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Continuity of the radius of convergence of differential equations on p-adic analytic curves

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0 Introduction

0.1 Radius of convergence of p-adic differential equations

It is well-known that any system of linear differential equations with complex analytic coefficients on a complex open disk admits a full set of solutions convergent on the whole disk.

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It is also well known that the same fact does not hold over a non-archimedean field k that contains the field of p-adic numbers. For example, solutions of the equation $\frac{df}{dT} = f$ are convergent only on an open disks of radius $|p|^{\frac{1}{p-1}}$. The question is actually of interest over any non-archimedean field k *of characteristic zero* (we do not exclude the case of a trivially valued k). What is the behaviour of the radius of convergence of a system of differential equations with k-analytic coefficients as a function on points of the k-analytic affine line ? Of course, one has to give a precise meaning to this question since points of k-analytic spaces are not necessarily k-rational. We recall in fact [6, 1.2.2] that to any point x of a k-analytic space X, one associates a completely valued extension field $\mathscr{H}(x)$ of k, called the *residue field at* x; the point x is k-*rational* (resp. *rigid*, resp. *separable* or Q-*rational*) iff $\mathscr{H}(x) = k$ (resp. $[\mathscr{H}(x) : k]$ is finite, resp. $\mathscr{H}(x)/k$ is finite and separable) ; $X(k)$ (resp. X_0) denotes the subset of k-rational (resp. rigid) points of X. On a smooth X, k-rational points admit a fundamental system of open neighborhoods which are open polydisks, but this is not the case for general points, not even using the étale topology [12, Corollary 2.3.3].

Let X be a relatively compact analytic domain in the k-analytic affine line $\mathbb{A}^1 = \mathbb{A}^1_k$, and suppose we are given a system

$$\Sigma : \frac{d\vec{y}}{dT} = G\,\vec{y}\,, \tag{0.1.0.1}$$

of linear differential equations, with G a $\mu \times \mu$ matrix of k-analytic functions on X. If $x \in X$ is a k-rational point, let $R(x) = R(x, \Sigma)$ denote the radius of the maximal open disk in X with center at x on which all solutions of Σ converge. If x is not necessarily k-rational, the $\mathscr{H}(x)$-analytic space $X_{\mathscr{H}(x)} := X \widehat{\otimes} \mathscr{H}(x) \subset \mathbb{A}^1_{\mathscr{H}(x)}$ contains a canonical $\mathscr{H}(x)$-rational point x' over x which corresponds to the induced character $\mathscr{H}(x)[T] \to \mathscr{H}(x)$, and we define $R(x)$ as the number $R(x')$ for the system of equations on $X_{\mathscr{H}(x)}$ induced by Σ. A more precise formulation of the above question is as follows. What is the behavior of the function $X \to \mathbb{R}_{>0}$, $x \mapsto R(x)$? For example, is it continuous ? The latter question is not trivial since a precise formula for the radius of convergence involves the infimum limit of an infinite number of continuous real valued functions on X (*cf.* §3).

A problem of the above type was considered for the first time in the p-adic case in a paper by Christol and Dwork (in a slightly different setting). Namely, let X be an affinoid domain in $\mathbb{A}^1$ that contains the open annulus

$$B(r_1, r_2) = \{x \in \mathbb{A}^1 \,|\, r_1 < |T(x)| < r_2\,\}\,. \tag{0.1.0.2}$$

Then there is a continuous embedding $[r_1, r_2] \hookrightarrow X$, $r \mapsto t_{0,r}$, where $t_{0,r}$ is the maximal point of the closed disk of radius r with center at zero. Christol and Dwork proved that, for any system of differential equations Σ on X, the function $[r_1, r_2] \to \mathbb{R}_{>0}$, $r \mapsto \inf(r, R(t_{0,r}))$ is continuous [20, Théorème 2.5].

The following example should convince the reader that the variation of the normalized radius of convergence presents, in general, some non obvious features.

Example 0.1.1. ([40, III.10.6]) Let k be a non-archimedean extension of $\mathbb{Q}_p$, $p > 2$, containing an element π such that $\pi^{p-1} = -p$. It is well-known that the *Dwork exponential* $F(x) = \exp \pi(T - T^p) = \exp[\pi T + \frac{(\pi T)^p}{p}]$ converges precisely for $|T| < p^{\frac{p-1}{p^2}}$. We can check this by comparison with the Artin-Hasse exponential series

$$E_p(T) = \exp(T + \frac{T^p}{p} + \frac{T^{p^2}}{p^2} + \dots) \in \mathbb{Z}_p[[T]]\,, \tag{0.1.1.1}$$

evaluated at πT. One then observes that the radius of convergence of each individual factor $\exp(\frac{(\pi T)^{p^n}}{p^n})$ is 1 for $n = 0, 1$, and $p^{\frac{1}{p-1} - \varepsilon_n}$, where $\varepsilon_n = \frac{1}{p^{n-1}}(\frac{1}{p-1} + n)$, for $n = 2, 3, \dots$, and that, aside for $n = 0, 1$, these numbers are all distinct. Consider the rank one differential equation whose solution at 0 is $F(T)$, on any sufficiently large affinoid disk X (say $X = D(0, p^+)$)

$$\Sigma \ : \frac{dy}{dT} = \pi(1 - pT^{p-1})\, y \ . \tag{0.1.1.2}$$

Notice that the corresponding differential module is a tensor product of the differential modules corresponding to

$$\Sigma_0 \ : \frac{dy}{dT} = \pi\, y \quad \text{and} \quad \Sigma_1 \ : \frac{dy}{dT} = -p\pi T^{p-1}\, y \ . \tag{0.1.1.3}$$

Obviously, $R(t_{0,r}, \Sigma_0) = 1\ \forall r$. A non-zero solution of Σ_1 at $t = t_{0,r}$ is

$$\exp[-\pi(T^p - t^p)] = \exp[-\pi(T - t)^p + \pi \sum_{i=1}^{p-1} \binom{p}{i} (-t)^i T^{p-i}] \ ,$$

which readily shows

$$R(t_{0,r}, \Sigma_1) = \begin{cases} 1 & \text{if} \quad r \leq p^{\frac{1}{p-1}} \\ p\, r^{1-p} & \text{if} \quad r \geq p^{\frac{1}{p-1}} \end{cases} \tag{0.1.1.4}$$

Therefore, $R(t_{0,r}, \Sigma) = p\, r^{1-p}$ for $r \geq p^{\frac{1}{p-1}}$. A basic remark on the behaviour of radii of convergence shows that $R(x, \Sigma) \geq R(t_{0,p^{\frac{1}{p-1}}}, \Sigma) = 1$, $\forall x \in D = D(0, (p^{\frac{1}{p-1}})^+)$, and that in fact equality holds on almost all residue classes in D. On the other hand, $R(0, \Sigma) = p^{\frac{p-1}{p^2}} > 1$, hence $R(x, \Sigma) = p^{\frac{p-1}{p^2}}$, $\forall x \in D(0, (p^{\frac{p-1}{p^2}})^+)$, shows that $D(0, (p^{\frac{1}{p-1}})^-)$ is an exceptional residue class of D. The investigation of the function $R(t_{0,r}, \Sigma)$ in the range $r \in [p^{\frac{p-1}{p^2}}, p^{\frac{1}{p-1}}]$ requires the calculation of $R(t_{0,r}, \Sigma_n)$, $\forall n$. The final result is

$$R(t_{0,r}, \Sigma) = \begin{cases} p^{\frac{p-1}{p^2}} & \text{if} \quad r \leq p^{\frac{1}{p-1} - \frac{1}{p^2}} \\ p\, r^{1-p} & \text{if} \quad r \geq p^{\frac{1}{p-1} - \frac{1}{p^2}} \end{cases} \tag{0.1.1.5}$$

Now, for every r, the function $x \mapsto R(x, \Sigma)$ is constant of value $R(t_{0,r}, \Sigma)$ on almost all residue classes of the disk $D(0, r^+)$, but there may exist exceptional residue classes, where $R(x, \Sigma)$ takes a bigger value. The complete picture is not understood.

A particular case of our main result states that for any relatively compact analytic domain X in $\mathbb{A}^1$ the function $x \mapsto R(x)$ is continuous on the whole space X. Our result in fact establishes the property of continuity for more general analytic curves. (Recall that in comparison with the complex analytic case a smooth k-analytic space is not necessarily locally isomorphic to an affine space.) To formulate the result, we first define a so-called "normalized" radius of convergence $\mathcal{R}(x)$, which is related to the function $R(x)$ via a simple formula but does not depend on the embedding of the analytic domain X into $\mathbb{A}^1$. After that we explain how the function $\mathcal{R}(x)$ can be defined in a more general setting on one-dimensional smooth[1] affinoids, which are not necessarily analytic subdomains of $\mathbb{A}^1$, and even in arbitrary dimensions.

Let X be an affinoid domain in $\mathbb{A}^1_k$, and assume at first that X is strict and that k is non-trivially valued and algebraically closed. Recall that the set of all points of X which

[1] in the sense of rigid geometry, *i.e. rig-smooth cf.* definition 0.1.2 below

have no open neighborhood isomorphic to an open disk, is a closed subset of X, called the *analytic skeleton* of X and denoted by $S(X)$ [6, §4]. The complement $X \setminus S(X)$ is a disjoint union of open sets $\mathcal{D}_y(X)$, where $y \in \mathcal{D}_y(X)$ is a k-rational point, isomorphic to the standard open unit disk

$$D_k(0,1^-) = \{x \in \mathbb{A}^1_k \,|\, |T(x)| < 1\}\ ,$$

via a *normalized* coordinate $T_y : \mathcal{D}_y(X) \xrightarrow{\sim} D_k(0,1^-)$ such that $T_y(y) = 0$. Furthermore, each of the above open disks has a unique boundary point at $S(X)$, and so there is a canonical continuous retraction map $\tau_X : X \to S(X)$. The topological space $S(X)$ carries a canonical structure of a finite polygon $\mathbf{S}(X)$ whose set of vertices $\mathcal{V}(X)$ consists of the points which have no neighborhood isomorphic to an open annulus: they are all of type (2) in the sense of [6, 1.4.4]. The set of open edges $\mathcal{E}(X)$ of $\mathbf{S}(X)$ is formed by the connected components of $S(X) \setminus \mathcal{V}(X)$; an open edge $E = E(u,v)$ connects precisely two vertices u, v and $E \cup \{u,v\}$ is a closed subset of X, canonically homeomorphic to the closed interval $[r,1]$ for a well defined $r \in |k| \cap (0,1)$, with $u \mapsto 1$, $v \mapsto r$. All points of the Shilov boundary $\Gamma(X)$ of X are among the vertices of $\mathbf{S}(X)$. For $x \in S(X)$, the set $\tau_X^{-1}(x) \setminus \{x\}$ is a disjoint union of maximal open disks $\mathcal{D}_y(X)$ with boundary point x in X, all of the same radius $r(x)$ with respect to the standard coordinate in $\mathbb{A}^1$. Then, $r(x)$ is the *radius* of the point $x \in \mathbb{A}^1$ defined on p. 78 of [6] and the function $r : S(X) \to \mathbb{R}_{>0}$ is continuous. Notice that if $r(x) \notin |k|$, $\tau_X^{-1}(x) \setminus \{x\} = \emptyset$. If k' is an algebraically closed non-archimedean field over k, then $\mathbf{S}(X\widehat{\otimes}k') \xrightarrow{\sim} \mathbf{S}(X)$ under the natural projection. For X not necessarily strict, we choose an algebraically closed non-archimedean field k' over k, such that $X\widehat{\otimes}k'$ is strict, and use the previous formula to *define* $\mathbf{S}(X)$. Now, for any non trivially valued non-archimedean field k, let $\widehat{k^{\rm alg}}$ be the completion of the algebraic closure $k^{\rm alg}$ of k, and $G_k = \mathrm{Gal}\,(k^{\rm alg}/k)$. For any affinoid domain X in $\mathbb{A}^1_k$, we define $\mathbf{S}(X)$ as the "quotient" of the polygon $\mathbf{S}(X\widehat{\otimes}\widehat{k^{\rm alg}})$ under the natural action of G_k, which preserves $\mathcal{V}(X\widehat{\otimes}\widehat{k^{\rm alg}})$ and $\mathcal{E}(X\widehat{\otimes}\widehat{k^{\rm alg}})$, inducing canonical homeomorphisms between corresponding edges. Recall [6, 4.2] that a disk (resp. an annulus) of $X \subset \mathbb{A}^1$ is, by definition, the image of a disk (resp. an annulus) of $X\widehat{\otimes}\widehat{k^{\rm alg}}$ under the natural (open) projection. The image $S(X)$ (resp. $\mathcal{V}(X)$) of $S(X\widehat{\otimes}\widehat{k^{\rm alg}})$ (resp. $\mathcal{V}(X\widehat{\otimes}\widehat{k^{\rm alg}})$) in X, consists of points of X which have no neighborhood which is an open disk (resp. an open annulus) of X, and we have a canonical continuous retraction $\tau_X : X \to S(X)$ induced by the similar retraction on $X\widehat{\otimes}\widehat{k^{\rm alg}}$. Elements of $\mathcal{V}(X)$ are called vertices of X. Open edges of $\mathbf{S}(X)$ are images in X of open edges of $X\widehat{\otimes}\widehat{k^{\rm alg}}$; the set $\mathcal{E}(X)$ of open edges of X is again the set of connected components of $S(X) \setminus \mathcal{V}(X)$. In this simple case, an open edge connects precisely two vertices u, v and their ordering "orients" E *i.e.* it canonically determines a homeomorphism $\rho : E \xrightarrow{\sim} (r,1)$. If k is trivially valued, we define $\mathbf{S}(X)$ as the image in X of $\mathbf{S}(X\widehat{\otimes}k')$, for any non-trivially valued non-archimedean extension field k'/k. We call the triple $(S(X), \mathcal{V}(X), \mathcal{E}(X))$, a "subpolygon" of X.

Suppose we are given a system of differential equations on the affinoid domain $X \subset \mathbb{A}^1$ with analytic coefficients, *i.e.* a free $\mathcal{O}_X$-Module of finite rank $\mathcal{F}$ with a connection ∇. If k is algebraically closed, X is strict, and $x \in X(k)$ is a k-rational point of X, we define the *normalized* radius of convergence $\mathcal{R}(x) = \mathcal{R}_X(x, (\mathcal{F}, \nabla))$ as the radius of convergence of $\mathcal{F}^\nabla$ around x in the corresponding maximal open disk neighborhood $\mathcal{D}_x(X) \cong D(0,1^-)$. If k, X and x are arbitrary, we consider an algebraically closed non-archimedean field k' over k such that $X\widehat{\otimes}_k k'$ is strict and so large that $\mathscr{H}(x)$ admits an isometric k-embedding φ in k'. Then, there is a canonical point $x' \in X\widehat{\otimes}_k k'$ and we set $\mathcal{R}(x) = \mathcal{R}(x')$, a quantity independent of the choice of k' and of φ. The function $\mathcal{R} : X \to \mathbb{R}_{>0}$ does not depend on the embedding of X in $\mathbb{A}^1$, either, and one has $R(x) = \mathcal{R}(x) \cdot r(\tau_X(x))$ for all $x \in X$. The function $r(\tau_X(x)) = \delta_{\widehat{\mathbb{P}^1}}(x, X)$ was called the *diameter of* X *at* x (with respect to the embedding $X \hookrightarrow \mathbb{A}^1 \hookrightarrow \mathbb{P}^1 = (\widehat{\mathbb{P}^1})_\eta$) in [4]; its continuity (at least under the present assumptions) was proven in *loc.cit.* [3.3]. One can easily show (see §3) that the function

$\mathcal{R}(x)$ is preserved by extensions of the ground field, and so it suffices to study its behavior in the case when X is strictly k-affinoid.

We now notice that, independently of the characteristic, if k is algebraically closed and non-trivially valued (resp. trivially valued), a strictly k-affinoid, as well as a projective, rig-smooth curve X is the generic fiber $\mathfrak{X}_\eta$ of a strictly semistable[2] formal scheme (resp. the analytification of a smooth scheme) $\mathfrak{X}$ over k° (resp. over $k^\circ = k$): we say that $\mathfrak{X}$ is a *strictly semistable model* (resp. a *smooth model*) of X. For a general non-trivially valued non-archimedean field k, and any compact rig-smooth strictly k-analytic curve X, there exists a finite galois extension k'/k such that $X \otimes k'$ admits a semistable model $\mathfrak{X}'$ equipped with an action of $H = \mathrm{Gal}\,(k'/k)$, whose restriction to the generic fiber $\mathfrak{X}'_\eta = X \otimes k'$ expresses the descent datum to X. The quotient k°-formal scheme $\mathfrak{X} := \mathfrak{X}'/H$ is unfortunately not, in general, topologically of finite type. Still, it is normal, flat over k°, and is endowed with a morphism of G-ringed spaces $\mathrm{sp}_{\mathfrak{X}} : X_G \to \mathfrak{X}$, where X_G denotes the G-analytic space associated to X *cf.* [7, 1.3], called the *specialization morphism.* We say that $\mathfrak{X}$ a *Q-semistable*[3] *model of* X and that X is its *generic fiber.* We obtain (see below definition 1.1.13) the category $\mathcal{FS}_k^Q$ of *Q-semistable k°-formal schemes* which is fibered over the category of compact rig-smooth strictly k-analytic curves. The fiber category $\mathcal{FS}^Q(X)$ of $\mathcal{FS}_k^Q$ over the curve X, is in fact an ordered set (*i.e.* a morphism $\mathfrak{Y} \to \mathfrak{X}$ inducing the identity on the generic fiber is necessarily unique and we set $\mathfrak{X} \leq \mathfrak{Y}$ if there is one) with nice reticular properties. Notice that, if the valuation of k is discrete, a Q-semistable k°-formal scheme is admissible and normal, but not necessarily semistable.

For a general non-trivially (resp. trivially) valued non-archimedean field k, to any Q-semistable formal (resp. to any smooth) scheme $\mathfrak{X}$ over k°, one associates (*cf.* [6, Chap. IV]), compatibly with galois actions, a triple of the previous type $\mathbf{S}(\mathfrak{X}) = (S(\mathfrak{X}), \mathcal{V}(\mathfrak{X}), \mathcal{E}(\mathfrak{X}))$ supported on a closed subset $S(\mathfrak{X}) \subset X := \mathfrak{X}_\eta$, called the *skeleton* of $\mathfrak{X}$. Here, an open edge E is a locally closed subset of X, canonically homeomorphic to either $[r, 1)$ or $(r, 1)$ (up to orientation, in the second case), for $r \in |k^{\mathrm{alg}}| \cap (0, 1)$; it may have one or two boundary points in $X \setminus E$, necessarily vertices, there may be loops, and two vertices may be connected by more (or less) than one open edge. Despite the fact that $\mathfrak{X}_s$ may have multiple components, we do not here attribute multiplicities to vertices nor to edges of $\mathbf{S}(\mathfrak{X})$. We call a structure of this type, a *Q-subpolygon* of X. We proceed in our general description under the assumption that k is non-trivially valued.

If $\mathfrak{X} \leq \mathfrak{Y}$ are Q-semistable models of X, $\mathbf{S}(\mathfrak{X})$ is a "Q-subpolygon" of $\mathbf{S}(\mathfrak{Y})$, in the sense that $S(\mathfrak{X}) \subset S(\mathfrak{Y})$, $\mathcal{V}(\mathfrak{X}) \subset \mathcal{V}(\mathfrak{Y})$, and an open edge of $\mathbf{S}(\mathfrak{X})$ is a union of vertices and open edges of $\mathbf{S}(\mathfrak{Y})$. There is a natural continuous retraction $\tau_{\mathfrak{X},\mathfrak{Y}} : S(\mathfrak{Y}) \to S(\mathfrak{X})$. Now, any strictly k-affinoid curve X, as much as any projective smooth curve which is neither rational nor a twisted Tate curve, admits a *minimum* Q-semistable model $\mathfrak{X}_0$ and the skeleton $\mathbf{S}(X)$ of X coincides with the skeleton $\mathbf{S}(\mathfrak{X}_0)$ of the formal scheme $\mathfrak{X}_0$[4]. Moreover, as a topological space, $X = \varprojlim_{\mathfrak{Y} \geq \mathfrak{Z}} (S(\mathfrak{Y}), \tau_{\mathfrak{Z},\mathfrak{Y}})$ where $\mathfrak{Y}$ and $\mathfrak{Z}$ run over a cofinal system of Q-semistable models of X, is a quasi-polyhedron (*cf.* [6, IV.1]) and, if the minimum Q-semistable model $\mathfrak{X}_0$ of X exists, the retraction $\tau_X : X \to S(X)$ is the natural projection to $S(\mathfrak{X}_0) = S(X)$. A semistable formal scheme $\mathfrak{X}$ over k° is an example of a nondegenerate polystable formal scheme over k° (see [10]). In the case of dimension one over k°, the class of such formal

[2]When k° is not noetherian, this definition is not completely standard *cf.* (1.1.5) below.

[3]We are grateful to Takeshi Saito for suggesting to us this name, in which "Q" stands for "quotient of".

