UNIQUE RANGE SET FOR MEROMORPHIC FUNCTIONS

For a non-constant meromorphic function f on $\mathbb{C}$ and a set $S \subset \mathbb{C} \cup \{\infty\}$ define

$$E_f(S) = \cup_{a \in S} \{(m, z) | f(z) = a \text{ with multiplicity } \; m\},$$

and

$$\bar{E}_f(S) = \cup_{a \in S} \{z | f(z) = a \text{ ignoring multiplicities } \; \}.$$

A set $S \subset \mathbb{C} \cup \{\infty\}$ is call *an unique range set for meromorphic functions* (URSM) if for any pair of non-constant meromorphic functions f and g on $\mathbb{C}$, the condition $E_f(S) = E_g(S)$ implies $f = g$. A set $S \subset \mathbb{C} \cup \{\infty\}$ is called *an unique range set for entire functions* (URSE) if for any pair of non-constant entire functions f and g on $\mathbb{C}$, the condition $E_f(S) = E_g(S)$ implies $f = g$. Classical theorems of Nevanlinna show that $f = g$ if $\bar{E}_f(a_j) = \bar{E}_g(a_j)$ for distinct values $a_1, \ldots, a_5$, and that f is a Möbius transformation of g if $E_f(a_j) = E_g(a_j)$ for distinct values $a_1, \ldots, a_4$. Gross and Yang show that the set

$$S = \{z \in C | z + e^z = 0\}$$

is an URSE. Notices that this set is infinite. Since 1995, URSE and also URSM with finitely many elements have been found by Yi ([Y1], [Y2]), Li and Yang ([LY1], [LY2]), Mues and Reinders [MR], Frank and Reinders [FR]. Li and Yang introduced the notation

$$\lambda_M = \inf\{\#S | S \text{ is a URSM}\},$$

$$\lambda_E = \inf\{\#S | S \text{ is a URSE}\},$$

where $\#S$ is the cardinality of the set S.The best lower and upper bounds known so far are

$$5 \leq \lambda_E \leq 7, \; 6 \leq \lambda_M \leq 11.$$

For p-adic meromorphic or entire function f on $\mathbb{C}_p$, we can similarly define $E_f(S)$ and $\bar{E}_f(S)$ for a set $S \subset \mathbb{C}_p \cup \{\infty\}$ and introduce the notation λ_M and λ_E. By using p-adic Nevanlinna theory and theory of singularities we can prove the following theorems:

Theorem 6.1. *Let P be a generic polynomial of degree at least 5. Let f and g be p-adic meromorphic functions such that $P(f) = CP(g)$ with a constant C. Then $f \equiv g$.*

Theorem 6.2. *Let $S = \{a_1, a_2, a_3, a_4\}$ be a generic set of 4 points in $\mathbb{C}_p$. Then for p-adic meromorphic functions f and g, the conditions $E_f(S) = E_g(S)$ and $E_f(\infty) = E_g(\infty)$ imply $f \equiv g$.*

For the proof, see [KA].

In [K-H] we proved that for a generic pair of sets $\{S, T\}$ with the total number of elements at leats 5, the condition $E_f(S) = E_g(S)$ and $E_f(T) = E_g(T)$ implies $f = g$ for p-adic meromorphic functions f, g (i.e., the pair $\{S, T\}$ is a bi-URS for p-adic meromorphic fucnctions). From this fact I would like to formulate the following

Conjecture 2. *A generic set of 6 points in* $\mathbb{C}_p \cup \{\infty\}$ *is a URS for p-adic meromorphic functions.*

As it is mentioned above, Li and Yang suggested that there exist a URS for complex meromorphic functions with 6 elements. However, from the result on bi-URS for p-adic meromorphic functions, I think, as in the Conjecture, a URS for p-adic meromorphic functions should have at least 6 elements, and then, a URS for complex meromorphic functions should have at least 7 elements.

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Institute of Mathematics, 18 Hoang Quoc Viet, 10307 Hanoi, VietNam

E-mail address: hhkhoai@math.ac.vn