[4]For a rational curve X, $S(X)$ is empty, while for any minimal Q-semistable model $\mathfrak{X}$ of X, $\mathbf{S}(X) = (\eta, \{\eta\}, \emptyset)$, for a point $\eta \in X$ with $\widetilde{\mathscr{H}(\eta)}$ of transcendence degree one over $\widetilde{k}$. For a twisted Tate curve X, a choice of $\eta \in S(X)$ determines either a homeomorphism $S(X) \xrightarrow{\sim} (\mathbb{R}_{>0}, \cdot)/r^{\mathbb{Z}} \approx S^1$, $\eta \mapsto 1$, up to orientation, or a homeomorphism $S(X) \xrightarrow{\sim} [r, 1]$, $\eta \mapsto 1$, for $r \in |k^{\mathrm{alg}}| \cap (0, 1)$. In this case, $\mathbf{S}(X) = (S(X), \emptyset, \{S(X)\})$. The choice of a minimal Q-semistable model $\mathfrak{X}$ of X corresponds to the choice of $\eta \in S(X)$: $\mathbf{S}(\mathfrak{X}) = (S(X), \{\eta\}, \{S(X) \setminus \{\eta\}\})$.

schemes coincides with the class of semistable formal schemes. Recall [10, 5.2] that every nondegenerate polystable formal scheme $\mathfrak{X}$ has a skeleton $S(\mathfrak{X}) \subset \mathfrak{X}_\eta$ and a retraction map $\tau_{\mathfrak{X}} : \mathfrak{X}_\eta \to S(\mathfrak{X})$ which are preserved under any ground field extension functor. Furthermore, if x is a Q-rational point of $\mathfrak{X}_\eta$, then there is a well-defined maximum open neighborhood $D_{\mathfrak{X}}(x, 1^-)$ of x in $\mathfrak{X}_\eta \setminus S(\mathfrak{X})$, which is isomorphic to a galois quotient of the standard open unit polydisk $D_{k'}(0, 1^-)/\mathrm{Gal}\,(k'/k)$ (with $x \mapsto 0$), for a sufficiently big finite galois extension k'/k, so that the open or closed polydisks $D_{\mathfrak{X}}(x, r^\pm)$, $r < 1$ with center at any Q-rational point of $\mathfrak{X}_\eta$ are well-defined (we are talking about polydisks with equal radii).

Let k be any non-archimedean field. *All k-analytic spaces considered in this paper are supposed to be separated.* We generalize a definition of [12, 1.1], as follows.

Definition 0.1.2. *A k-analytic space X is said to be* rig-smooth *(resp.* of pure dimension *n) if, for any non-archimedean field k' over k, and any connected strictly affinoid domain $V \subset X\widehat{\otimes}k'$, Ω^1_V is a locally free $\mathcal{O}_V$-Module (resp. of rank n).*

Assume now k is of characteristic zero, let X be a rig-smooth k-analytic space, and let X_G be the associated G-analytic space. We denote by $\mathbf{MIC}(X/k)$ the category of pairs consisting of a locally free $\mathcal{O}_{X_G}$-Module $\mathcal{F}$ of finite type and of an integrable k-linear connection ∇ on $\mathcal{F}$, with the usual (horizontal) morphisms. Notice that, unlike in the classical case, for an object $(\mathcal{F}, \nabla)$ of $\mathbf{MIC}(X/k)$, the abelian sheaf

$$\mathcal{E}^\nabla = \mathrm{Ker}\,(\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_{X/k})$$

for the G-topology of X is not in general locally constant.

Definition 0.1.3. *Let k be a non-archimedean field of characteristic zero, and X be a rig-smooth k-analytic space. We say that an object $(\mathcal{F}, \nabla)$ of $\mathbf{MIC}(X/k)$ is a* Robba connection on X *if for any algebraically closed complete extension k' of k, and any open polydisk $D' \subset X' = X\widehat{\otimes}k'$, if we denote by $(\mathcal{E}', \nabla')$ the k'-linear extension of $(\mathcal{E}, \nabla)$ on X', the sheaf $(\mathcal{E}'^{\nabla'})_{|D'} = (\mathcal{E}'_{|D'})^{\nabla'_{|D'}}$ is constant. We denote by $\mathbf{MIC}^{Robba}(X/k)$ the full abelian subcategory of $\mathbf{MIC}(X/k)$, consisting of Robba connections.*

Example 0.1.4. Let $\mathbb{C}$ be the complex field, equipped with the trivial valuation. On the $\mathbb{C}$-analytic space $\mathbb{A}^1_{\mathbb{C}}$, we consider a system of linear differential equations Σ of the form (0.1.0.1), with G a matrix of rational functions in $\mathbb{C}(T)$. To analyze the singularity of Σ at $T = 0$, we consider the open analytic domain

$$X = \{x \in \mathbb{A}^1_{\mathbb{C}} \,|\, 0 < |T(x)| < 1\,\} \subset \mathbb{A}^1_{\mathbb{C}} \,.$$

Notice that $X = S(X) \cong (0, 1)$ via $\rho \mapsto t_{0,\rho}$. The field $\mathscr{H}(t_{0,\rho})$ is the non-archimedean field over $\mathbb{C}$, $\mathcal{K}_\rho = \mathbb{C}((T))$, equipped with the absolute value $||\ ||_\rho$, such that $||0||_\rho = 0$ and $||\sum_n a_n T^n||_\rho = \rho^m$, if $a_n = 0$, for $n < m$ and $a_m \neq 0$. Let $X' := X\widehat{\otimes}\mathcal{K}_\rho$, and let $t'_{0,\rho}$ be the canonical $\mathcal{K}_\rho$-rational point in X' above $t_{0,\rho}$. Then, writing $t = T(t_{0,\rho})$, the maximal open disk containing $t'_{0,\rho}$ in X' is $D_{\mathcal{K}_\rho}(t'_{0,\rho}, \rho^-)$, with normalized coordinate $\dfrac{T-t}{t}$, and the fact that Σ is a Robba connection is equivalent to the property that the singularity at $T = 0$ is regular. In fact, by the formal Turrittin theory [3], it is enough to check this statement for the rank one differential equations

$$\frac{dy}{dT} = \frac{\alpha}{T}\, y \;,\; \alpha \in \mathbb{C} \;,$$

and

$$\frac{dy}{dT} = \frac{\alpha}{T^N}\, y \;,\; \alpha \in \mathbb{C}^\times \;,\; N = 2, 3, \dots \;.$$

In the former case the solution at $t'_{0,\rho}$ is $y = \sum_i \binom{\alpha}{i}(\frac{T-t}{t})^i$, converging for $|\frac{T-t}{t}| < 1$. In the latter case a simple calculation shows that the solution at $t'_{0,\rho}$ is a power series $\sum a_i(t)(\frac{T-t}{t})^i \in \mathcal{K}_\rho[[\frac{T-t}{t}]]$, having the same radius of convergence as the series $\exp(t^{-N}\frac{T-t}{t})$, that is converging for $|\frac{T-t}{t}| < |t|^N$. We recall from *loc.cit.* that N is the Poincaré-Katz rank of irregularity of the corresponding differential equation at $T = 0$.

Notice that if $(\mathcal{F}, \nabla)$ is any connection, the natural map

$$\mathcal{E}^\nabla \otimes_k \mathcal{O}_X \longrightarrow \mathcal{E} \ , \tag{0.1.4.1}$$

is an isomorphism. So, If D is an open polydisk[5] in X $(\mathcal{F}, \nabla)_{|D}$ is isomorphic to the trivial connection $(\mathcal{O}_D, d_{D/k})^\mu$, for $\mu = \mathrm{rk}\, \mathcal{F}_{|D}$.

Assume now that the k-analytic space X is isomorphic to the generic fiber $\mathfrak{X}_\eta$ of a nondegenerate polystable formal scheme $\mathfrak{X}$ over k°. For any object $(\mathcal{F}, \nabla)$ of $\mathbf{MIC}(X/k)$, and for a k-rational point $x \in \mathfrak{X}_\eta$, we may define the ($\mathfrak{X}$-*normalized*) *radius of convergence* $\mathcal{R}(x) = \mathcal{R}_\mathfrak{X}(x, (\mathcal{F}, \nabla))$ (3.1.3.5). For $x \in X(k)$, it is the supremum of $r \in (0, 1]$ such that the restriction of $\mathcal{F}^\nabla$ to $D_\mathfrak{X}(x, r^-)$ is constant. Notice that the restriction of $\mathcal{F}^\nabla$ to $D_\mathfrak{X}(x, \mathcal{R}(x)^-)$ is then locally constant *for the usual topology*[6], and is therefore constant, since an open disk is simply connected for the usual topology. If x is arbitrary, we set $\mathcal{R}(x) = \mathcal{R}(x')$, where x' is the canonical $\mathscr{H}(x)$-rational point in $\mathfrak{X}_\eta \widehat{\otimes} \mathscr{H}(x) = (\mathfrak{X} \widehat{\otimes}_{k^\circ} \mathscr{H}(x)^\circ)_\eta$ over x. Let $\varphi : \mathfrak{Y} \to \mathfrak{X}$ be an étale morphism of nondegenerate polystable formal schemes over k°, and let $\varphi_\eta : Y \to X$ be the generic fiber of φ. For any object $(\mathcal{F}, \nabla)$ of $\mathbf{MIC}(X/k)$, $\varphi_\eta^*(\mathcal{F}, \nabla)$ is an object of $\mathbf{MIC}(Y/k)$ and for any point $y \in Y$ we have

$$\mathcal{R}_\mathfrak{Y}(y, \varphi_\eta^*(\mathcal{F}, \nabla)) = \mathcal{R}_\mathfrak{X}(\varphi_\eta(y), (\mathcal{F}, \nabla)) \ . \tag{0.1.4.2}$$

Under the same assumptions, if k'/k is any extension of non-archimedean fields, the k'°-formal scheme $\mathfrak{X}' := \mathfrak{X}\widehat{\otimes} k'^\circ$ is nondegenerate polystable, and, $\psi : \mathfrak{X}' \to \mathfrak{X}$ is the natural projection

$$\mathcal{R}_{\mathfrak{X}'}(x, \psi_\eta^*(\mathcal{F}, \nabla)) = \mathcal{R}_\mathfrak{X}(\psi_\eta(x), (\mathcal{F}, \nabla)) \ . \tag{0.1.4.3}$$

Conjecture 0.1.5. *Let $X = \mathfrak{X}_\eta$ be the generic fiber $\mathfrak{X}_\eta$ of a nondegenerate polystable formal scheme $\mathfrak{X}$ over k° and $(\mathcal{F}, \nabla)$ be an object of* $\mathbf{MIC}(X/k)$*. The function $X \to \mathbb{R}_{>0}$, $x \mapsto \mathcal{R}_\mathfrak{X}(x, (\mathcal{F}, \nabla))$ is continuous.*

Notice that, by formula 0.1.4.2, the conjecture holds if it holds under the restrictive condition that $\mathfrak{X}$ is strictly polystable over k° (base change to a finite extension field k'/k is irrelevant). Let $\mathbf{MIC}_\mathfrak{X}(X/k)$ be the full subcategory of $\mathbf{MIC}(X/k)$ consisting of the objects $(\mathcal{F}, \nabla)$, such that $\mathcal{F}$ is the $\mathcal{O}_X$-Module associated to some locally free $\mathcal{O}_\mathfrak{X}$-Module of finite type $\mathfrak{F}$, called a *formal model* of $\mathcal{F}$ *over* $\mathfrak{X}$. In case $\mathcal{F}$ admits a formal model $\mathfrak{F}$, the calculation of $\mathcal{R}(x)$ may be performed as follows. We pick an open affine neighborhood $\mathfrak{Y}$ of the specialization of x in $\mathfrak{X}$ such that $\mathfrak{F}$ is free on $\mathfrak{Y}$. Then $x \in \mathfrak{Y}_\eta$, $D_\mathfrak{X}(x, 1^-) = D_\mathfrak{Y}(x, 1^-)$, and $\mathcal{R}_\mathfrak{X}(x, (\mathcal{F}, \nabla)) = \mathcal{R}_\mathfrak{Y}(x, (\mathcal{F}, \nabla)_{|\mathfrak{Y}_\eta})$ is the maximum common radius of convergence of all sections of $\mathcal{F}^\nabla$ in $D_\mathfrak{Y}(x, 1^-)$ around x, when expressed in terms of a basis of global sections of $\mathfrak{F}$ on $\mathfrak{Y}$. In dimension one, this amounts to solving a system of differential equations as in (0.1.0.1), *over an affinoid domain in a smooth projective curve.*

[5] more precisely, any open analytic domain in X which becomes isomorphic to a standard open polydisk over a finite separable extension of k

[6] This means for the topology of a Berkovich analytic space.

Conjecture 0.1.6. *Let $X = \mathfrak{X}_\eta$ be the generic fiber $\mathfrak{X}_\eta$ of a nondegenerate polystable formal scheme $\mathfrak{X}$ over k° and $(\mathcal{F}, \nabla)$ be an object of $\mathbf{MIC}_\mathfrak{X}(X/k)$. Then $(\mathcal{F}, \nabla)$ is a Robba connection on X if and only if the function $x \mapsto \mathcal{R}_\mathfrak{X}(x, (\mathcal{F}, \nabla))$ is identically 1 on X.*

And here is our main result, where we are able to replace "semistable" by the more general notion of "Q-semistable": for the precise definition of the category $\mathbf{MIC}_\mathfrak{X}(X/k)$ in this more general context, see the body of this paper.

Theorem 0.1.7. *Let k be a non-archimedean field extension of $\mathbb{Q}_p$ and X be a rig-smooth compact strictly k-analytic curve. Conjectures 0.1.5 and 0.1.6 hold for objects of $\mathbf{MIC}_\mathfrak{X}(X/k)$, where $\mathfrak{X}$ is any Q-semistable model of X.*

Surprisingly enough, before our joint paper with Di Vizio [4], conjecture 0.1.5 seemed to be open even in the case when $\mathfrak{X} = \mathrm{Spf}\, k^\circ\{T\}$, hence $\mathfrak{X}_\eta = X$ is the closed unit disk $D_k(0, 1^+)$, a case extensively discussed in the literature (*cf.* [35], [26] and [21] for reference, and [40, 10.4.3] for a partial result in our direction). A direct proof, in the case of an affinoid domain X of $\mathbb{A}^1$, was given in [4, 5.3]. Notice that, always by (0.1.4.2), it suffices to prove theorem 0.1.7 in the special case when $\mathfrak{X}$ is either a "formal disk" or a "formal annulus", *i.e.* is affine connected and admits a dominant étale morphism φ to $\mathrm{Spf}\, k^\circ\{X\}$ or to $\mathrm{Spf}\, k^\circ\{X, Y\}/(XY - a)$, for $a \in k^{\circ\circ} \setminus \{0\}$, respectively. The previously mentioned result of [4], implies the theorem when φ is an open immersion. In the general case however, we must appeal to the more powerful statement in section 2 below.

Our detailed description of the function $x \mapsto \mathcal{R}_\mathfrak{X}(x, (\mathcal{F}, \nabla))$ will in fact show the following

Theorem 0.1.8. *Let k, X and $\mathfrak{X}$ be as in theorem 0.1.7, and let $(\mathcal{F}, \nabla)$ be an object of $\mathbf{MIC}_\mathfrak{X}(X/k)$. If $\mathcal{R}_\mathfrak{X}(\xi, (\mathcal{F}, \nabla)) = 1$ at any vertex ξ of the skeleton $\mathbf{S}(\mathfrak{X})$ of $\mathfrak{X}$, then $x \mapsto \mathcal{R}_\mathfrak{X}(x, (\mathcal{F}, \nabla))$ is identically 1 on X. Moreover, $(\mathcal{F}, \nabla)$ is a Robba connection on X.*

Definition 0.1.9. *Let k, X be as in 0.1.7 and let $\mathfrak{X}$ be a Q-semistable formal model of X. An object $(\mathcal{F}, \nabla)$ of $\mathbf{MIC}(X/k)$ is said to be* $\mathfrak{X}$-convergent *if it is an object of $\mathbf{MIC}_\mathfrak{X}(X/k)$ and $\mathcal{R}_\mathfrak{X}(x, (\mathcal{F}, \nabla)) = 1$ identically on X. We denote by $\mathbf{MIC}^{\mathfrak{X}-conv}(X/k)$ the full subcategory of $\mathbf{MIC}(X/k)$ consisting of $\mathfrak{X}$-convergent objects.*

Let $\varphi : \mathfrak{Y} \to \mathfrak{X}$ be a Q-étale[7] morphism of Q-semistable k°-formal schemes and let $\varphi_\eta : Y \to X$ be its generic fiber. It follows from formula 0.1.4.2 that an object $(\mathcal{F}, \nabla)$ of $\mathbf{MIC}_\mathfrak{X}(X/k)$ is $\mathfrak{X}$-convergent if and only if the object $\varphi_\eta^*(\mathcal{F}, \nabla)$ of $\mathbf{MIC}_\mathfrak{Y}(Y/k)$ is $\mathfrak{Y}$-convergent.

Corollary 0.1.10. *The category $\mathbf{MIC}^{Robba}(X/k)$ is the 2-colimit category of its full subcategories $\mathbf{MIC}^{\mathfrak{X}-conv}(X/k)$, where $\mathfrak{X}$ runs over the Q-semistable models of X.*

Notice that, despite the previous result, the introduction of formal models is necessary for several reasons:

1. The analytic definition of $\mathcal{D}_x(X)$ given above for x a k-rational point of the smooth affinoid X, as the maximum open neighborhood of x in X isomorphic to the polydisk $D(0, 1^-)$, does not work in higher dimensions, where such a *maximum* neighborhood need not exist.

2. Even in the case of an affinoid $X \subset \mathbb{A}^1$, and $x \in X(k)$, the above analytic definition of $\mathcal{D}_x(X)$, does not globalize well. If $X = X_1 \cup X_2$ is a union of two affinoids, and x is a k-rational point of X_1, one can only say that $\mathcal{D}_x(X_1)$ and $\mathcal{D}_x(X_2)$ are contained in $\mathcal{D}_x(X)$. The assignment of a semistable model $\mathfrak{X}$ of X, remedies to this problem, *i.e.* $D_{\mathfrak{X}_1}(x, 1^-) = D_\mathfrak{X}(x, 1^-)$, provided the decomposition $X = X_1 \cup X_2$ originates from a Zariski open covering $\mathfrak{X} = \mathfrak{X}_1 \cup \mathfrak{X}_2$ of formal schemes. Notice that if X is the analytic projective line, there is no maximum disk containing a given point.

[7]defined below in the paper *cf.* definition 1.1.15.

3. According to conjecture 0.1.6, consideration of the formal model $\mathfrak{X}$ of X should be of lesser importance in the study of Robba equations. It seems instead to be unavoidable in the case of connections with weaker convergence properties.

We finally propose, for future use, a more general definition, where we consider various possible topologies on categories of analytic spaces over X. We have in mind, in particular, the natural (Berkovich) topology on X, the G-topology on X described in [7, 1.3], the étale topology of [7, 4.1], the quasi-étale topology of [8, §3].

Definition 0.1.11. *Let k and X be as in definition 0.1.3 and let τ be a topology on X. An object $(\mathcal{F}, \nabla)$ of $\mathbf{MIC}(X/k)$ will be said to be* τ-convergent *if, τ-locally, $(\mathcal{F}, \nabla)$ is a Robba connection. We will denote by $\mathbf{MIC}^{\tau-conv}(X/k)$ the full subcategory of $\mathbf{MIC}(X/k)$ consisting of τ-convergent objects. For τ the natural (resp. the G-, resp. the étale, resp. the quasi-étale) topology, $\mathbf{MIC}^{\tau-conv}(X/k)$ will be denoted by $\mathbf{MIC}^{conv}(X/k)$ (resp. $\mathbf{MIC}^{G-conv}(X/k)$, resp. $\mathbf{MIC}^{et-conv}(X/k)$, resp. $\mathbf{MIC}^{qet-conv}(X/k)$).*

0.2 Basic definitions and description of contents

The organization of the paper is as follows. In section 1 we review the structure of a rig-smooth compact connected strictly k-analytic curve X over a non-archimedean field k with non-trivial valuation and the parallel theory of semistable reduction of k-algebraic curves. Most, but not all, of these results may be extracted from [6, Chap. IV] or [15]. A source of complication in our treatment, which makes it slightly non-standard, is the fact that we do not assume k to be algebraically closed nor its valuation to be discrete. This causes the systematic appearance of formal schemes that are not topologically of finite type. We sketch the equivalence between the notions of Q-semistable model of X, of formal affinoid covering of X with potentially semistable reduction, of *Q-semistable partition*[8] of X, and of complete Q-subpolygon of X. A consequence of these equivalences is that all results on Q-semistable reduction of curves admit in principle both an algebraic and an analytic proof.

The algebraic theory of semistable models of families of algebraic curves [23] has recently made important progress thanks to Temkin [41], who has eliminated in particular the properness assumption, and has clarified minimality of the stable model. It should be possible to apply Temkin's method to completely describe the reticular structure of the category $\mathcal{FS}^Q(X)$ of Q-semistable models of X, and in particular to showing that, unless X is either rational or a twisted Tate curve, a minimum Q-semistable model $\mathfrak{X}_0$ of X exists, and its skeleton coincides with the analytically defined skeleton of X. We have not seriously attempted to follow this algebraic strategy. Instead, we approach the problem via an analytic discussion of intersections and unions of disks and annuli in the style of [15, Prop. 5.4]: we content ourselves with a precise account of the proofs. Details, which would lead us astray of the main path, have been collected in [5].

An open or closed *k-disk* (resp. *k-annulus*) is a k-analytic curve isomorphic to a standard open or closed disk (resp. annulus) in $\mathbb{A}^1_k$ centered at 0. A *k-punctured open disk* is an open k-disk punctured at a k-rational point. For X as above, we say that an open or closed connected analytic domain V in X is an open (or closed) *Q-disk* (resp. *Q-annulus*) if there exists a finite galois extension k'/k such that $V \otimes k'$ is a disjoint union of a finite number of open (or closed) k'-disks (resp. annuli) in $X' := X \otimes k'$, with radius (resp. radii) in $|k'|$. Similarly for a *Q-punctured open disk* in X.

We discuss some properties of disks and annuli. In particular, open Q-annuli in X may be *non-split* or *split*, depending on whether, viewed as simply connected quasi-polyhedra, they have one or two endpoints. An *open Q-segment* (resp. *open Q-half-line*) in X is the skeleton of an open Q-annulus (resp. of an open Q-punctured disk) in X: the Q-annulus (resp. the

[8] See the definition below.

Q-punctured disk) is then uniquely defined. In general, an open Q-segment E is a locally closed subset of X and it admits a continuous $(1:1)$ parametrization $\rho \mapsto \eta_\rho$ by an open interval $(r,1) \subset (0,1)$ (resp. $[r,1) \subset (0,1)$), with $r \in \sqrt{|k^\times|}$, which is canonical up to the inversion $\rho \mapsto r/\rho$ (resp. canonical, with η_r an interior point of the corresponding annulus) if E is split (resp. non-split). Similarly, an open Q-half-line is canonically parametrized by $(0,1)$, where η_ρ tends to a point in X_0, as $\rho \to 0$.

We define the category $\mathcal{GP}(X)$ (resp. $\mathcal{GP}^c(X)$) of (resp. complete) Q-subpolygons $\Gamma = (|\Gamma|, \mathcal{V}_\Gamma, \mathcal{E}_\Gamma)$ of X. Here $|\Gamma|$ is a closed connected subset of X, $\mathcal{V}_\Gamma$ is a finite subset of points of type (2) of X ("vertices" of Γ) and $\mathcal{E}_\Gamma$ is a finite set of open Q-segments of X ("open edges" of Γ), such that $|\Gamma|$ is the disjoint union of all vertices and open edges of Γ.

A Q-subpolygon Γ of X is "complete" if $X \setminus |\Gamma|$ is a union of open Q-disks. The category $\mathcal{GP}^c(X)$ is really a directed partially ordered set, where $\Gamma \leq \Gamma'$ if $|\Gamma| \subset |\Gamma'|$, $\mathcal{V}_\Gamma \subset \mathcal{V}_{\Gamma'}$, so that an open edge of Γ is a union of open edges and vertices of Γ'. We prove (more details in [5]) that $\mathcal{GP}^c(X)$ is isomorphic to the category $\mathcal{FS}^Q(X)$ of Q-semistable formal models of X and morphisms inducing the identity on the generic fiber (they turn out to be precisely quotients of the "admissible blow-ups" of [16], [13]). The isomorphism $\mathbf{S} : \mathcal{FS}^Q(X) \xrightarrow{\sim} \mathcal{GP}^c(X)$ is the *skeleton functor*. Unless X is a rational projective curve or a twisted form of a Tate curve, the category $\mathcal{FS}^Q(X)$ admits a minimum $\mathfrak{X}_0$. The skeleton $\mathbf{S}(\mathfrak{X}_0) = (S(\mathfrak{X}_0), \mathcal{V}(\mathfrak{X}_0), \mathcal{E}(\mathfrak{X}_0))$ is the skeleton $\mathbf{S}(X) = (S(X), \mathcal{V}(X), \mathcal{E}(X))$ of the analytic curve X, in the sense of the previous subsection.

We define below *cf.* (1.3.2) a "basic formal disk" (resp. a "basic formal annulus") $(\mathfrak{B}, T)$ (resp. or $(\mathfrak{B}, S = a/T)$, for $a \in k^{\circ\circ}\setminus\{0\}$). We write $(\mathcal{B}, T)$ for the corresponding affinoid disk (resp. annulus). The coordinate T in $\mathfrak{B}$ defines, for any $z \in \mathcal{B}(k)$, a normalized coordinate $T_z : D_{\mathfrak{B}}(z, 1^-) \xrightarrow{\sim} D_k(0, 1^-)$, with $z \mapsto 0$, by

$$T_z(y) = \begin{cases} T(y) - T(z)\ , & \text{if } (\mathcal{B}, T) \text{ is a disk}\ , \\ \\ (T(y) - T(z))/T(z)\ , & \text{if } (\mathcal{B}, T) \text{ is an annulus}\ , \end{cases} \qquad (0.2.0.1)$$

$\forall y \in D_{\mathfrak{B}}(z, 1^-)$. Notice that here $\mathfrak{B}$ is the minimum semistable model of $\mathcal{B}$, so that $\mathbf{S}(\mathfrak{B}) = \mathbf{S}(\mathcal{B})$, $D_{\mathfrak{B}}(z, 1^-) = D_{\mathcal{B}}(z, 1^-)$, $\tau_{\mathcal{B}} = \tau_{\mathfrak{B}}$ and $|T(x)| = |T(\tau_{\mathcal{B}}(x))| = r(T(\tau_{\mathcal{B}}(x)))$, for any $x \in \mathcal{B}$.
The main point for us is the fact that any Q-semistable model $\mathfrak{X}$ of the curve X is a quotient of a semistable model $\mathfrak{X}'$ of $X \widehat{\otimes} k'^{\circ}$ for some finite galois extension k'/k and $\mathfrak{X}'$ admits an étale covering by formal disks or annuli.

This allows us to recover and globalize most results of the classical non-archimedean theory of linear differential systems [26].

In section 2 we prove a general criterion due to Berkovich (private communication) to test whether a function $X \to \mathbb{R}_{>0}$, on a smooth k-affinoid curve X is continuous: it represents an abstract and more efficient version of the method of proof used in [4, 5.3]. In section 3, we start assuming that k has characteristic zero. We pick a semistable model $\mathfrak{X}$ of X and introduce the function "normalized radius of convergence at x" $\mathcal{R}_{\mathfrak{X}}(x, (\mathcal{F}, \nabla))$, for an object $(\mathcal{F}, \nabla)$ of $\mathbf{MIC}(X/k)$. Actually, in view of the most common applications, we also consider the following more general situation: X is the complement of a finite reduced divisor $\mathcal{Z} = \{z_1, \dots, z_r\}$ in a compact rig-smooth connected k-analytic curve $\overline{X}$. So, if $\overline{X} = \mathcal{M}(\mathscr{A})$, $z_1, \dots, z_r$ correspond to r distinct maximal ideals of the k-affinoid algebra $\mathscr{A}$. We assume that $\overline{X}$ is the generic fiber of a Q-semistable k°-formal scheme $\overline{\mathfrak{X}} = \overline{\mathfrak{X}}'/H$, represented over a finite galois extension k'/k of group H, by the semistable k'°-formal scheme $\overline{\mathfrak{X}}'$. It is well known that the inverse image $\mathcal{Z}'$ of $\mathcal{Z}$ in $\overline{X}' = \overline{X} \otimes k'$, determines a finite flat generically étale closed reduced subscheme $\mathfrak{Z}'$ of $\overline{\mathfrak{X}}'$. We denote by $\mathfrak{Z}$ the quotient $\mathfrak{Z}'/H$, a flat normal generically étale closed formal subscheme of $\overline{\mathfrak{X}}$ of relative dimension zero. We will assume that $\mathfrak{Z}$ is Q-*étale* over k°, meaning that, for a suitable choice of k'/k, $\mathfrak{Z}'$ is étale over k'°. For the limited purposes of this paper, consideration of $\mathfrak{Z}$ will not be necessary, and the previous assumption will concealed under the requirement that the disks $D_{\overline{\mathfrak{X}}}(z_1, 1^-), \dots, D_{\overline{\mathfrak{X}}}(z_r, 1^-)$ are distinct and that their boundary points $\tau_{\overline{\mathfrak{X}}}(z_1), \dots, \tau_{\overline{\mathfrak{X}}}(z_r)$ are vertices of $\mathbf{S}(\overline{\mathfrak{X}})$. We keep the notational distinction mainly for future use.

For any object $(\mathcal{F}, \nabla)$ of $\mathbf{MIC}(X/k)$, we define the function $x \mapsto \mathcal{R}_{\overline{\mathfrak{X}},\mathfrak{Z}}(x, (\mathcal{F}, \nabla))$, $X \to (0, 1]$. We restrict our attention to the abelian tannakian subcategory $\mathbf{MIC}(\overline{X}(*\mathcal{Z})/k)$ of $\mathbf{MIC}(X/k)$, consisting of the objects $(\mathcal{F}, \nabla)$ *having meromorphic singularities at* $\mathcal{Z}$. This means that $\mathcal{F}$ is the restriction to X of a coherent and locally free $\mathcal{O}_{\overline{X}}$-Module $\overline{\mathcal{F}}$ and for any open $U \subset \overline{X}$ and any section $e \in \Gamma(U, \overline{\mathcal{F}})$, there is a non-zero $f \in \mathcal{O}_{\overline{X}}(U)$, such that $f\nabla(e) \in \Gamma(U, \overline{\mathcal{F}} \otimes \Omega^1_{\overline{X}})$. Morphisms are horizontal $\mathcal{O}_X$-linear maps, with meromorphic poles at $\{z_1, \dots, z_r\}$.

We are mostly concerned with the full abelian tannakian subcategory $\mathbf{MIC}_{\overline{\mathfrak{X}}}(\overline{X}(*\mathcal{Z})/k)$ of $\mathbf{MIC}(\overline{X}(*\mathcal{Z})/k)$. It consists of objects $(\mathcal{F}, \nabla)$ of $\mathbf{MIC}(\overline{X}(*\mathcal{Z})/k)$, for which there exists a finite galois extension k'/k of group H and a semistable model $\overline{\mathfrak{X}}'$ of $\overline{X}' = X \otimes k'$ with $\mathfrak{X} = \overline{\mathfrak{X}}'/H$, and such that the $\mathcal{O}_X$-Module $\mathcal{F}$ is the restriction to X of the quotient $\overline{\mathfrak{F}} := (\overline{\mathfrak{F}}')^H$ of a coherent and locally free $\mathcal{O}_{\overline{\mathfrak{X}}'}$-Module $\overline{\mathfrak{F}}'$, equipped with an equivariant action of H inducing the trivial galois action on its generic fiber $\overline{\mathcal{F}}' = \overline{\mathcal{F}} \otimes k'$ (*cf.* definition 1.1.16 below). The category $\mathbf{MIC}_{\overline{\mathfrak{X}}}(\overline{X}(*\mathcal{Z})/k)$ admits natural internal tensor product and internal $\mathcal{H}om$ and $\mathcal{E}nd$.

Let us assume, from now on in this section, that k is an extension of $\mathbb{Q}_p$. We say that an object $(\mathcal{F}, \nabla)$ of $\mathbf{MIC}_{\overline{\mathfrak{X}}}(\overline{X}(*\mathcal{Z})/k)$ *satisfies condition* **NL** *at* $z \in \mathcal{Z}$, if the formal Fuchs exponents of (the regular part of) $\mathcal{E}nd((\mathcal{F}, \nabla))$ at z are p-adic non-Liouville numbers. An old result of the author [3, Prop. 4], which applies under condition **NL**, relates the asymptotic behavior of $x \mapsto \mathcal{R}_{\overline{\mathfrak{X}},\mathfrak{Z}}(x, (\mathcal{F}, \nabla))$ for $x \to z_i$ to the algebraic irregularity $\rho_{z_i}(\mathcal{F}, \nabla)$ of the connection at z_i, namely

$$\mathcal{R}_{\overline{\mathfrak{X}},\mathfrak{Z}}(x, (\mathcal{F}, \nabla)) \sim |T_{z_i}(x)|^{\rho_{z_i}(\mathcal{F}, \nabla)} \quad , \text{ as } x \to z_i \ , \tag{0.2.0.2}$$

where T_{z_i} is the normalized coordinate on $D_{\overline{\mathfrak{X}}}(z_i, 1^-)$ with $T_{z_i}(z_i) = 0$. Our main result is that, for any object $(\mathcal{F}, \nabla)$ of $\mathbf{MIC}_{\overline{\mathfrak{X}}}(\overline{X}(*\mathcal{Z})/k)$, the function $x \mapsto \mathcal{R}(x) = \mathcal{R}_{\overline{\mathfrak{X}},\mathfrak{Z}}(x, (\mathcal{F}, \nabla))$ is continuous on X. We first reduce the problem of continuity of this function to the standard situation of a connection $(\mathcal{F}, \nabla)$ with $\mathcal{F}$ free of finite rank μ over a basic affinoid disk or annulus.

For the convenience of the reader, we recall in section 4 the classical theory of differential systems on an annulus, largely due to Dwork, Robba and Christol. One should also refer to the elegant account [35], for more recent results. These authors treat the case of a system of ordinary differential equations (0.1.0.1) defined on the open annulus $B(r_1, r_2)$ (0.1.0.2), with coefficients in the Banach k-algebra $\mathscr{H}(r_1, r_2)$ of *analytic elements* on $B(r_1, r_2)$: $\mathscr{H}(r_1, r_2)$ is the completion of the k-subalgebra of $k(T)$, consisting of rational functions with no pole in $B(r_1, r_2)$, under the supnorm on $B(r_1, r_2)$.

Our main technical tool is explained in section 5.1: it is a generalization of the Dwork-Robba theorem [26, IV.3.1] on effective bounds for the growth of local solutions (theorem (5.1.1) and its corollaries). Essentially, we must replace in the original formulation analytic elements in the previous sense on the open annulus $B(r_1, r_2) \subset \mathbb{A}^1$, with functions which extend to analytic functions on a basic affinoid annulus containing $B(r_1, r_2)$ as an open analytic subspace. The Dwork-Robba theorem is the essential step in our proof of the upper semicontinuity of the radius of convergence (*cf.* §5.2) and in generalizing the continuity result of Christol-Dwork [20, 2.5] to the present situation §5.3. Section 4 of the joint work with Di Vizio [4] contains a several variable generalization of this result. We point out that the upper semicontinuity is precisely the non-trivial part of [20, 2.5], and that our proof (as given here and in [4]) differs from the one of Christol-Dwork. We conclude the proof of continuity of $x \mapsto \mathcal{R}(x)$ in section 6. As in the classical case, we obtain a more precise description of

$x \mapsto \mathcal{R}(x)$. Namely, for any Q-semistable model $\mathfrak{Y} \geq \overline{\mathfrak{X}}$ of $\overline{X}$ and any oriented open edge E of $\mathbf{S}(\mathfrak{Y})$, which is a split Q-open segment, let $(r_E, 1) \xrightarrow{\sim} E$, $\rho \mapsto \eta_\rho$, be the canonical parametrization, with $r_E \in \sqrt{|k^\times|} \cap (0,1)$. Then the restriction to $(r_E, 1)$ of $x \mapsto \mathcal{R}(x)$, namely $\rho \mapsto \mathcal{R}(\rho) = \mathcal{R}(\eta_\rho)$, is the infimum of the constant 1 and a finite set of functions of the form

$$|p|^{1/(p-1)p^h} |b|^{1/jp^h} \rho^{s/j} \,, \tag{0.2.0.3}$$

where $j \in \{1, 2, \dots, \mu - 1\}$, $s \in \mathbb{Z}$, $h \in \mathbb{Z} \cup \{\infty\}$, $b \in k^\times$. The statement for non-split Q-open segments may be easily deduced by consideration of the galois action. Arbitrarily high h can appear even in the simplest rank 1 case of the equation killing x^α *cf.* [26, IV.7.3 (*iv*)].

This statement in the classical situation appeared in [38]: we provide here a hopefully more convincing proof[9]. We point out the novelty of using Dwork's technique of descent by Frobenius on basic affinoid annuli which are not necessarily affinoid subdomains of $\mathbb{A}^1$, *cf.* §5.4.

Remark 0.2.1. The expression above for $\mathcal{R}(\rho)$ is bound to be invariant under the substitution $\rho \mapsto r_E/\rho$. This puts some constraint on r_E and leads us to believe that $r_E^{p^h} \in |k^\times|$, for some $h \in \mathbb{Z}$. More generally we believe the

Conjecture 0.2.2. *A tamely ramified open k-analytic Q-annulus is a k-annulus.*

A similar statement for open Q-disks has recently been proven by A. Ducros [25].

If $(\mathcal{F}, \nabla)$ has meromorphic singularity at a $z_i \in \mathcal{Z}$, and the previous p-adic non-Liouvilleness assumption holds at z_i, then a similar result holds on the half-line E connecting z_i with the point $\tau_{\overline{\mathfrak{X}}}(z_i)$ at the boundary of $D_{\overline{\mathfrak{X}}}(z_i, 1^-)$. In terms of the canonical parametrization via the function "radius of a point" $\rho : E \cup \{z_i, \tau_{\overline{\mathfrak{X}}}(z_i)\} \xrightarrow{\sim} [0,1]$, the restriction of $x \mapsto \mathcal{R}(x)$ to $\rho^{-1}((\varepsilon, 1))$, for any $\varepsilon \in (0,1) \cap \sqrt{|k^\times|}$, is precisely of the form (0.2.0.3), with moreover $s/j \leq \rho_{z_i}(\mathcal{F}, \nabla)$. The conclusion is

Theorem 0.2.3. *Let $(\mathcal{F}, \nabla)$ be an object of $\mathbf{MIC}_{\overline{\mathfrak{X}}}(\overline{X}(*\mathcal{Z})/k)$ satisfying condition* **NL** *at each $z \in \mathcal{Z}$. Then $\mathcal{R}_{\overline{\mathfrak{X}},\mathfrak{Z}}(x, (\mathcal{F}, \nabla)) = 1$ identically for $x \in X$ if and only if both*

1. *$\mathcal{R}_{\overline{\mathfrak{X}},\mathfrak{Z}}(\xi, (\mathcal{F}, \nabla)) = 1$, for each vertex ξ of $\mathbf{S}(\overline{\mathfrak{X}})$,*
2. *$(\mathcal{F}, \nabla)$ has regular singularities along $\mathcal{Z}$.*

Under the previous assumptions $(\mathcal{F}, \nabla)$ is a Robba connection on X.

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The collaboration with Lucia over the past few years has greatly contributed to shape the ideas appearing in this paper.

[9] The treatment of [19] is unfortunately restricted to the solvable case, where negative slopes do not appear.

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1 The structure of rig-smooth compact strictly k-analytic curves

Notation 1.0.4. All over this section, the valuation of k is assumed to be non-trivial. All k-analytic spaces are assumed to be separated. As in [22, §3], a k-analytic curve is a k-analytic space pure of dimension 1. All over this and the next section, X denotes a compact connected rig-smooth strictly k-analytic curve. For any field L an *L-algebraic curve* is a separated L-scheme of finite type, of pure dimension 1.

Any k-analytic curve is a good analytic space, *i.e.* every point of it has an affinoid neighborhood, and is paracompact [22, §3]. An irreducible compact k-analytic curve is either the analytification of a projective curve or it is affinoid [29], [22, Prop. 3.2]. So, in our case, X is either the analytification $\mathcal{X}^{\rm an}$ of a smooth projective k-algebraic curve $\mathcal{X}$ or it is strictly k-affinoid; its underlying topological space is a quasipolyhedron [6, 4.1.1].

1.1 Semistable models

Classically, one gives the following definition.

Notation 1.1.1. Let L be a non-archimedean field over k. An L°-scheme (resp. L°-formal scheme) is *admissible* if it is reduced, quasi-compact, separated, of (resp. topologically) finite presentation, and flat over L°. The algebra of an affine admissible L°-scheme (resp. L°-formal scheme) is also said to be *admissible.*

Our schemes and formal schemes will always be separated and quasi-compact. *Schemes or formal schemes over k° will not be assumed to be admissible, in general.*

For an admissible k°-scheme $\mathscr{X}$, we define the *completion along the closed fiber* of $\mathscr{X}$ as the admissible k°-formal scheme

$$\widehat{\mathscr{X}} = \widehat{\mathscr{X}_{/\mathscr{X}_s}} := \varinjlim_n \mathscr{X} \otimes K^\circ/(\pi^n) \tag{1.1.1.1}$$

where $\pi \in k^{\circ\circ} \setminus \{0\}$. The definition is clearly independent of the choice of π and is functorial.

Definition 1.1.2. *For an admissible formal scheme $\mathfrak{Y}$, we denote by* $\mathrm{sp} : \mathfrak{Y}_\eta \to \mathfrak{Y}_s$ *the* set-theoretic specialization map *and by* $\mathrm{sp}_{\mathfrak{Y}} : (\mathfrak{Y}_\eta)_G \to \mathfrak{Y}$ *the* specialization map *viewed as a morphism of locally ringed G-topological spaces.*

Definition 1.1.3. *A morphism $\varphi : \mathfrak{X} \to \mathfrak{Y}$ of k°-formal schemes, is* proper *if for any $\pi \in k^{\circ\circ} \setminus \{0\}$ the morphism of $k^\circ/(\pi)$-schemes $\varphi_0 : \mathfrak{X} \otimes k^\circ/(\pi) \to \mathfrak{Y} \otimes k^\circ/(\pi)$ is proper.*

This definition agrees with [27, III.1, 3.4.1] when k° is noetherian and $\mathfrak{X}$, $\mathfrak{Y}$ are admissible. We recall [8, §2] that a morphism $\varphi : \mathfrak{X} \to \mathfrak{Y}$ of admissible k°-formal schemes, is said to be *étale* if φ_0 is étale, for any choice of π.

We do not distinguish between "Tate curve" and "twisted Tate curve", according to the following definition :

Definition 1.1.4. *A (*twisted*)* Tate curve *over k is a smooth connected projective k-algebraic curve Y, such that any connected component of $Y \otimes \widehat{k^{\rm alg}}$ is a $\widehat{k^{\rm alg}}$-curve of genus 1, which does not admit a smooth projective model over the ring of integers of $\widehat{k^{\rm alg}}$. Similarly, a* rational *curve over k is a smooth connected projective k-algebraic curve Y, such that any connected component of $Y \otimes \widehat{k^{\rm alg}}$ is isomorphic to $\mathbb{P}^1_{\widehat{k^{\rm alg}}}$. The analytification of a rational (resp. of a Tate) curve is also called* rational *(resp. a* Tate curve*).*

For a rational (resp. Tate) curve X over k, there exists a finite galois extension k'/k, such that every irreducible component of $X \otimes k'$ is isomorphic to $\mathbb{P}^1_{\widehat{k^{\rm alg}}}$ (resp. k'-analytically isomorphic to $\mathbb{G}_{m,k'}/q^{\mathbb{Z}}$, where $\mathbb{G}_{m,k'}$ is the k'-analytic multiplicative group, while $q \in k'$, $0 < |q| < 1$).

We further abuse the classical terminology as follows.

Definition 1.1.5. *An admissible formal scheme $\mathfrak{X}$ over k° is* strictly semistable *if, locally for the Zariski topology, it is of the form*

$$\mathfrak{X} \xrightarrow{\phi} \operatorname{Spf} k^\circ\{S,T\}/(ST-a) \to \operatorname{Spf} k^\circ \tag{1.1.5.1}$$

with $a \in k^{\circ\circ} \setminus \{0\}$, where ϕ is étale. Similarly, a reduced separated $\widetilde{k}$-scheme of finite type $\mathcal{Y}$ is strictly semistable *if, locally for the Zariski topology, it is of the form*

$$\mathcal{Y} \xrightarrow{\phi} \operatorname{Spec} \widetilde{k}[x,y]/(xy) \to \operatorname{Spec} \widetilde{k}\ , \tag{1.1.5.2}$$

with ϕ étale.
The k°-formal scheme $\mathfrak{X}$ (resp. the $\widetilde{k}$-scheme $\mathcal{Y}$) is semistable *if there is a surjective étale morphism $\mathfrak{X}' \to \mathfrak{X}$ (resp. $\mathcal{Y}' \to \mathcal{Y}$) with $\mathfrak{X}'$ (resp. $\mathcal{Y}'$) strictly semistable.*

The generic fiber of a semistable k°-formal scheme is rig-smooth. It follows from [10, Prop. 1.4 and step (1) in its proof] that a semistable k°-formal scheme is normal.

The notion of (resp. strict) semistability we use is the one-dimensional case of (resp. strict) nondegenerate polystability [10]. We have

Theorem 1.1.6. *An integral admissible formal scheme $\mathfrak{X}$ over k° is (resp. strictly) semistable if and only if its closed fiber $\mathfrak{X}_s$ is a (resp. strictly) semistable scheme over $\widetilde{k}$.*

Proof. We only have to show the sufficiency of the condition. So, let $\mathfrak{X}$ be a formal scheme over k° whose closed fiber $\mathfrak{X}_s$ is a semistable scheme over $\widetilde{k}$. Then, there exists a surjective étale morphism $Y \to \mathfrak{X}_s$ of $\widetilde{k}$-schemes, such that Y is a strictly semistable $\widetilde{k}$-scheme. By [8, 2.1], Y is the closed fiber of a formal scheme $\mathfrak{Y}$ étale over $\mathfrak{X}$, so we are reduced to prove that $\mathfrak{Y}$ is strictly semistable, that is the "resp." case of our statement. We may assume that $\mathfrak{X} = \operatorname{Spf} A$ is affine and that $\mathfrak{X}_s$ is in one of the two cases allowed by formula 1.1.5.2.

Lemma 1.1.7. *Let A be an integral topological k°-algebra topologically of finite presentation, flat over k°. Let $B = A/k^{\circ\circ}A$ be the residue $\widetilde{k}$-algebra. Assume that one of the following assumptions holds*

1. *B is a domain étale over a subring of the form $\widetilde{k}[x]$;*

2. *B is étale over a subring of the form $\widetilde{k}[x,y]/(xy)$, and x,y are contained in a maximal ideal of B,*

where $\widetilde{k}[x]$ (resp. $\widetilde{k}[x,y]$) is the affine algebra of the affine space of dimension 1 (resp. 2) over $\widetilde{k}$. Then, in case 1, for any lifting T of x in A, A is étale over $k^\circ\{T\}$, while in case 2, for any open maximal ideal $\mathfrak{P}$ of A above (x,y), there exist $a \in k^{\circ\circ} \setminus \{0\}$ and liftings T of x and S of y in $\mathfrak{P}$, such that A contains $k^\circ\{S,T\}/(ST-a)$. For any such choice of a, S and T in A, A is étale over $k^\circ\{S,T\}/(ST-a)$.

Proof. The existence of S, T, a in case (2) is proven in [15, Prop. 2.3 (ii)]. We bypass the proof of generic étaleness in [Prop. 2.2] and [Prop. 2.3] of [15], as follows[10]. So, we have already seen that

Sublemma 1.1.8.

1. *The morphism of case (1) of the lemma extends to a dominant morphism*

$$\varphi : \mathfrak{X} = \mathrm{Spf}\, A \to \widehat{\mathbb{A}}^1_{k^\circ} = \mathrm{Spf}\, k^\circ\{T\} \; ; \tag{1.1.8.1}$$

2. *the morphism of case (2) of the lemma extends to a dominant morphism*

$$\varphi : \mathfrak{X} = \mathrm{Spf}\, A \to \mathrm{Spf}\, k^\circ\{S,T\}/(ST-a) \; , \tag{1.1.8.2}$$

   for a suitable $a \in k^{\circ\circ} \setminus \{0\}$.

To conclude the proof of lemma 1.1.7 we need to show that the morphism φ in both formulas 1.1.8.1 and 1.1.8.2 is étale. By definition [8, §2], this means that for any $\pi \in k^{\circ\circ} \setminus \{0\}$, the morphism of finite presentation φ_0 obtained from φ by reduction modulo π is étale, *i.e.* flat and unramified. For flatness, let π be as before, $R_0 := k^\circ/\pi k^\circ$ and let $\mathfrak{m}$ be the maximal ideal of R_0. The fiberwise flatness criterion [27, Thm. 11.3.10] may be spelled out in our case as follows.

Proposition 1.1.9. *Let $C \to A$ be a morphism of flat R_0-algebras of finite presentation. Then the following facts are equivalent:*

1. *A is a flat C-Module;*
2. *$A/\mathfrak{m}A$ is a flat $C/\mathfrak{m}C$-Module.*

So we are done with flatness; we still have to show that the extension $k^\circ\{T\}\otimes R_0 \subset A \otimes R_0$ (resp. $k^\circ\{S,T\}/(ST-a) \otimes R_0 \subset A \otimes R_0$) of the sublemma, is *unramified.* This is a consequence of the following

Proposition 1.1.10. *Let π, R_0 be as above, and let $\varphi : C \to A$ be a morphism of R_0-algebras of finite presentation. Assume that the morphism of $\widetilde{k}$-algebras of finite type $\varphi_s : C \otimes \widetilde{k} \to A \otimes \widetilde{k}$ is unramified. Then φ is unramified.*

Proof. We have to show that the map $\Phi : A \otimes \Omega^1_{C/R_0} \to \Omega^1_{A/R_0}$, $x \otimes dy \longmapsto x\, d(\varphi(y))$, is surjective. Let $\mathfrak{m} = k^{\circ\circ} \otimes R_0$ be the maximal ideal of R_0. Let $\omega_1, \dots, \omega_N \in \Omega^1_{A/R_0}$ be a system of generators of the A-Module Ω^1_{A/R_0} and let $\eta_1, \dots, \eta_M \in \Omega^1_{C/R_0}$ and $a^{(1)}_1, \dots, a^{(M)}_N \in A$ be such that $\omega_i = \sum_{j=1}^M a^j_{(i)}\, \Phi(\eta_j) + \epsilon_i$ with $\epsilon_1, \dots, \epsilon_N \in \mathfrak{m}\,\Omega^1_{A/R_0}$. There exists $c \in \mathfrak{m}$ such that $\epsilon_i \in c\,\Omega^1_{A/R_0}$, $\forall i = 1, \dots, N$. But then $\Omega^1_{A/R_0} = \mathrm{Im}\,(\Phi) + c\,\Omega^1_{A/R_0}$ implies $\Omega^1_{A/R_0} = \mathrm{Im}\,(\Phi)$. □

So the proof of lemma 1.1.7 is complete. □

Theorem 1.1.6 follows. □

Definition 1.1.11. *A normal admissible k°-scheme $\mathscr{X}$ is said to be (resp.* strictly*)* semistable *if its special fiber is (resp. strictly) semistable.*

It follows from theorem 1.1.6 that a normal admissible k°-scheme $\mathscr{X}$ is (resp. strictly) semistable if and only if its completion is a (resp. strictly) semistable k°-formal scheme.

[10] We are indebted to Lorenzo Ramero for this shortcut.

Definition 1.1.12. *Let k be a fixed non-archimedean field with non-trivial valuation. The k-analytic curve X and, in case, the projective curve $\mathcal{X}$ such that $X = \mathcal{X}^{\mathrm{an}}$, are as before. Let $\mathcal{FA}_{/k}$ (resp. $\mathcal{SA}_{/k}$, resp. $\mathcal{AC}_{/k}$, resp. $\mathcal{SC}_{/k}$) be the category of pairs $(\tau : k \to K, \mathfrak{Y})$ (resp. $(\tau : k \to K, \mathscr{Y})$, resp. $(\tau : k \to K, Y)$, resp. $(\tau : k \to K, \mathcal{Y})$), where $\tau : k \to K$ is a finite separable extension of non-archimedean fields, and $\mathfrak{Y}$ is an admissible formal scheme of relative dimension 1 over K° (resp. $\mathscr{Y}$ is an admissible K°-scheme of relative dimension 1, resp. Y is a strictly K-analytic curve, resp. $\mathcal{Y}$ is a K-algebraic curve). A morphism $(\tau : k \to K, \mathfrak{Y}) \to (\sigma : k \to K', \mathfrak{Z})$ in $\mathcal{FA}_{/k}$ (resp. $(\tau : k \to K, \mathscr{Y}) \to (\sigma : k \to K', \mathscr{Z})$ in $\mathcal{SA}_{/k}$, resp. $(\tau : k \to K, Y) \to (\sigma : k \to K', Z)$ in $\mathcal{AC}_{/k}$, resp. $(\tau : k \to K, \mathcal{Y}) \to (\sigma : k \to K', \mathcal{Z})$ in $\mathcal{SC}_{/k}$) consists of k-embeddings $K \to L$ $K' \to L$, and of a morphism of L°-formal schemes $\mathfrak{Y}\widehat{\otimes}L^\circ \to \mathfrak{Z}\widehat{\otimes}L^\circ$ (resp. of L°-schemes $\mathscr{Y} \otimes L^\circ \to \mathscr{Z} \otimes L^\circ$, resp. of L-analytic curves $Y \otimes L \to Z \otimes L$, resp. of L-algebraic curves $\mathcal{Y} \otimes L \to \mathcal{Z} \otimes L$). The functor* generic fiber *$\mathcal{FA}_{/k} \to \mathcal{AC}_{/k}$ (resp. $\mathcal{SA}_{/k} \to \mathcal{SC}_{/k}$) exhibits $\mathcal{FA}_{/k}$ (resp. $\mathcal{SA}_{/k}$) as a fibered category over $\mathcal{AC}_{/k}$ (resp. $\mathcal{SC}_{/k}$) which in turns is fibered over the category of finite separable extensions of k. The subcategory of $\mathcal{FA}_{/k}$ (resp. $\mathcal{SA}_{/k}$) of objects whose generic fiber is isomorphic to the object $(\tau : k \to K, X \otimes K)$ of $\mathcal{AC}_{/k}$ (resp. $(\tau : k \to K, \mathcal{X} \otimes K)$ of $\mathcal{SC}_{/k}$), for some choice of $\tau : k \to K$, will be denoted by $\mathcal{FA}_{/X}$ (resp. $\mathcal{SA}_{/\mathcal{X}}$). We define $\mathcal{FS}_{/k}$ (resp. $\mathcal{FS}_{/X}$) to be the full subcategory of $\mathcal{FA}_{/k}$ (resp. of $\mathcal{FA}_{/X}$) consisting of pairs $(\tau : k \to K, \mathfrak{Y})$ in which $\mathfrak{Y}$ is a semistable K°-formal scheme. Similarly for $\mathcal{SS}_{/k}$ (resp. $\mathcal{SS}_{/\mathcal{X}}$) and $\mathcal{SA}_{/k}$ (resp. $\mathcal{SA}_{/\mathcal{X}}$). We denote by $\mathcal{FS}(X) \subset \mathcal{FA}(X)$ (resp. $\mathcal{FS}_K \subset \mathcal{FA}_K$) the fiber categories over $(k \to k, X)$ (resp. over the identity embedding of the finite separable extension K of k). Similarly for $\mathcal{SS}(\mathcal{X}) \subset \mathcal{SA}(\mathcal{X})$ (resp. $\mathcal{SS}_K \subset \mathcal{SA}_K$). So, $\mathcal{FA}_k$ (resp. $\mathcal{FA}(X)$) is the category of admissible k°-formal schemes of relative dimension 1 (resp. of* admissible formal models of *X), with morphisms of k°-formal schemes (resp. commuting with the specialization map,* i.e. *inducing the identity on the generic fiber X), and $\mathcal{FS}_k$ (resp. $\mathcal{FS}(X)$) is the full subcategory of semistable k°-formal schemes (resp. of* semistable formal models of *X). Similarly for $\mathcal{SA}_k$ (resp. $\mathcal{SA}(\mathcal{X})$) and $\mathcal{SS}_k$ (resp. $\mathcal{SS}(\mathcal{X})$). We will also denote by $\mathcal{PSA}(\mathcal{X})$ (resp. $\mathcal{PSS}(\mathcal{X})$) the full subcategory of $\mathcal{SA}(\mathcal{X})$ (resp. $\mathcal{SS}(\mathcal{X})$) consisting of projective admissible (resp. semistable) models of $\mathcal{X}$ over k°. Finally, we denote by $\mathcal{FS}^{\mathrm{st}}(X)$ (resp. $\mathcal{SS}^{\mathrm{st}}(\mathcal{X})$, resp. $\mathcal{PSS}^{\mathrm{st}}(\mathcal{X})$) the full subcategory of $\mathcal{FS}(X)$ (resp. $\mathcal{SS}(\mathcal{X})$, resp. $\mathcal{PSS}(\mathcal{X})$) consisting of strictly semistable models of X (resp. $\mathcal{X}$, resp. $\mathcal{X}$).*

Definition 1.1.13. *A (resp. formal) scheme $\mathscr{Y}$ (resp. $\mathfrak{Y}$) over k° is* Q-semistable *if it is flat, reduced, normal, and there exists a finite galois extension k'/k of group H and a semistable k'°-scheme (resp. k'°-formal scheme) $\mathscr{Y}'$ (resp. $\mathfrak{Y}'$) equipped with a galois action[11] of H such that $\mathscr{Y} = \mathscr{Y}'/H$ (resp. $\mathfrak{Y} = \mathfrak{Y}'/H$). The rig-smooth k-analytic curve $\mathfrak{Y}'_\eta/H$ is called (by analogy to the algebraic case) the* generic fiber of $\mathfrak{Y}$ *and is denoted by $\mathfrak{Y}_\eta$. If in particular $\mathscr{Y}'$ (resp. $\mathfrak{Y}'$) is smooth or étale, we say that $\mathscr{Y}$ (resp. $\mathfrak{Y}$) is* Q-smooth *or* Q-étale. *The morphism of sets (resp. of ringed G-spaces) $\mathrm{sp} : Y \to \mathfrak{Y}_s = \mathfrak{Y}'/H$ (resp. $\mathrm{sp}_{\mathfrak{Y}} : Y_G \to \mathfrak{Y}$) induced by the H-equivariant specialization morphism of $\mathfrak{Y}'$ is called the* set-theoretic specialization map *(resp. the* specialization map*) of the Q-semistable k°-formal scheme $\mathfrak{Y}$.*

Remark 1.1.14. Notice that a flat normal k°-scheme $\mathcal{Y}$ is Q-semistable if and only if there exists a finite galois extension k'/k such that the normalization of $\mathcal{Y}$ in $\mathcal{Y} \otimes k'$ is semistable. In the formal case, it is still true that, in the notation of the definition, $\mathfrak{Y}'$ is the normalization of $\mathfrak{Y}$ in the generic fiber of $\mathfrak{Y}_\eta \otimes k'$. In order to use this as a definition though, we should start with a definition of generic fiber of a formal scheme more general than the one given classically.

We continue this discussion only for formal schemes. Let $\phi : \mathfrak{Y} \to \mathfrak{X}$ be a morphism of Q-semistable k°-formal schemes. Let k'/k be a finite galois extension, of group H, such

[11] This means that the structural morphism is equivariant.

that $\mathfrak{Y} = \mathfrak{Y}'/H$, $\mathfrak{X} = \mathfrak{X}'/H$, with $\mathfrak{Y}'$ and $\mathfrak{X}'$ two k'°-semistable formal schemes equipped with an action of H and $\phi = (\phi')^H$, for an H-equivariant morphism of k'°-formal schemes. Then, ϕ'_η is an H-equivariant morphism of k'-analytic curves, and we set $\phi_\eta := (\phi'_\eta)^H$, the *generic fiber of* ϕ. We obtain the category $\mathcal{FS}_k^Q$ of *Q-semistable k°-formal schemes* (resp. $\mathcal{SS}_k^Q$ of *Q-semistable k°-schemes*) which is fibered over the category of compact rig-smooth strictly k-analytic curves (resp. of smooth k-schemes). The category $\mathcal{FS}_k^Q$ is equivalent to the subcategory of $\mathcal{FS}_{/k}$ consisting of pairs $(\tau : k \to k', \mathfrak{Y}')$, where k'/k is galois and $H = \mathrm{Gal}\,(k'/k)$ acts on $\mathfrak{Y}'$, in such a way that $\mathfrak{Y}'_\eta/H$ is rig-smooth and $\mathfrak{Y}'_\eta = (\mathfrak{Y}'_\eta/H) \otimes k'$. Then the k°-formal scheme $\mathfrak{Y} := \mathfrak{Y}'/H$ exists[12] (in particular, it is separated) and is an object of $\mathcal{FS}_k^Q$. The object $(\tau : k \to k', \mathfrak{Y}')$ is a *representative of* $\mathfrak{Y}$ *in* $\mathcal{FS}_{/k}$ or, more precisely, *in* $\mathcal{FS}_{k'}$. In particular, a morphism in the fiber category $\mathcal{FS}^Q(X)$ of $\mathcal{FS}_k^Q$ over the curve X, consists of a morphism of k°-formal schemes, between two Q-semistable models of X, which commutes with the specialization maps. A morphism $f : \mathfrak{Y} \to \mathfrak{Z}$ in the category $\mathcal{FS}_k^Q$ may be viewed, for some finite galois extension k'/k of group H, as an H-equivariant a morphism $f' : \mathfrak{Y}' \to \mathfrak{Z}'$ in $\mathcal{FS}_{k'}$. We call such an f' a *representative* of the morphism f in $\mathcal{FS}_{k'}$. Similarly for objects and morphisms of $\mathcal{SS}_k^Q$ or $\mathcal{SS}^Q(\mathcal{X})$. An object of $\mathcal{SS}_k^Q$ is *Q-projective* if, for some finite galois extension k'/k, it admits a representative which is an admissible projective k'°-scheme.

Definition 1.1.15. *A morphism $f : \mathfrak{Y} \to \mathfrak{Z}$ in the category $\mathcal{FS}_k^Q$ is called* Q-étale *(resp.* an admissible Q-blow-up*) if, for some finite galois extension k'/k, it admits a representative morphism in $\mathcal{FS}_{k'}$ which is étale (resp. an admissible blow-up). A morphism $f : \mathscr{Y} \to \mathscr{X}$ in the category $\mathcal{SS}_k^Q$ is called* Q-projective *if, for some finite galois extension k'/k, it admits a representative morphism in $\mathcal{SS}_{k'}$ which is projective. For $\mathcal{X}$ a projective smooth k-algebraic curve, we denote by $\mathcal{PSS}^Q(\mathcal{X})$ the subcategory of $\mathcal{SS}^Q(\mathcal{X})$ consisting of Q-projective objects and Q-projective morphisms: a morphism in $\mathcal{PSS}^Q(\mathcal{X})$ may be represented by a morphism of $\mathcal{SS}(\mathcal{X} \otimes k')$, over a suitable finite galois extension k' of k.*

Definition 1.1.16. *Let $\mathfrak{X}$ be an object of $\mathcal{FS}_k^Q$ and let $\mathrm{sp}_{\mathfrak{X}} : X_G \to \mathfrak{X}$ be its specialization morphism. An $\mathcal{O}_{\mathfrak{X}}$-Module $\mathfrak{F}$ is* Q-coherent *(resp.* Q-locally free of rank μ*) if $\mathcal{F} := \mathrm{sp}_{\mathfrak{X}}^*(\mathfrak{F})$ is a coherent (resp. locally free of rank μ) $\mathcal{O}_{X_G}$-Module and there exist a finite galois extension k'/k of group H, a representative $\mathfrak{X}'$ of $\mathfrak{X}$ in $\mathcal{FS}_{k'}$ and a coherent (resp. locally free of rank μ) $\mathcal{O}_{\mathfrak{X}'}$-Module $\mathfrak{F}'$, equipped with an H-equivariant action of H, such that $\mathfrak{F} = (\mathfrak{F}')^H$.*

Definition 1.1.17. [41] *Let $\mathscr{X}$, $\mathscr{Y}$ (resp. $\mathfrak{X}$, $\mathfrak{Y}$) be either admissible or Q-semistable k°-schemes (resp. k°-formal schemes) of pure relative dimension 1, with smooth (resp. rig-smooth) generic fibers. A proper dominant morphism $\varphi : \mathscr{X} \to \mathscr{Y}$ (resp. $\varphi : \mathfrak{X} \to \mathfrak{Y}$) is called an* η-modification of $\mathfrak{Y}$ *(resp.* of $\mathfrak{Y}$*) if the generic fiber $\varphi_\eta : \mathscr{X}_\eta \to \mathscr{Y}_\eta$ (resp. $\varphi_\eta : \mathfrak{X}_\eta \to \mathfrak{Y}_\eta$) is an isomorphism of k-algebraic curves (resp. of k-analytic curves).*

In the special case when the valuation of k is discrete, a Q-semistable k°-formal scheme is necessarily topologically of finite type (and presentation) over k°. So, it is admissible and normal. But it is not necessarily semistable.

The following strong form of the semistable reduction theorem for curves over k (in fact over a general basis) has recently been made available by Temkin [41].

Theorem 1.1.18. *(Strong Semistable Reduction Theorem) Let $\mathscr{Y}$ be an admissible k°-scheme of pure relative dimension one over k°. Let us assume that the generic fiber $\mathcal{Y} := \mathscr{Y}_\eta$ of $\mathscr{Y}$ is a smooth k-algebraic curve. Then, there is a finite galois extension $\kappa : k \hookrightarrow k'$, a strictly semistable k'°-scheme $\mathscr{Y}'$ and an η-modification $\varphi' : \mathscr{Y}' \to \mathscr{Y} \otimes_{(k^\circ,\kappa)} k'^{\circ}$ of $\mathscr{Y} \otimes_{(k^\circ,\kappa)} k'^{\circ}$. Moreover, the pair $(\kappa : k \to k', \mathscr{Y}')$ with the above property may be chosen to be* minimal *in the following sense. If $\lambda : k \hookrightarrow k''$ is a finite galois extension, $\mathscr{Y}''$ is a*

[12]In fact, every point of $\mathfrak{Y}'$ has an affine H-stable neighborhood.

semistable k''°-scheme and a morphism $\varphi'' : \mathscr{Y}'' \to \mathscr{Y} \otimes_{(k^\circ,\kappa)} k''^\circ$ is an η-modification of $\mathscr{Y} \otimes_{(k^\circ,\kappa)} k''^\circ$, then for any choice (μ,ν) of k-embeddings $\mu : k' \hookrightarrow k'''$ and $\nu : k'' \hookrightarrow k'''$ in a finite galois extension k'''/k, there is a unique k'''°-morphism $\chi : \mathscr{Y}'' \otimes_{(k''^\circ,\nu)} k'''^\circ \to \mathscr{Y}' \otimes_{(k'^\circ,\mu)} k'''^\circ$, such that $\varphi'' \otimes_{(k''^\circ,\nu)} \mathrm{id}_{k'''^\circ} = (\varphi' \otimes_{(k'^\circ,\mu)} \mathrm{id}_{k'''^\circ}) \circ \chi$. For any such minimal choice of $(\kappa : k \hookrightarrow k', \mathscr{Y}', \varphi')$, the morphism φ' is projective and is an isomorphism over the maximal semistable (open) subscheme of $\mathscr{Y} \otimes_{(k^\circ,\kappa)} k'^\circ$.

Remark 1.1.19. Any triple $(\kappa : k \hookrightarrow k', \mathscr{Y}', \varphi')$ (not necessarily minimal) as considered in the theorem, may be seen as an object $(\kappa : k \to k', \mathscr{Y}')$ of the category $\mathcal{SS}_{/\mathcal{Y}}$ equipped with a proper, dominant morphism $\varphi' : \mathscr{Y}' \to \mathscr{Y} \otimes_{(k^\circ,\kappa)} k'^\circ$, whose generic fiber is an isomorphism. A "morphism" of triples $(\kappa : k \hookrightarrow k', \mathscr{Y}', \varphi') \to (\lambda : k \hookrightarrow k'', \mathscr{Y}'')$, determined by the χ considered in the theorem, is precisely a morphism in the category $\mathcal{SS}_{/\mathcal{Y}}$. Therefore, a minimal triple $(\kappa : k \hookrightarrow k', \mathscr{Y}', \varphi')$ as described in the theorem, is *minimum* in the full subcategory of $\mathcal{SS}_{/\mathcal{Y}}$ consisting of objects $(\kappa : k \to k', \mathscr{Y}')$, admitting an η-modification to $\mathscr{Y} \otimes_{(k^\circ,\kappa)} k'^\circ$.

The following result is based on the more classical, and weaker, version of theorem 1.1.18, in which $\mathscr{Y}$ is assumed to be proper.

Proposition 1.1.20. *(Berkovich) Let $V = \mathscr{M}(\mathscr{A})$ be a rig-smooth strictly k-affinoid curve. There exists a finite galois extension k'/k, a projective strictly semistable scheme $\mathscr{Y}'$ over k'° and an embedding of $V' := V \otimes k'$ in the generic fiber $\mathfrak{Y}'_\eta$ of the formal completion $\mathfrak{Y}'$ of $\mathscr{Y}'$ along its closed fiber, which identifies V' to the affinoid $\mathrm{sp}^{-1}(\mathscr{Z}')$ for an affine open subset $\mathscr{Z}'$ of $\mathfrak{Y}'_s$. In particular, V' is the generic fiber of the strictly semistable k'°-formal scheme $\widehat{\mathscr{Y}'}_{/\mathscr{Z}'}$, formal completion of $\mathscr{Y}'$ along $\mathscr{Z}'$.*

Proof. (Steps 1 and 2 hold for a higher dimensional affinoid V).

Step 1. *V is isomorphic to a strictly affinoid domain in the analytification $\mathscr{V}^{\mathrm{an}}$ of an affine scheme $\mathscr{V}$ over k.* By the theory of Raynaud [13, §2.8], the closed k°-subalgebra $\mathscr{A}^\circ$ of $\mathscr{A}$ consisting of elements of spectral norm ≤ 1, contains an admissible k°-subalgebra A, such that $A \otimes_{k^\circ} k \xrightarrow{\sim} \mathscr{A}$. Applying Elkik's theorem [28, Thm. 7 and Rmk. 2 on p. 587], one can find a finitely generated k°-algebra B such that $\widehat{B} \xrightarrow{\sim} A$ and that $B \otimes_{k^\circ} k$ is smooth over k. Then the claim holds for $\mathscr{V} = \mathrm{Spec}\,(B \otimes_{k^\circ} k)$.

Step 2. By [10, Lemma 9.4], there is an open embedding of $\mathscr{V}$ in $\mathscr{Y}_\eta$, where $\mathscr{Y}$ is an integral scheme proper finitely presented and flat over k°, and an open subscheme $\mathscr{Z} \subset \mathscr{Y}_s$ such that $V = \mathrm{sp}_{\mathscr{Y}}^{-1}(\mathscr{Z})$, where $\mathrm{sp} : (\mathscr{Y}_\eta)^{\mathrm{an}} \to \mathscr{Y}_s$ underlies the specialization map $\mathrm{sp}_{\widehat{\mathscr{Y}}} : (\widehat{\mathscr{Y}})_\eta \to \widehat{\mathscr{Y}}$ of the formal completion $\widehat{\mathscr{Y}}$ of $\mathscr{Y}$ along $\mathscr{Y}_s$, and $(\widehat{\mathscr{Y}})_\eta$ is identified with $(\mathscr{Y}_\eta)^{\mathrm{an}}$, by properness. It follows that $\mathfrak{V}$ coincides with the admissible k°-formal scheme $\widehat{\mathscr{Y}_{/\mathscr{Z}}}$, formal completion of $\mathscr{Y}$ along $\mathscr{Z}$.

Step 3. Assume now that V is one-dimensional. Then both $\mathscr{V}$ and $\mathscr{Y}$ are one-dimensional. By the strong semistable reduction theorem for curves (1.1.18), there is a finite separable extension k' of k and a projective strictly semistable η-modification $\mathscr{Y}'$ of $\mathscr{Y} \otimes k'^\circ$. If $\mathscr{Z}'$ is the preimage of $\mathscr{Z}$ in $\mathscr{Y}'_s$, then the lemma holds for $\mathfrak{V}' = \widehat{\mathscr{Y}'}_{/\mathscr{Z}'}$, formal completion of $\mathscr{Y}'$ along $\mathscr{Z}'$. □

For any minimal triple $(\kappa : k \hookrightarrow k', \mathscr{Y}', \varphi')$ determined by $\mathscr{Y}$, as described in the theorem, and any $\sigma \in H := \mathrm{Gal}\,(k'/k)$, the triple $(\sigma \circ \kappa : k \hookrightarrow k', (\mathscr{Y}')^{(\sigma)}, (\varphi')^{(\sigma)})$, where

$$(\mathscr{Y}')^{(\sigma)} = \mathscr{Y}' \otimes_{(k'^\circ,\sigma)} k'^\circ ,$$

and

$$(\varphi')^{(\sigma)} = \varphi' \otimes_{(k'^\circ,\sigma)} \mathrm{id}_{k'^\circ} : (\mathscr{Y}')^{(\sigma)} \to \mathscr{Y} \otimes_{(k^\circ,\sigma\circ\kappa)} k'^\circ ,$$

is also minimal. Therefore, for any $\sigma \in H = \mathrm{Gal}(k'/k)$, there is a unique isomorphism $\iota_\sigma : \mathscr{Y}' \to (\mathscr{Y}'_{\mathrm{st}})^{(\sigma)}$ of k'°-formal schemes fitting in the commutative diagram

$$
\begin{array}{ccc}
\mathscr{Y}' & \xrightarrow{\varphi'} & \mathscr{Y} \otimes_{(k^\circ,\kappa)} k^\circ \\
\iota_\sigma \Big\downarrow & & \Big\downarrow \mathrm{id}_{\mathscr{Y}} \otimes \sigma \\
(\mathscr{Y}')^{(\sigma)} & \xrightarrow{(\varphi')^{(\sigma)}} & \mathscr{Y} \otimes_{(k^\circ,\sigma\circ\kappa)} k'^\circ
\end{array} \qquad (1.1.20.1)
$$

The action of $H = \mathrm{Gal}(k'/k)$ on $\mathscr{Y}'$ is not in general a descent datum for the faithfully flat extension $k^\circ \hookrightarrow k'^\circ$. Still, the quotient k°-scheme $\mathscr{Y}_{\mathrm{st}} := \mathscr{Y}'/H$ exists as a Q-semistable k°-scheme, and is endowed with an η-modification $\varphi_{\mathrm{st}} : \mathscr{Y}_{\mathrm{st}} \to \mathscr{Y}$.

Definition 1.1.21. *The pair $(\mathscr{Y}_{\mathrm{st}}, \varphi_{\mathrm{st}})$ will be called* the minimal Q-semistable η-modification of $\mathscr{Y}$.

The minimal Q-semistable η-modification $(\mathscr{Y}_{\mathrm{st}}, \varphi_{\mathrm{st}})$ of $\mathscr{Y}$ is is unique up to unique isomorphism inducing the identity on $\mathscr{Y}$. The previous notions have a formal counterpart as follows.

Theorem 1.1.22. *Let $\mathfrak{Y}$ be an admissible k°-formal scheme of pure relative dimension 1. Assume the generic fiber $\mathfrak{Y}_\eta$ is rig-smooth. Then, there is a finite galois extension $\kappa : k \hookrightarrow k'$, a strictly semistable formal scheme $\mathfrak{Y}'$ over k'° and η-modification $\phi' : \mathfrak{Y}' \to \mathfrak{Y} \widehat{\otimes}_{(k^\circ,\kappa)} k'^\circ$. Moreover, k' and the k'°-semistable formal scheme $\mathfrak{Y}'$ may be chosen to be* minimal *in the following sense. If $\lambda : k \hookrightarrow k''$ is a finite galois extension, $\mathfrak{Y}''$ is a k''°-semistable formal scheme and a morphism $\varphi'' : \mathfrak{Y}'' \to \mathfrak{Y}\widehat{\otimes}_{(k^\circ,\kappa)} k''^\circ$ is an η-modification of $\mathfrak{Y}\widehat{\otimes}_{(k^\circ,\kappa)} k''^\circ$, then for any choice (μ, ν) of k-embeddings $\mu : k' \hookrightarrow k'''$ and $\nu : k'' \hookrightarrow k'''$ in a finite galois extension k'''/k, there is a unique k'''°-morphism $\chi : \mathfrak{Y}'' \widehat{\otimes}_{(k''^\circ,\nu)} k'''^\circ \to \mathfrak{Y}' \widehat{\otimes}_{(k'^\circ,\mu)} k'''^\circ$, such that $\varphi''\widehat{\otimes}_{(k''^\circ,\nu)} \mathrm{id}_{k'''^\circ} = (\varphi'\widehat{\otimes}_{(k'^\circ,\mu)} \mathrm{id}_{k'''^\circ}) \circ \chi$.*

Proof. By uniqueness over a suitable finite galois extension k'/k, it is enough to prove the statement for $\mathfrak{Y} = \mathrm{Spf}\, A$ affine. Under this assumption, by Elkik's theorem [28, Thm. 7 and Rmk. 2 on p. 587], one can find a finitely generated k°-algebra B such that $\widehat{B} \xrightarrow{\sim} A$ and $B \otimes_{k^\circ} k$ is smooth over k. We now apply theorem 1.1.18 to the admissible affine k°-scheme $\mathscr{Y} = \mathrm{Spec}\, B$. We obtain a minimal semistable η-modification $\varphi' : \mathscr{Y}' \to \mathscr{Y} \otimes k'^\circ$, $\kappa : k \hookrightarrow k'$. Then (a)-adic completion $\widehat{\varphi'} : \mathfrak{Y}' \to \mathfrak{Y}\widehat{\otimes} k'^\circ$ of φ', for any $a \in k^{\circ\circ} \setminus \{0\}$, satisfies the requirements of the theorem. It is in fact clearly semistable. It is also minimal, because if $\lambda : k \hookrightarrow k''$ and $\phi'' : \mathfrak{Y}'' \to \mathfrak{Y}\widehat{\otimes}_{(k^\circ,\kappa)} k''^\circ$ have the properties of the statement, we may again apply Elkik's result, in the form quoted in [18, Lemma 5.14], to show that ϕ'' is the formal (a)-adic completion of a morphism $\varphi'' : \mathscr{Y}'' \to \mathscr{Y} \otimes_{(k^\circ,\kappa)} k''^\circ$ as in theorem 1.1.18. So, for any k-embeddings $\mu : k' \hookrightarrow k'''$ and $\nu : k'' \hookrightarrow k'''$ where k'''/k is a finite galois extension, there is a unique k'''°-morphism $\chi : \mathscr{Y}'' \otimes_{(k''^\circ,\nu)} k'''^\circ \to \mathscr{Y}' \otimes_{(k'^\circ,\mu)} k'''^\circ$, such that $\varphi'' \otimes_{(k''^\circ,\nu)} \mathrm{id}_{k'''^\circ} = (\varphi' \otimes_{(k'^\circ,\mu)} \mathrm{id}_{k'''^\circ}) \circ \chi$. Then the (a)-adic completion of χ satisfies the requirements in our statement. □

We deduce as above from theorem 1.1.22 the notion of the *minimal Q-semistable η-modification of* $\mathfrak{Y}$, which is a pair $(\mathfrak{Y}_{\mathrm{st}}, \varphi_{\mathrm{st}})$, consisting of a Q-semistable k°-formal scheme $\mathfrak{Y}_{\mathrm{st}}$, and of a morphism $\varphi_{\mathrm{st}} : \mathfrak{Y}_{\mathrm{st}} \to \mathfrak{Y}$: it is again unique up to unique isomorphism inducing the identity on $\mathfrak{Y}$.

The main result of this section is

Theorem 1.1.23. *(Q-semistable reduction of compact k-analytic curves) Let Y be a compact rig-smooth strictly k-analytic curve. The category $\mathcal{FS}^Q(Y)$ of Q-semistable models of Y*

and morphisms of k°-formal schemes inducing the identity on the generic fiber, is a non-empty partially ordered set, where $x \to y \Leftrightarrow x \geq y$. For any objects x, y of $\mathcal{FS}^{\mathrm{Q}}(Y)$, $x \vee y = \sup\{x, y\}$ exists. Every morphism in $\mathcal{FS}^{\mathrm{Q}}(Y)$ is an admissible Q-blow-up (in particular, is proper). For any object x of $\mathcal{FS}^{\mathrm{Q}}(Y)$, there is a unique minimal object m_x of $\mathcal{FS}^{\mathrm{Q}}(Y)$, such that $x \geq m_x$. Unless Y is a rational or a Tate projective curve, for any objects x, y of $\mathcal{FS}^{\mathrm{Q}}(Y)$, $x \wedge y = \inf\{x, y\}$ exists, hence $\mathcal{FS}^{\mathrm{Q}}(Y)$ admits a minimum.

The proof will be sketched in the next subsections. For more details, see [5].

In the following lemma we use the *completion functor*

(1.1.23.1)
$$\begin{array}{rccc} \widehat{\ }: & \mathcal{SA}_{/k} & \longrightarrow & \mathcal{FA}_{/k} \\ & (\tau : k \to K, \mathscr{X}) & \longmapsto & (\tau : k \to K, \widehat{\mathscr{X}}) \,. \end{array}$$

We admit here the following relatively standard result discussed in [5] (*cf.* [27, III.1, Th. 5.4.5], [15, Cor. 2.7].)

Lemma 1.1.24. *Let $\mathcal{Y}$ be a smooth projective k-algebraic curve and let $Y = \mathcal{Y}^{\mathrm{an}}$ be the associated compact k-analytic curve. The completion functor $\mathscr{X} \mapsto \widehat{\mathscr{X}}$ induces equivalences of categories*

(1.1.24.1)
$$\mathcal{PSS}(\mathcal{Y}) \xrightarrow{\sim} \mathcal{FS}(Y)$$

and

(1.1.24.2)
$$\mathcal{PSS}^{\mathrm{st}}(\mathcal{Y}) \xrightarrow{\sim} \mathcal{FS}^{\mathrm{st}}(Y) \,.$$

Corollary 1.1.25. *The completion functor $\mathscr{X} \mapsto \widehat{\mathscr{X}}$ induces an equivalence of categories*

(1.1.25.1)
$$\mathcal{PSS}^{\mathrm{Q}}(\mathcal{Y}) \xrightarrow{\sim} \mathcal{FS}^{\mathrm{Q}}(Y) \,.$$

The following corollary of theorem 1.1.23 and lemma 1.1.24 gives a complete description of the category of proper semistable models of a given smooth projective curve. We are not aware of a direct algebraic proof in the style of Temkin's theorem 1.1.18.

Theorem 1.1.26. *(Q-Semistable Reduction of Projective Curves) Let $\mathcal{Y}$ be a smooth projective k-algebraic curve. A proper Q-semistable model of $\mathcal{Y}$ is necessarily Q-projective. The category $\mathcal{PSS}^{\mathrm{Q}}(\mathcal{Y})$ of Q-projective, Q-semistable models of $\mathcal{Y}$, and morphisms of k°-schemes inducing the identity on the generic fiber, is a non-empty partially ordered set, where $x \to y \Leftrightarrow x \geq y$. For any objects x, y of $\mathcal{PSS}^{\mathrm{Q}}(\mathcal{Y})$, $x \vee y = \sup\{x, y\}$ exists. Every morphism in $\mathcal{PSS}^{\mathrm{Q}}(\mathcal{Y})$ is a Q-blow-up. For any object x of $\mathcal{PSS}^{\mathrm{Q}}(\mathcal{Y})$, there is a unique minimal object m_x of $\mathcal{PS}^{\mathrm{Q}}(\mathcal{Y})$, such that $x \geq m_x$. Unless a component of $\mathcal{Y} \otimes \widehat{k^{\mathrm{alg}}}$ is rational or is a Tate curve, for any objects x, y of $\mathcal{PSS}^{\mathrm{Q}}(\mathcal{Y})$, $x \wedge y = \inf\{x, y\}$ exists, hence $\mathcal{PSS}^{\mathrm{Q}}(\mathcal{Y})$ admits a minimum.*

1.2 Formal coverings and formal models

For the definition and basic properties of formal affinoid coverings (always assumed to be *strictly affinoid* in this paper) we refer to the first part of [6, §4.3]. We want to describe a functorial equivalence between (resp. strictly) semistable models and the (resp. strictly) semistable coverings of X described below.

The set of formal coverings of X forms a category $\mathcal{C}ov(X)$ in which an arrow $\mathcal{U} \to \mathcal{V}$ is a *refinement* of coverings, *i.e.* a map of sets $\Phi : \mathcal{U} \to \mathcal{V}$ such that $U \subset \Phi(U)$, $\forall U \in \mathcal{U}$. If $\Phi : \mathcal{U} \to \mathcal{V}$ is a refinement of formal coverings of X, the inclusions $U \hookrightarrow \Phi(U) = V$ induce morphisms of canonical reductions $\widetilde{U} \to \widetilde{V}$ which patch together to a morphism of $\widetilde{k}$-curves $\widetilde{\varphi}_\Phi : \widetilde{X}_\mathcal{U} \to \widetilde{X}_\mathcal{V}$. We obtain a natural functor $\mathcal{U} \mapsto \widetilde{X}_\mathcal{U}$, $\Phi \mapsto \widetilde{\varphi}_\Phi$, from $\mathcal{C}ov(X)$ to the category $\mathcal{S}ch_{\widetilde{k}}$ of $\widetilde{k}$-schemes.

For any formal covering $\mathcal{U}$ of X there is a canonical *specialization map* $\mathrm{sp}_\mathcal{U} : X \to \widetilde{X}_\mathcal{U}$, obtained by patching together the canonical reduction maps for all affinoids in the covering (their compatibility being precisely the definition of a *formal* affinoid covering). Specialization maps also have a functorial behaviour: if $\Phi : \mathcal{U} \to \mathcal{V}$ is a refinement, $\mathrm{sp}_\mathcal{V} = \widetilde{\varphi}_\Phi \circ \mathrm{sp}_\mathcal{U}$.

Two formal coverings $\mathcal{U}$ and $\mathcal{V}$ of X are *equivalent* if $\mathcal{U} \cup \mathcal{V}$ is a formal covering of X. Equivalence of formal coverings is an equivalence relation: we then write $\mathcal{U} \sim \mathcal{V}$, and denote by $[\mathcal{U}]$ the equivalence class of $\mathcal{U}$.

Definition 1.2.1. *A refinement of equivalent coverings $\Phi : \mathcal{U} \to \mathcal{V}$ is called a* quasi-isomorphism.

It is easy to check that the class of quasi-isomorphisms in $\mathcal{C}ov(X)$ admits calculus of right fractions. The corresponding localized category is denoted $\mathcal{C}ov(X)/\sim$ and is called *the category of formal affinoid coverings of X up to equivalence.* Its objects are equivalence classes $[\mathcal{U}]$ of formal coverings of X, and, for any pair $\mathcal{U}$, $\mathcal{V}$ of formal coverings of X,

$$\mathrm{Hom}_{\mathcal{C}ov(X)/\sim}([\mathcal{U}],[\mathcal{V}]) = \lim_{\substack{\longrightarrow \\ \mathcal{U}' \to \mathcal{U}}} \mathrm{Hom}_{\mathcal{C}ov(X)}(\mathcal{U}',\mathcal{V}) \ , \tag{1.2.1.1}$$

where $\mathcal{U}' \to \mathcal{U}$ is a quasi-isomorphism of formal coverings. If Φ is a quasi-isomorphism, $\widetilde{\varphi}_\Phi$ canonically identifies $\widetilde{X}_\mathcal{U}$ to $\widetilde{X}_\mathcal{V}$. In particular, if $\mathcal{U}$ and $\mathcal{V}$ are any pair of equivalent formal coverings of X, the inclusions $\Phi : \mathcal{U} \hookrightarrow \mathcal{U} \cup \mathcal{V}$ and $\Psi : \mathcal{V} \hookrightarrow \mathcal{U} \cup \mathcal{V}$ are quasi-isomorphisms and $\widetilde{\varphi}_\Phi$ and $\widetilde{\varphi}_\Psi$ are isomorphisms which canonically identify $\mathrm{sp}_\mathcal{U}$ with $\mathrm{sp}_\mathcal{V}$. In other words, the natural functor $\mathcal{U} \mapsto \widetilde{X}_\mathcal{U}$ factors through the *reduction functor* $\mathcal{R} : \mathcal{C}ov(X)/\sim \to \mathcal{S}ch_{\widetilde{k}}$.

Let us now assume that the formal covering $\mathcal{U}$ is distinguished. One then defines a sheaf of topological rings, flat and topologically of finite type [14, §6.4.1 Cor. 5], hence topologically of finite presentation [13, §2.3 Cor. 5] over k°, $\mathcal{O}_\mathcal{U}$ on $\widetilde{X}_\mathcal{U}$ by setting $\mathcal{O}_\mathcal{U}(V) = ((\mathrm{sp}_* \mathcal{O}_{X_G})(V))^\circ$, for any open subset V of $\widetilde{X}_\mathcal{U}$. The topologically ringed space $\mathfrak{X}_\mathcal{U} = (\widetilde{X}_\mathcal{U}, \mathcal{O}_\mathcal{U})$ is a formal model of X whose specialization map of ringed spaces $\mathrm{sp}_{\mathfrak{X}_\mathcal{U}} : X_G \to \mathfrak{X}_\mathcal{U}$ set-theoretically identifies with $\mathrm{sp}_\mathcal{U}$. It is easy to check that if $\Phi : \mathcal{U} \to \mathcal{V}$ is a refinement of formal coverings of X, there is a natural morphism of sheaves of topological rings $\varphi_\Phi^\flat : \mathcal{O}_\mathcal{V} \to (\widetilde{\varphi}_\Phi)_* \mathcal{O}_\mathcal{U}$ lifting $\widetilde{\varphi}_\Phi$ to a morphism of formal schemes $\varphi_\Phi = (\widetilde{\varphi}_\Phi, \varphi_\Phi^\flat) : \mathfrak{X}_\mathcal{U} \to \mathfrak{X}_\mathcal{V}$ whose generic fiber is the identity of the k-analytic curve X. Moreover, if Φ is a quasi-isomorphism of distinguished coverings, $\varphi_\Phi : \mathfrak{X}_\mathcal{U} \to \mathfrak{X}_\mathcal{V} = \mathfrak{X}_\mathcal{U}$ is the identity of the formal scheme $\mathfrak{X}_\mathcal{U}$.

Definition 1.2.2. *We denote by $\mathcal{C}ov^{\mathrm{dis}}(X)$ the full subcategory of $\mathcal{C}ov(X)$ whose objects are distinguished formal coverings of X .*

The class of quasi-isomorphisms in $\mathcal{C}ov^{\mathrm{dis}}(X)$ admits calculus of right fractions. The corresponding localized category $\mathcal{C}ov^{\mathrm{dis}}(X)/\sim$ is a full subcategory of $\mathcal{C}ov^{\mathrm{dis}}(X)/\sim$, and is called *the category of distinguished formal coverings of X up to equivalence.* Its objects are equivalence classes $[\mathcal{U}]$ of distinguished formal coverings of X. The following result will be detailed in [5].

Theorem 1.2.3. *The construction $\mathcal{U} \mapsto \mathfrak{X}_\mathcal{U}$, $\Phi \mapsto \varphi_\Phi$ extends to a functor $\mathcal{C}ov^{\mathrm{dis}}(X) \to \mathcal{FA}(X)$, which induces an equivalence of categories $\mathcal{C}ov^{\mathrm{dis}}(X)/\sim \xrightarrow{\sim} \mathcal{FA}(X)$.*

As explained in [13, §2.8, Step (b) in the proof of Thm. 3], for two admissible formal models $\mathfrak{X}$ and $\mathfrak{Y}$ of X, there is at most one morphism $\varphi : \mathfrak{X} \to \mathfrak{Y}$ inducing the identity on generic fibers. Therefore, the category $\mathcal{FA}(X)$ is really a partially ordered directed set. It then follows that, for any pair $\mathcal{U}$, $\mathcal{V}$ of distinguished formal coverings of X,

$$\text{(1.2.3.1)} \qquad \mathrm{Hom}_{\mathcal{C}ov^{\mathrm{dis}}(X)/\sim}([\mathcal{U}],[\mathcal{V}]) = \begin{cases} \{\geq\} & \text{if } \exists\, \Phi : \mathcal{U}' \to \mathcal{V}'\,,\ \mathcal{U}' \sim \mathcal{U}\,,\ \mathcal{V}' \sim \mathcal{V}; \\ \emptyset & \text{otherwise.} \end{cases} ,$$

where $\Phi : \mathcal{U}' \to \mathcal{V}'$ is a refinement of formal coverings.

Definition 1.2.4. *Let X be as before, and let $\mathcal{U}$ be a distinguished formal strictly affinoid covering of X. We say that $\mathcal{U}$ is a (resp.* strictly*) semistable affinoid covering *of X, if $\widetilde{X}_{\mathcal{U}}$ is a (strictly) semistable curve over $\widetilde{k}$. A formal strictly affinoid covering $\mathcal{U}$ of X is* potentially semistable *if there is a finite separable extension k'/k such that*

$$\mathcal{U}' := \{V \mid V \textit{is a connected component of} \, U\widehat{\otimes}_k k'\}_{U\in\mathcal{U}}$$

is a semistable affinoid covering of $X' := X\widehat{\otimes}_k k'$. It is clear that these notions are compatible with equivalence of formal coverings. We define $\mathcal{CS}(X)$ (resp. $\mathcal{CS}^{\mathrm{st}}(X)$, resp. $\mathcal{CS}^{\mathrm{pt}}(X)$) as the full subcategory of $\mathcal{C}ov^{\mathrm{dis}}(X)/\sim$ whose objects are equivalence classes of (resp. strictly, resp. potentially) semistable affinoid coverings of X.

Remark 1.2.5. Notice that a potentially semistable affinoid covering $\mathcal{U}$ of X may not be distinguished. Let k'/k be a finite galois extension of group H such that $\mathcal{U}'$, defined as in (1.2.4) is a semistable covering of $X' = X\widehat{\otimes}_k k'$. The action of the group H on the $\widetilde{k}'$-scheme $\widetilde{X'}_{\mathcal{U}'}$ is compatible with the action of H on $\widetilde{k}'$ by $\widetilde{k}$-automorphisms. The reduced k-scheme $\widetilde{X}_{\mathcal{U}}$ is the *defined* as the quotient $\widetilde{X'}_{\mathcal{U}'}/H$.

Remark 1.2.6. A semistable covering $\mathcal{U}$ of X is distinguished by assumption and the reduction $\widetilde{X}_{\mathcal{U}}$ is geometrically reduced. We deduce from [15, Prop. 1.2] that for any non-archimedean extension field K of k and for $\mathcal{U}\widehat{\otimes}K = \{U\widehat{\otimes}K\}_{U\in\mathcal{U}}$, one has

$$\widetilde{X\widehat{\otimes}K}_{\mathcal{U}\widehat{\otimes}K} = \widetilde{X}_{\mathcal{U}} \otimes \widetilde{K}\ .$$

In particular, $\mathcal{U}\widehat{\otimes}K$ is a semistable covering of $X\widehat{\otimes}K$.

It follows from Theorem 1.1.6 that if the covering $\mathcal{U}$ is (strictly) semistable, then $\mathfrak{X}_{\mathcal{U}}$ is a (strictly) semistable model of X.

Corollary 1.2.7. *The construction $\mathcal{U} \mapsto \mathfrak{X}_{\mathcal{U}}$ induces an equivalence of the category $\mathcal{CS}(X)$ (resp. $\mathcal{CS}^{\mathrm{st}}(X)$, resp. $\mathcal{CS}^{\mathrm{pt}}(X)$) of (resp. strictly, resp. potentially) semistable coverings up to equivalence, and the category $\mathcal{FS}(X)$ (resp. $\mathcal{FS}^{\mathrm{st}}(X)$, resp. $\mathcal{FS}^{\mathrm{Q}}(X)$). It has the property that* $\mathrm{Hom}(\mathfrak{X}_{\mathcal{U}},\mathfrak{X}_{\mathcal{V}})$ *is nonempty if and only if there are formal coverings $\mathcal{U}'$ and $\mathcal{V}'$ respectively equivalent to $\mathcal{U}$ and $\mathcal{V}$, and a refinement $\Phi : \mathcal{U}' \to \mathcal{V}'$. In that case* $\mathrm{Hom}(\mathfrak{X}_{\mathcal{U}},\mathfrak{X}_{\mathcal{V}}) = \{\varphi_\Phi\}$, *via the canonical identifictions of $\mathfrak{X}_{\mathcal{U}'}$ and $\mathfrak{X}_{\mathcal{V}'}$ with $\mathfrak{X}_{\mathcal{U}}$ and $\mathfrak{X}_{\mathcal{V}}$, respectively.*

1.3 Disks and annuli

For $a \in \mathbb{A}^1(k)$ and $r \in \mathbb{R}_{>0}$ we denote by

$$D(a,r^-) = \{x \in \mathbb{A}^1 : |T(x) - a| < r\} \subset \mathbb{A}^1$$

(resp.

$$D(a,r^+) = \{x \in \mathbb{A}^1 : |T(x) - a| \leq r\} \subset \mathbb{A}^1)\ ,$$

the *standard open* (resp. *closed*) *k-disk* of radius $r > 0$ centered at a. A real number r is *k-rational* if $r \in |k|$. The maximal point point $t_{a,r}$ of $D(a, r^+)$ is defined by the multiplicative norm

$$|\sum_{i=0}^{m} a_i(T-a)^i(t_{a,r})| = \max_{i=0,\dots,m} |a_i| r^i$$

on $k[T]$. We also denote by

$$B(a; r_1, r_2) = \{x \in \mathbb{A}^1 : r_1 < |T(x) - a| < r_2\} \subset \mathbb{A}^1$$

(resp.

$$B[a; r_1, r_2] = \{x \in \mathbb{A}^1 : r_1 \le |T(x) - a| \le r_2\} \subset \mathbb{A}^1) \ ,$$

the *standard open* (resp. *closed*) *k-annulus* of radii $0 < r_1 < r_2$ (resp. $0 < r_1 \le r_2$), centered at $a \in \mathbb{A}^1(k)$. We write $B(r_1, r_2)$ (resp. $B[r_1, r_2]$) for the standard annulus $B(0; r_1, r_2)$ (resp. $B[0; r_1, r_2]$). The ratio $r(V) = r_1/r_2 \in (0, 1)$ is the *height* of $V = B(a; r_1, r_2)$. It is well-known that a standard open k-disk $D(a, r^-)$ (resp. k-annulus $V = B(a; r_1, r_2)$) is isomorphic to the standard unit open disk $D(0, 1^-)$ (resp. to the standard open annulus $B(r(V), 1)$, if and only if r (resp. either r_1 or r_2) is k-rational. Moreover, if a k-analytic curve V is isomorphic via the coordinate T to $B(r(V), 1)$, for any automorphism $\varphi \in \mathrm{Aut}(V)$, and any $x \in V$, $|T(\varphi(x))|$ equals either $|T(x)|$ or $r(V)/|T(x)|$: φ is *direct* in the former case and *inverse* in the latter. We denote by $\mathrm{Aut}^+(V)$ the subgroup of $\mathrm{Aut}(V)$ consisting of direct automorphisms. Obviously, $\mathrm{Aut}(V)/\mathrm{Aut}^+(V) = \{\pm 1\}$.

Definition 1.3.1. *An (open or closed)* k-disk *(resp.* k-annulus*) is a k-analytic curve V isomorphic to a standard (open or closed) k-disk (resp. k-annulus). It is* k-rational *if the corresponding standard disk (resp. annulus) has k-rational radius (resp. radii). The* height *$r(V)$ of a k-annulus V is the height of the corresponding annulus in $\mathbb{A}^1$. An open or closed (k-analytic)* disk *(resp.* annulus*) is a connected rig-smooth (k-analytic) curve V for which there exists a finite separable extension k'/k such that one (hence any) connected component of $V \otimes k'$ is an (open or closed) k'-disk (resp. k'-annulus). Let V be an open k-analytic annulus, let k'/k be a finite galois extension such that $V \otimes k'$ is the disjoint union of its connected components $B_1, \dots, B_g$, each a standard open k'-annulus. Let $D_i \subset \mathrm{Gal}\,(k'/k)$ be the stabilizer of B_i, for $i = 1, \dots, g$. The open k-analytic annulus V is* split *if, for some (hence for any) $i = 1, \dots, g$, the corresponding cohomology class in $H^1(D_i, \mathrm{Aut}(B_i))$ comes from a cohomology class in $H^1(D_i, \mathrm{Aut}^+(B_i))$. Otherwise V is* non-split. *The* height $r(V)$ *of a (k-analytic) annulus V is the height $r(B_i)$ of any component B_i of $V \otimes_k k'$, as before.*

Let Y be a rig-smooth strictly k-analytic curve, and let V be a relatively compact open (resp. a compact) analytic domain in Y. Then, V is an open (resp. closed) disk or annulus if and only if every connected component of $V \widehat{\otimes} \widehat{k^{\mathrm{alg}}}$ is an open (resp. closed) $\widehat{k^{\mathrm{alg}}}$-disk or $\widehat{k^{\mathrm{alg}}}$-annulus in the $\widehat{k^{\mathrm{alg}}}$-analytic curve $Y \widehat{\otimes} \widehat{k^{\mathrm{alg}}}$. If, on the other hand, V' is an open (resp. closed) $\widehat{k^{\mathrm{alg}}}$-disk or $\widehat{k^{\mathrm{alg}}}$-annulus in $Y \widehat{\otimes} \widehat{k^{\mathrm{alg}}}$ and V is the image of V' in Y under the canonical projection $Y \widehat{\otimes} \widehat{k^{\mathrm{alg}}} \to Y$, we prove in [5] that either V is the analytification of a rational projective curve (resp. of a Tate curve) or it is an open k-analytic disk (resp. annulus) in Y. An open disk (resp. annulus) in X is a simply-connected quasi-polyhedron. An open disk has precisely one endpoint. An open annulus has one (resp. two) endpoint(s) if and only if it is non-split (resp. split).

Definition 1.3.2. *The* standard formal disk *over k° is the formal affine line $\widehat{\mathbb{A}}^1 := \widehat{\mathbb{A}^1_{k^\circ}} = \mathrm{Spf}\, k^\circ\{T\}$. The* standard formal annulus of height $r \in |k| \cap (0, 1)$ *over k° is the k°-formal scheme $\mathfrak{B}[r, 1] := \mathrm{Spf}\, k^\circ\{S, T\}/(ST - a)$, where $a \in k^\circ$, $|a| = r$. A* basic formal disk *(resp. a* basic formal annulus*), is an affine connected k°-formal scheme $\mathfrak{B} = \mathrm{Spf}\, A$ equipped with a dominant étale morphism T to the standard formal disk (resp. to a standard formal annulus) over k°, with the following property.*

- *the generic fiber of T induces an isomorphism of k-analytic spaces $T^{-1}(B(|a|,1)) \xrightarrow{\sim} B(|a|,1)$ (resp. $T^{-1}(D(0,1^-)) \xrightarrow{\sim} D(0,1^-)$)*

A basic affinoid disk *(resp.* basic affinoid annulus*) over k is the generic fiber of a basic formal disk (resp. annulus) over k°. We write $(\mathfrak{B},T)$ for a formal disk (resp. annulus) and $(\mathcal{B},T)$ for the corresponding affinoid disk (resp. annulus).*

We will make essential use of the following immediate consequence of definition 1.1.5.

Lemma 1.3.3. *Every semistable k°-formal scheme has a finite covering by étale neighborhoods which are basic formal annuli or disks over k°. For every Q-semistable k°-formal scheme $\mathfrak{X}$, there is a finite galois extension k'/k of group H, a semistable k'°-formal scheme $\mathfrak{X}'$ such that $\mathfrak{X} = \mathfrak{X}'/H$, and a finite covering of $\mathfrak{X}'$ by étale neighborhoods which are basic formal annuli or disks over k'°.*

In our discussion below, it will be more convenient to use a different normalization for the coordinate function T on a formal annulus. Namely, for $r_1, r_2 \in |k| \cap (0,1]$, we define,

(1.3.3.1) $$A[r_1,r_2] = k^\circ\{S,T,U\}/(a_2S - T, TU - a_1) \ , \ \mathfrak{B}[r_1,r_2] = \mathrm{Spf}\, A[r_1,r_2] \ ,$$

where $a_i \in k^\circ$, $|a_i| = r_i$, $i = 1,2$. So, $\mathfrak{B}[r_1,r_2]$ is isomorphic to the formal annulus $\mathfrak{B}[r_1/r_2,1]$, via the coordinate T/a_2. Let $\mathfrak{B}$ be a k°-formal scheme and $T : \mathfrak{B} \to \mathfrak{B}[r_1,r_2]$ be an étale morphism, such that the composite $\mathfrak{X} \xrightarrow{T} \mathfrak{B}[r_1,r_2] \xrightarrow{T/a_2} \mathfrak{B}[r_1/r_2,1]$ is a basic formal annulus of height r_1/r_2 over k°. We still say that $(\mathfrak{B},T)$ (resp. its generic fiber $(\mathcal{B},T)$) is a *basic formal (resp. affinoid) annulus of radii r_1, r_2 over k°.*

1.4 Semistable partitions

We introduce here a very simple notion, equivalent to the ones discussed in the previous subsections of semistable formal model and of semistable affinoid covering of X. This is the notion of a "semistable partition" of X. It has the virtue of making completely obvious two well-known statements for which there is no canonical reference in the literature. Namely, the existence of a minimum semistable model of X (unless X is the analytification of either a rational or a Tate curve), and the existence of a common minimum semistable refinement of two semistable formal coverings. Again, we need such explicit descriptions, in view of application to differential systems on X. Again we refer to [5] for full proofs.

Definition 1.4.1. *Let X be a connected compact rig-smooth strictly k-analytic curve over a non-trivially valued non-archimedean field k. A partition of X into a disjoint union of a finite set of open Q-annuli $\mathcal{B} = \{B_1,\dots,B_N\}$ and a finite set $\mathcal{C} = \{C_1,\dots,C_r\}$ of connected strictly affinoid domains having potentially (*i.e. *after base-change to a finite separable extension k'/k) good canonical reduction will be called a Q*-semistable partition *of X, and will be denoted by $\mathscr{P}(\mathcal{B},\mathcal{C})$. The Q-semistable partition $\mathscr{P}(\mathcal{B},\mathcal{C})$ is a* strictly semistable partition *if every annulus $B \in \mathcal{B}$ is a split k-annulus with two distinct boundary points in $X \setminus B$ and every affinoid $C \in \mathcal{C}$ has good canonical reduction. If X is the analytification of a smooth projective curve with (resp. potentially) good reduction, we also regard $\mathscr{P}(\emptyset,\{X\})$ as a strictly semistable (resp. Q-semistable) partition. For two Q-semistable partitions $\mathscr{P} = \mathscr{P}(\mathcal{B},\mathcal{C})$ and $\mathscr{P}' = \mathscr{P}(\mathcal{B}',\mathcal{C}')$ of X, we say that $\mathscr{P}' \geq \mathscr{P}$ if for every $B \in \mathcal{B}$ (resp. $C \in \mathcal{C}$), the elements of $\mathcal{B}'$ and $\mathcal{C}'$ contained in B (resp. C) give a partition of B (resp. C). We denote by $\mathcal{PS}^{\mathrm{Q}}(X)$ (resp. $\mathcal{PS}^{\mathrm{st}}(X)$) the category, in fact a partially ordered set, of Q-semistable (resp. strictly semistable) partitions of X.*

Remark 1.4.2. If $\mathscr{P}(\mathcal{B},\mathcal{C})$ is a strictly semistable partition of X, and $B \in \mathcal{B}$, the two boundary points of B in $X \setminus B$ are the maximal points of two distinct affinoids $C_{B,1}$ and $C_{B,2} \in \mathcal{C}$.

Let $\mathfrak{X}$ be a Q-semistable (resp. a strictly semistable) formal model of X and let sp : $X \to \mathfrak{X}_s$ be the corresponding specialization map to a curve over $\widetilde{k}$. Let $\mathrm{Sing}(\mathfrak{X}_s)$ be the singular locus of $\mathfrak{X}_s$, and let $\mathfrak{X}_s^{\mathrm{sm}} = \mathfrak{X}_s \setminus \mathrm{Sing}(\mathfrak{X}_s)$ be the smooth part of $\mathfrak{X}_s$. Then $\mathrm{Sing}(\mathfrak{X}_s) = \{z_1, \dots, z_N\}$ where each z_i is a closed point of $\mathfrak{X}_s$, and $B_i := \mathrm{sp}^{-1}(z_i)$ is an open annulus in X. Similarly, let $\mathfrak{c}_1, \dots, \mathfrak{c}_r$ be the connected (hence irreducible) components of the smooth locus $\mathfrak{X}_s^{\mathrm{sm}}$, and let $\widetilde{\eta}_{\mathfrak{c}_j}$ be the generic point of $\mathfrak{c}_j$. Then either $\mathfrak{X}$ is smooth (resp. Q-smooth) in which case $r = 1$ and $\mathfrak{c}_1 = \mathfrak{X}_s$, or $\mathrm{sp}^{-1}(\mathfrak{c}_j)$ is a connected affinoid domain C_j in X with (resp. potentially) good canonical reduction, whose Shilov boundary consists of the single point η_{C_j}, unique inverse image of $\widetilde{\eta}_{\mathfrak{c}_j}$. Let $\mathcal{B}_{\mathfrak{X}} = \{B_1, \dots, B_N\}$ and $\mathcal{C}_{\mathfrak{X}} = \{C_1, \dots, C_r\}$. We conclude that $\mathscr{P}_{\mathfrak{X}} = \mathscr{P}(\mathcal{B}_{\mathfrak{X}}, \mathcal{C}_{\mathfrak{X}})$ is a Q-semistable (resp. strictly semistable) partition of X. If $\varphi : \mathfrak{Y} \to \mathfrak{X}$ is a morphism of Q-semistable (resp. a strictly semistable) formal models of X, then $\mathscr{P}_{\mathfrak{Y}} \geq \mathscr{P}_{\mathfrak{X}}$: we have defined a functor $\mathfrak{X} \mapsto \mathscr{P}_{\mathfrak{X}}$ from the category of Q-semistable (resp. strictly semistable) formal models of X to the category of Q-semistable (resp. strictly semistable) partitions of X.

We prove in [5]

Theorem 1.4.3. *The functor $\mathfrak{X} \mapsto \mathscr{P}_{\mathfrak{X}}$ induces an equivalence of categories between $\mathcal{FS}^{\mathrm{Q}}(X)$ (resp. $\mathcal{FS}^{\mathrm{st}}(X)$) and $\mathcal{PS}^{\mathrm{Q}}(X)$ (resp. $\mathcal{PS}^{\mathrm{st}}(X)$).*

In order to construct a quasi-inverse of the functor $\mathfrak{X} \mapsto \mathscr{P}_{\mathfrak{X}}$, we first associate to a Q-semistable (resp. strictly semistable) partition $\mathscr{P} = \mathscr{P}(\mathcal{B}, \mathcal{C})$ of X, a potentially (resp. strictly) semistable covering $\mathcal{U}_{\mathscr{P}}$ of X. We show that the composite functor $\mathfrak{X} \mapsto \mathscr{P} := \mathscr{P}_{\mathfrak{X}} \mapsto \mathcal{U}_{\mathscr{P}} \mapsto \mathfrak{X}_{\mathcal{U}_{\mathscr{P}}}$ is the identity of $\mathcal{FS}^{\mathrm{Q}}(X)$ (resp. of $\mathcal{FS}^{\mathrm{st}}(X)$) and that $\mathscr{P} \mapsto \mathcal{U}_{\mathscr{P}} \mapsto \mathfrak{X} := \mathfrak{X}_{\mathcal{U}_{\mathscr{P}}} \mapsto \mathscr{P}_{\mathfrak{X}}$ is the identity of $\mathcal{PS}^{\mathrm{Q}}(X)$ (resp. $\mathcal{PS}^{\mathrm{st}}(X)$).

The following results are proven in [5].

Proposition 1.4.4. *The partially ordered set of Q-semistable partitions of X is directed: if $\mathscr{P} = \mathscr{P}(\mathcal{B}, \mathcal{C})$ and $\mathscr{P}' = \mathscr{P}(\mathcal{B}', \mathcal{C}')$ are Q-semistable partitions of X, there exists a minimum Q-semistable partition $\mathscr{P}''$ of X with $\mathscr{P}'' \geq \mathscr{P}$ and $\mathscr{P}'' \geq \mathscr{P}'$.*

Proposition 1.4.5. *Assume X is neither the analytification of a rational nor of a Tate curve. Then the partially ordered set of Q-semistable partitions of X has a minimum $\mathscr{P}_0 = \mathscr{P}(\mathcal{B}_0, \mathcal{C}_0)$.*

1.5 Q-subpolygons of an analytic curve

There is a fourth equivalent viewpoint we need to master for dealing with differential equations on X. This is the notion of a "Q-subpolygon of X". Actually, this notion has been used implicitly by Dwork, Robba, Christol,... since the infancy of the theory of p-adic differential equations. The framework of Berkovich analytic spaces gives substance to what has been for a long time a somewhat artificial tool to describe the "generic" behaviour of radii of convergence of power series solutions to differential equations. The global description of radii of convergence on the full Berkovich analytic space is the main novelty of this paper. This section is entirely due to Berkovich, but we are unable to give precise references for most of the material we have collected here. Full details are given in [5].

Let k be any non-archimedean field.

Definition 1.5.1. *Let Y be any rig-smooth strictly k-analytic curve. The* analytic skeleton *$S(Y)$ of Y is the minimum closed subset T of Y, such that $Y \setminus T$ is a union of open Q-disks. A* vertex *of Y is a point contained in no open Q-annulus. We denote by $\mathcal{V}(Y) \subset Y$ the set of vertices of Y.*

Notice that a vertex of Y necessarily belongs to $S(Y)$ and that, if Y is affinoid, any point in the Shilov boundary of Y is a vertex. Points of $S(Y)$ are of type (2) or (3). We denote by $\mathbb{H}(Y) \subset Y$ the subset consisting of points of type (2) or (3), and call it the *Berkovich hyperbolic subspace* of the curve Y. Moreover, for any non-archimedean extension field k'/k, $S(Y)$ (resp. $\mathcal{V}(Y)$) is the image of $S(Y\widehat{\otimes}k')$ (resp. of $\mathcal{V}(Y\widehat{\otimes}k')$) in Y under the canonical projection $Y\widehat{\otimes}k' \to Y$.

Definition 1.5.2. *An* open segment *(resp.* open Q-segment*)* E *in* Y *is the skeleton of an open* k*-annulus (resp.* Q*-annulus) in* Y*. Its (*multiplicative*)* length $\ell(E)$ *is the inverse of the height of the corresponding annulus. An* open half-line *(resp.* open Q-half-line*) in* Y *is the skeleton of a* k*-punctured (resp.* Q*-punctured) open disk in* Y*. We say that it has* infinite length.

Example 1.5.3. For the standard k-annulus $B(r_1, r_2)$, the skeleton is the subset $\{t_{0,\rho}|r_1 < \rho < r_2\}$, homeomorphic to (r_1, r_2) via $\rho \mapsto t_{0,\rho}$. The elements of $\mathrm{Aut}(B(r_1, r_2))$ act on $S(B(r_1, r_2))$ via the identity, if they are in $\mathrm{Aut}^+(B(r_1, r_2))$, and via $\rho \mapsto r_1r_2/\rho$, otherwise. Similarly for pointed open disks.

We show in [5], with the help of [15, 5.4], that for an open segment (resp. an open half-line) $E \subset Y$, the open k-annulus (resp. the open k-punctured disk) $A \subset Y$ such that $S(A) = E$ is uniquely determined as the union of open k-disks D in $Y \setminus E$ such that the closure of D in Y intersects E. We denote it by A_E. Similarly for the Q-variants. An open segment E is *k-rational* if A_E is a k-rational open k-annulus in Y. An open Q-segment E is *split* (resp. *non-split*) if A_E is a split (resp. non-split) open annulus in Y.

Let E be an open Q-segment of Y of length $1/r(E)$ let and k'/k be a non-archimedean field extension such that every component of $A_E\widehat{\otimes}_k k'$ is an open k'-rational k'-annulus. Let us identify one such component B' with $B_{k'}(r(E), 1)$. Then, if A_E is split, the homeomorphism $S(B') \xrightarrow{\sim} S(B_{k'}(r(E), 1)) \xrightarrow{\sim} (r(E), 1)$, induces a homeomorphism $\rho = |T(-)| : E \xrightarrow{\sim} (r(E), 1)$, which is canonical up to the inversion $\rho \mapsto r(E)/\rho$. If on the other hand A_E is non-split, the homeomorphism $S(B') \xrightarrow{\sim} (r(E), 1)$, induces a canonical homeomorphism $\rho = |T(-)| : E \xrightarrow{\sim} [r(E)^{1/2}, 1)$.

Notice that the previous construction associates canonically, on any simply connected sub-quasi-polyhedron U of $\mathbb{H}(X)$, a (multiplicative) *length* $\ell_U([x, y]) \in (1, \infty)$ where the *link* $[x, y]$, with $x, y \in U$ is the unique path in U joining x to y. In fact, $[x, y]$ is a disjoint union of a finite set $E_1, \dots, E_N$ of open Q-segments of X contained in U, and $\ell([x, y]) = \prod_{i=1}^{N} \ell(E_i)$. The function $(x, y) \mapsto \ell([x, y])$ is clearly continuous on $U \times U$.

Definition 1.5.4. *A* Q-subpolygon *of* X *is a triple* $\mathbf{S} = (S, \mathcal{V}, \mathcal{E})$*, where* S *is a compact connected subset of* $\mathbb{H}(X)$ *and* $\mathcal{V}$ *is a finite subset of points of type* (2) *of* S *called the* vertices *of* $\mathbf{S}$*. We assume that the connected components of* $S \setminus \mathcal{V}$ *are a finite set of open* Q*-segments* $E_1, \dots, E_N$ *of* X*. We call them* open edges *of* $\mathbf{S}$*, write* $\mathcal{E} = \{E_1, \dots, E_N\}$*. For two* Q*-subpolygons* $\mathbf{S} = (S, \mathcal{V}, \mathcal{E})$ *and* $\mathbf{S}' = (S', \mathcal{V}', \mathcal{E}')$ *of* X*, we write* $\mathbf{S} \le \mathbf{S}'$*, and say that* $\mathbf{S}$ *is a* Q-subpolygon of $\mathbf{S}'$ *if* $S \subset S'$*,* $\mathcal{V} \subset \mathcal{V}'$*, and every open edge* $E \in \mathcal{E}$ *is a union of vertices and open edges of* $\mathbf{S}'$*. A* morphism $\varphi : \mathbf{S}' \to \mathbf{S}$ *of* Q*-subpolygons of* X *is a continuous retraction* $\varphi : S' \to S$ *such that for any open edge* $E' \in \mathcal{E}'$*, either* $\varphi(E')$ *is a vertex of* $\mathbf{S}$ *or* φ *induces the inclusion of* E *in an open edge of* $\mathbf{S}$*. A* Q*-subpolygon* $\mathbf{S} = (S, \mathcal{V}, \mathcal{E})$ *of* X *is* complete *if* $X \setminus S$ *is a union of open disks. We denote by* $\mathcal{GP}(X)$ *the category of* Q*-subpolygons of* X*, and by* $\mathcal{GP}^c(X)$ *the full subcategory consisting of complete* Q*-subpolygons of* X*.*

Notice that, if a morphism $\varphi : \mathbf{S}' \to \mathbf{S}$ in $\mathcal{GP}(X)$ exists, $\mathbf{S}$ is a subpolygon of $\mathbf{S}'$, and the fibers of φ over a vertex are simply connected trees. We also denote by $(\mathcal{GP}(X), \le)$ the category associated to the order relation $\le$: a morphism $\mathbf{S} \le \mathbf{S}'$ in this category is called an *inclusion* $\mathbf{S} \to \mathbf{S}'$ of Q-subpolygons of X. Let $\iota : \mathbf{S} \to \mathbf{S}'$ be an inclusion of subpolygons of X. We say that ι is a *subdivision* (and that $\mathbf{S}'$ is "obtained by subdivision of $\mathbf{S}$") if $S = S'$.

A subdivision is visibly the composition of a finite number of subdivisions obtained by the addition of one vertex inside an open edge. We call *1-step subdivision* such a subdivision. In general, a subdivision $\mathbf{S} \leq \mathbf{S}'$ in which $\mathbf{S}'$ has N vertices more than $\mathbf{S}$, *i.e.* an *N-step subdivision*, is a product of N 1-step subdivisions. An inclusion $\mathbf{S} \leq \mathbf{S}'$ is called *exact* if every vertex (resp. open edge) of $\mathbf{S}'$ contained in S is a vertex (resp. open edge) of $\mathbf{S}$. An inclusion $\iota : \mathbf{S} \to \mathbf{S}'$ canonically decomposes into a composition $\iota = \varepsilon \circ \sigma$, where $\varepsilon : \mathbf{S}^s \to \mathbf{S}'$ is an exact inclusion and $\sigma : \mathbf{S} \to \mathbf{S}^s$ is a subdivision.

For a morphism $\varphi : \mathbf{S}' \to \mathbf{S}$ in $\mathcal{GP}(X)$, we say that it is a *trivial retraction* if $\mathbf{S} \leq \mathbf{S}'$ is a subdivision, and a *neat retraction* if $\mathbf{S} \leq \mathbf{S}'$ is an exact inclusion. Any morphism $\varphi : \mathbf{S}' \to \mathbf{S}$ in $\mathcal{GP}(X)$ canonically decomposes as $\varphi^{\rm triv} \circ \varphi^{\rm neat}$, where $\varphi^{\rm neat} : \mathbf{S}' \to \mathbf{S}^s$ is neat and $\varphi^{\rm triv} : \mathbf{S}^s \to \mathbf{S}$ is trivial. Notice that a neat retraction sends vertices to vertices. A neat retraction $\varphi : \mathbf{S}' \to \mathbf{S}$ is *simple* if there are precisely one open edge and one vertex of $\mathbf{S}'$ which are not, respectively, an open edge and a vertex of $\mathbf{S}$. Every neat retraction $\varphi : \mathbf{S}' \to \mathbf{S}$ is a composition of a finite number N of simple retractions. The number N is the number of open edges of $\mathbf{S}'$ which are not open edges of $\mathbf{S}$. Similarly, every trivial retraction is a product of a finite number of trivial 1-step retractions, each corresponding to a 1-step subdivision.

To any Q-semistable partition $\mathscr{P} = \mathscr{P}(\mathcal{B} = \{B_1, \dots, B_N\}, \mathcal{C} = \{C_1, \dots, C_r\})$ of X, we naturally associate a complete Q-subpolygon $\mathbf{S}(\mathscr{P})$, whose vertices are the maximal points of the affinoids $C_1, \dots, C_r$ and whose open edges are the skeleta of the open annuli $B_1, \dots, B_N$. If $\mathscr{P}' \geq \mathscr{P}$ in $\mathcal{PS}^Q$, it is clear that $\mathbf{S}(\mathscr{P}') \geq \mathbf{S}(\mathscr{P})$ as Q-subpolygons of X, and that there is a morphism $\mathbf{S}(\mathscr{P}') \to \mathbf{S}(\mathscr{P})$ in $\mathcal{GP}^c$.

The following results are proven in [5].

Theorem 1.5.5. *The functor* $\mathbf{S}$ *establishes an equivalence of categories between* $\mathcal{PS}^Q(X)$ *and* $\mathcal{GP}^c(X)$. *Moreover,* $\mathcal{GP}^c(X)$ *is the opposite category to* $(\mathcal{GP}^c(X), \leq)$. *We also denote by* $\mathbf{S}$ *the composite functor* $\mathfrak{X} \mapsto \mathscr{P}_{\mathfrak{X}} \mapsto \mathbf{S}(\mathscr{P}_{\mathfrak{X}}) =: \mathbf{S}(\mathfrak{X})$*, which induces an equivalence of categories between* $\mathcal{FS}^Q(X)$ *and* $\mathcal{GP}^c(X)$.

Notice that [10, 1.7] the generic point of a component $\mathfrak{c}_i$ of $\mathfrak{X}_s$ has a unique inverse image η_i in X under ${\rm sp}_{\mathfrak{X}}$, and the set of vertices $\mathcal{V}(\mathfrak{X})$ of $\mathbf{S}(\mathfrak{X})$ is precisely the set of those inverse images. The set of open edges $\mathcal{E}(\mathfrak{X})$ is the set of analytic skeleta of the inverse images under ${\rm sp}_{\mathfrak{X}}$ of the singular points of $\mathfrak{X}_s$. For a morphism $\varphi : \mathfrak{Y} \to \mathfrak{X}$ in $\mathcal{FS}^Q(X)$, we denote by $\tau_{\mathfrak{X},\mathfrak{Y}} : \mathbf{S}(\mathfrak{Y}) \to \mathbf{S}(\mathfrak{X})$ the corresponding morphism in $\mathcal{GP}^c(X)$. Then, as a topological space,

$$X = \varprojlim_{\mathfrak{Y} \geq \mathfrak{Z}} (S(\mathfrak{Y}), \tau_{\mathfrak{Z},\mathfrak{Y}}) \ , \tag{1.5.5.1}$$

while the subset $\mathbb{H}(X) \subset X$ of points of type (2) or (3) is

$$\mathbb{H}(X) = \bigcup S(\mathfrak{Y}) \ , \tag{1.5.5.2}$$

where $\mathfrak{Y}$ varies among Q-semistable models of X. In particular, we get, for any Q-semistable model $\mathfrak{X}$ of $\mathfrak{X}$, a retraction $\tau_{\mathfrak{X}} : X \to S(\mathfrak{X})$, as the natural map

$$\tau_{\mathfrak{X}} := \varprojlim_{\mathfrak{Y} \geq \mathfrak{X}} (S(\mathfrak{Y}), \tau_{\mathfrak{X},\mathfrak{Y}}) \ , \tag{1.5.5.3}$$

to $S(\mathfrak{X})$. If $\mathfrak{X} = \mathfrak{X}_0$ is the minimum Q-semistable model of X, then $\mathbf{S}(\mathfrak{X}_0) = \mathbf{S}(X)$ and $\tau_{\mathfrak{X}_0} = \tau_X$.

Definition 1.5.6. *For any Q-semistable model $\mathfrak{X}$ of X, and any Q-rational point $y \in X$ (necessarily, $y \in X \setminus S(\mathfrak{X})$), we denote by $D_{\mathfrak{X}}(y, 1^-)$ the maximal open Q-disk neighborhood of y contained in $X \setminus S(\mathfrak{X})$.*

Notice that if, in the previous definition, k'/k is a finite galois extension of group H such that $\mathfrak{X}$ is represented by the semistable k'°-formal scheme $\mathfrak{X}'$, and $y' \in X \otimes k'$ is a k'-rational point projecting to y, there exist H-equivariant isomorphisms $D_{\mathfrak{X}'}(y', 1^-) \xrightarrow{\sim} D_{k'}(0, 1^-)$, $y' \mapsto 0$.

Definition 1.5.7. *Assumptions as in definition 1.5.6. The induced isomorphism*

$$D_{\mathfrak{X}}(y, 1^-) = D_{\mathfrak{X}'}(y', 1^-)/H \xrightarrow{\sim} D_{k'}(0, 1^-)/H = D_k(0, 1^-) \tag{1.5.7.1}$$

is called an $\mathfrak{X}$-normalized coordinate at y.

Notice that the map $\tau_{\mathfrak{X}}$ takes the Q-disk $D_{\mathfrak{X}}(y, 1^-)$ to the unique boundary point of $D_{\mathfrak{X}}(y, 1^-)$ in X, not in $D_{\mathfrak{X}}(y, 1^-)$, $\tau_{\mathfrak{X}}(y) \in S(\mathfrak{X})$. The fiber $\tau_{\mathfrak{X}}^{-1}(\tau_{\mathfrak{X}}(y)) \setminus \{\tau_{\mathfrak{X}}(y)\}$ is the disjoint union of a family of Q-disks having the same limit point $\tau_{\mathfrak{X}}(y) \in S(\mathfrak{X})$. It follows from theorem 1.1.23 that for any compact rig-smooth strictly k-analytic curve X, which is not rational, and any Q-rational point $y \in X$ (necessarily, $y \in X \setminus S(X)$), there exists a maximal open Q-disk neighborhood $D_X(y, 1^-)$ of y in X. It generalizes the neighborhood $\mathcal{D}_y(X)$ defined in the introduction for an affinoid X in $\mathbb{A}^1$.

1.6 Admissible blow-ups

Let us recall that for any admissible k°-formal scheme $\mathfrak{X}$, any $a \in k^{\circ\circ} \setminus \{0\}$, and any open ideal of finite presentation $\mathfrak{A} \subset \mathcal{O}_{\mathfrak{X}}$, one defines the blow-up of $\mathfrak{X}$ along $\mathfrak{A}$ as the morphism $\varphi : \mathfrak{Y} \to \mathfrak{X}$ of formal schemes, inductive limit as $n \to \infty$, of the blow-up $\varphi_n : \mathfrak{Y}_n \to \mathfrak{X}_n$ of the scheme $\mathfrak{X}_n = (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/(a)^{n+1})$ along the ideal $\mathfrak{A} \otimes \mathcal{O}_{\mathfrak{X}}/(a)^{n+1}$. Such a morphism is called an *admissible blow-up*, and is independent of the choice of a.

Assume now, for simplicity, that k is algebraically closed and consider two strictly semistable partitions $\mathscr{P} = \mathscr{P}(\mathcal{B}, \mathcal{C})$ and $\mathscr{P}' = \mathscr{P}(\mathcal{B}', \mathcal{C}')$ of X, with $\mathscr{P}' \geq \mathscr{P}$. So, there is a morphism in $\mathcal{FS}(X)$, $\varphi : \mathfrak{X}' := \mathfrak{X}_{\mathscr{P}'} \to \mathfrak{X} := \mathfrak{X}_{\mathscr{P}}$, which we now describe. Suppose that an affinoid with good canonical reduction $C' \in \mathcal{C}'$ is contained in the affinoid $C \in \mathcal{C}$. If the maximal point $\eta_{C'}$ of C' coincides with the maximal point η_C of C, then C' is the complement in C of a finite number of residue classes $D_1, \dots, D_r$ of C. For each $i = 1, \dots, r$, D_i must contain a finite number of disjoint affinoids with good canonical reduction belonging to $\mathcal{C}'$. Therefore, the partition $\mathscr{P}'$ is completely determined from the family $\mathscr{F} = \mathscr{F}(\mathscr{P}', \mathscr{P})$ of the elements of $\mathcal{C}'$ contained in either an open disk $B \in \mathcal{B}$, or in a maximal disk D of some affinoid with good canonical reduction $C \in \mathcal{C}$. Now B (resp. D) is an open k-annulus (resp k-disk). So, the description of connected strictly affinoid domains with good canonical reduction in B or D is elementary. They are the complement of a finite number of maximal open disks in a closed strictly affinoid k-disk. We define C_{φ} as the union of all affinoids $V \in \mathscr{F}$ and $\mathfrak{A}_{\varphi}$ as the sheaf of ideals of $\mathcal{O}_{\mathfrak{X}}$ consisting of sections f of $\mathcal{O}_{\mathfrak{X}}$ whose pull-back under $\mathrm{sp}_{\mathfrak{X}}$ vanishes on C_{φ}. It is clear (by explicit description of generators) that $\mathfrak{A}_{\varphi}$ is an open ideal of finite presentation of $\mathcal{O}_{\mathfrak{X}}$. We will show in [5]

Theorem 1.6.1. *The morphism φ is the admissible blow-up of $\mathfrak{X}$ along $\mathfrak{A}_{\varphi}$. So, φ is a composition $\mathfrak{X}' =: \mathfrak{X}^{(N)} \to \cdots \to \mathfrak{X}^{(i+1)} \to \mathfrak{X}^{(i)} \to \cdots \to \mathfrak{X}^{(0)} := \mathfrak{X}$, where $\mathfrak{X}^{(i+1)} \to \mathfrak{X}^{(i)}$ is the blow-up of a single closed disk contained in an open disk $D_{\mathfrak{X}^{(i)}}(x, 1^-)$, for some $x \in X \setminus S(\mathfrak{X}^{(i)})$. In particular, every morphism $\varphi : \mathfrak{X}' \to \mathfrak{X}$ in $\mathcal{FS}(X)$ is an admissible blow-up and each non-empty fiber of $\mathfrak{X}'_s \to \mathfrak{X}_s$ is a connected union of rational smooth projective curves whose graph is a tree.*

1.7 Étale morphisms of formal schemes

For any morphism $\varphi : \mathfrak{Y} \to \mathfrak{X}$ of Q-semistable k°-formal schemes, with generic fiber $\varphi_\eta : Y \to X$, the following diagram is commutative [11, 4.4.2]

$$\begin{array}{ccc} S(\mathfrak{Y}) & \xrightarrow{\ \tau_{\mathfrak{X}} \circ \varphi_\eta\ } & S(\mathfrak{X}) \\ {\scriptstyle \tau_{\mathfrak{Y}}} \big\uparrow & & \big\uparrow {\scriptstyle \tau_{\mathfrak{X}}} \\ Y & \xrightarrow{\ \varphi_\eta\ } & X \end{array} \tag{1.7.0.1}$$

If, moreover, $\varphi : \mathfrak{Y} \to \mathfrak{X}$ is Q-étale, then $\varphi_\eta(S(\mathfrak{Y})) \subset S(\mathfrak{X})$, and $\varphi_\eta^{-1}(S(\mathfrak{X})) = S(\mathfrak{Y})$, so that the previous diagram becomes

$$\begin{array}{ccc} S(\mathfrak{Y}) & \xrightarrow{\ (\varphi_\eta)_{|S(\mathfrak{Y})}\ } & S(\mathfrak{X}) \\ {\scriptstyle \tau_{\mathfrak{Y}}} \big\uparrow & & \big\uparrow {\scriptstyle \tau_{\mathfrak{X}}} \\ Y & \xrightarrow{\ \varphi_\eta\ } & X \end{array} \tag{1.7.0.2}$$

and φ_η induces isomorphisms $D_{\mathfrak{Y}}(y, r^-) \xrightarrow{\sim} D_{\mathfrak{X}}(x, r^-)$, for any $y \in Y \setminus S(\mathfrak{Y})$, $\varphi_\eta(y) = x$, $r \in (0, 1]$ [4.3.2 *loc.cit.*].

Suppose the point $y \in Y$ is a vertex of $\mathbf{S}(\mathfrak{Y})$. This means that it is the unique inverse image under $\mathrm{sp}_{\mathfrak{Y}}$ of the generic point of a component $\mathfrak{c}$ of $\mathfrak{Y}_s$. But, passing to a finite galois extension k'/k, φ induces an étale morphism from the smooth part $\mathfrak{c}^{\mathrm{sm}}$ to a component $\mathfrak{c}'$ of $\mathfrak{X}_s$ [10, 2.2 (*i*)]. Therefore, $\varphi_\eta(\mathcal{V}(\mathfrak{Y})) \subset \mathcal{V}(\mathfrak{X})$.

2 Continuity of real valued functions on rig-smooth k-analytic curves

Lemma 2.0.1. *Let k be any non-archimedean field, Y be any k-analytic space, L a non-archimedean field over k and $Y_L = Y \widehat{\otimes}_k L$ be the extension of Y to L. Then the natural topology of Y is the quotient topology of the natural topology of Y_L via the projection map $\psi_L = \psi_{Y,L/k} : Y_L \to Y$.*

Proof. We first prove that the map ψ_L is closed. Let C be a closed subset of Y_L. Let y be a point of $Y \setminus \psi_L(C)$, and let D_2 be a compact neighborhood of y in Y. Then $D_1 = \psi_L^{-1}(D_2)$ is a compact subset of Y_L. The intersection $C \cap D_1$ is then compact; its image $\psi_L(C \cap D_1)$ is then closed, so that $D_2 \setminus \psi_L(C \cap D_1)$ is a neighborhood of y in Y not intersecting $\psi_L(C)$. The conclusion follows from [30, 2.4]. □

We assume from now on in this section that the non-archimedean field k is non-trivially valued.

Definition 2.0.2. *Let $\mathfrak{X}$ be a Q-semistable k°-formal scheme, Γ a topological space, and let $X = \mathfrak{X}_\eta$ be the generic fiber of $\mathfrak{X}$. Let $f : X \to \Gamma$ be any function and, for any non-archimedean field extension k'/k and Q-étale morphism of Q-semistable k'°-formal schemes $\psi : \mathfrak{Y} \to \mathfrak{X} \widehat{\otimes} k'^\circ$, let $f_\psi : \mathfrak{Y}_\eta \to \Gamma$ be the composite*

$$f_\psi = f \circ \psi_{\mathfrak{X}_\eta, k'/k} \circ \psi_\eta \ .$$

We say that $(f_\psi)_\psi$ is the étale-local system of functions on $\mathfrak{X}$ with values in Γ *associated to the function f. We identify $f = f_{\mathrm{id}_{\mathfrak{X}}}$ with the system $(f_\psi)_\psi$.*

Notice that if $f : X \to \Gamma$ is continuous, every component f_ψ of the étale-local system of functions associated with f is also continuous. We present some basic results in the opposite direction. The first lemma is a consequence of [10, Lemma 5.11].

Lemma 2.0.3. *Let $\mathfrak{X}$ be a Q-semistable k°-formal scheme and $f = (f_\psi)_\psi$ be an étale-local system of functions on $\mathfrak{X}$ with values in the topological space Γ, associated to the function $f : X = \mathfrak{X}_\eta \to \Gamma$. Assume there is a non-archimedean extension field k'/k such that $\mathfrak{X}\widehat{\otimes}k'^\circ$ is semistable and an étale covering $\{\psi_\alpha : \mathfrak{Y} \to \mathfrak{X}\widehat{\otimes}k'^\circ\}_\alpha$ of $\mathfrak{X}\widehat{\otimes}k'^\circ$, such that $\forall\alpha$, f_{ψ_α} is continuous. Then f is continuous.*

Notation 2.0.4. Let $[a, b]$ be an interval in $\mathbb{R}$, and let Γ be a topological subspace of the metric space of all functions $[a, b] \to \mathbb{R}$, where $d(h, g) = \sup_{a\le x\le b} |h(x) - g(x)|$. We consider from now on étale-local systems of functions on $\mathfrak{X}$ with values in a topological space Γ of this type. Notice that Γ has a partial ordering $\le$, where $h \le g$ if $h(x) \le g(x)$ $\forall x \in [a, b]$, and that if $g - \varepsilon < h < g + \varepsilon$, with $\varepsilon > 0$ a constant function on $[a, b]$, then $d(g, h) < 2\,\varepsilon$.

We recall for completeness that a function $\varphi : T \to \Gamma$, where T is any topological space is *upper semicontinuous* or USC (resp. *lower semicontinuous* or LSC) if $\forall t_0 \in T$ and $\varepsilon > 0$, there exists a neighborhood $U_{t_0,\varepsilon}$ of t_0 in T such that

$$\varphi(t) < \varphi(t_0) + \varepsilon \ \ (\text{resp. } \varphi(t) > \varphi(t_0) - \varepsilon\ )$$

$\forall t \in U_{t_0,\varepsilon}$. If $\forall\alpha \in I$, φ_α is USC (resp. LSC), then

$$\varphi = \inf_{\alpha\in I} \varphi_\alpha \ \ (\text{resp. } \varphi = \sup_{\alpha\in I} \varphi_\alpha\ )$$

is USC (resp. LSC). Notice that if φ is both USC and LSC at $t_0 \in T$, then it is continuous at t_0.

Theorem 2.0.5. *Assume k is algebraically closed and let $\mathfrak{X}$ be a semistable k°-formal scheme of generic fiber X. Let $f = (f_\psi)_\psi$ be the étale-local system of functions on $\mathfrak{X}$ with values in Γ as in (2.0.4) associated to f. For any non-archimedean field k' over k, let $X_{k'} := X\widehat{\otimes}k'$, $\psi_{X,k'/k} : X_{k'} \to X$ be the canonical projection, and $f_{k'} := f \circ \psi_{X,k'/k}$. Assume the following five conditions hold:*

1. *for any algebraically closed non-archimedean field k' over k, $f_{k'}$ is continuous at every k'-rational point of $X_{k'}$;*
2. *for any algebraically closed non-archimedean field k' over k and any étale morphism $\psi : \mathfrak{Y} \to \mathfrak{X}\widehat{\otimes}k'^\circ$, where $\mathfrak{Y}$ is either a basic formal disk or a basic formal annulus over k'°, $f_\psi(x) \ge \min_{y\in\Gamma(\mathfrak{Y}_\eta)} f_\psi(y)$ for all points $x \in \mathfrak{Y}_\eta$.*
3. *for any algebraically closed non-archimedean field k' over k, the restriction of $f_{k'}$ to any open segment of $X_{k'}$ is continuous;*
4. *the restriction of f to $S(\mathfrak{X})$ is continuous at all vertices of $\mathbf{S}(\mathfrak{X})$;*
5. *f is* USC.

Then f is continuous.

Notice that the properties (3) and (4) of the theorem imply that the restriction of f to $S(\mathfrak{X})$ is continuous.

Proof. First of all, we observe that, given an algebraically closed non-archimedean field k' over k, the function $f_{k'} : X_{k'} \to \Gamma$ possesses the properties $(1) - (5)$ with respect to $\mathfrak{X}\widehat{\otimes}k'^{\circ}$.

By lemma 2.0.1 we may increase the field k and to assume that it is maximally complete. By the property (5), it suffices to verify that, for every point $x_0 \in X$ and every $\varepsilon > 0$, the set $\{x \in X \mid f(x) > f(x_0) - \varepsilon\}$ contains a neighborhood of the point x_0.

Consider first the case when $x_0 \notin S(\mathfrak{X})$. We may then assume that $x_0 \in D_{\mathfrak{X}}(y, 1^-)$, for some $y \in X(k)$. Then $D_{\mathfrak{X}}(y, 1^-) \cong D_k(0, 1^-)$, $y \mapsto 0$, and $x_0 \mapsto t_{0,r} \in D_k(0, 1^-)$ for some $0 < r < 1$. Take now a number $r < R < 1$. It follows from the property (3) that the restriction of f to the interval $\{t_{0,r'} | r \leq r' \leq R\}$ is continuous and, therefore, we can find R sufficiently close to r with $f(t_{0,R}) > f(x_0) - \varepsilon$. Let V be the closed disk $D(0, R^+)$. It is an affinoid neighborhood of $x_0 = t_{0,r}$ in $D(0, 1^-)$ with $\Gamma(V) = \{t_{0,R}\}$ and, by the property (2), we get $f(x) \geq f(t_{0,R}) > f(x_0) - \varepsilon$ for all points $x \in V$.

Suppose now that $x_0 \in S(\mathfrak{X})$. First of all, if x lies in an open edge of $\mathbf{S}(\mathfrak{X})$, then the preimage of that edge under the retraction $\tau_{\mathfrak{X}} : X \to S(\mathfrak{X})$ is isomorphic to an open k-annulus isomorphic to $B(r_1, r_2)$, where $x_0 \mapsto t_{0,r}$, with $r_1 < r < r_2$. By the property (3), the function $(r_1, r_2) \to \Gamma : r' \mapsto f(t_{0,r'})$ is continuous, and so we can find numbers $r_1 < R_1 < r < R_2 < r_2$ sufficiently close to r such that $f(y_1)$, $f(y_2) > f(x_0) - \varepsilon$, where $y_1 = t_{0,R_1}$ and $y_2 = t_{0,R_2}$. Let V be the closed annulus $\tau_{\mathfrak{X}}^{-1}([R_1, R_2]) \cong D(0, R_2^+) \backslash D(0, R_1^-)$. Then V is a neighborhood of x_0, and $\Gamma(V) = \{y_1, y_2\}$. Property (2) implies that $f(x) \geq \min(f(y_1), f(y_2)) > f(x_0) - \varepsilon$ for all $x \in V$.

Furthermore, suppose that x_0 is a vertex of $\mathbf{S}(\mathfrak{X})$. If $S(\mathfrak{X}) = \{x_0\}$, then $\mathfrak{X}$ is smooth. Hence it is covered by a finite number of étale neighborhoods $\psi : \mathfrak{Y} \to \mathfrak{X}$ which are basic formal disks. Of course, X itself is covered by the images of the ψ_η. The Shilov boundary of $\mathfrak{Y}_\eta$ consists of the inverse image $\psi_\eta^{-1}(x_0) = \{y_0\}$ of the point x_0. It then follows from property (2) that $\forall y \in \mathfrak{Y}_\eta$, $f_\psi(y) \geq f_\psi(y_0)$. So, $f(\psi_\eta(y)) \geq f(x_0)$ $\forall y \in \mathfrak{Y}_\eta$, and, since the union of the images of the various ψ_η is all of X, we obtain $f(x) \geq f(x_0)$ for all $x \in X$. Assume therefore that $S(\mathfrak{X}) \neq \{x_0\}$. In any case, x_0 is the unique maximal point of an affinoid domain $C \subset X$ with good canonical reduction, compatible with the specialization map $\mathrm{sp}_{\mathfrak{X}} : X \to \mathfrak{X}_s$. But C is not a neighborhood of x_0 in X. Consider a connected neighborhood L of x_0 in $S(\mathfrak{X})$ which does not contain vertices of $\mathbf{S}(\mathfrak{X})$ other than x_0. One has $L = \cup_{i=1}^n L_i$, where each L_i is homeomorphic to $[r_i, 1]$ with $r_i \in |k^\times|$ and x_0 corresponds to the point 1 of $[r_i, 1]$. Let V_i be the affinoid domain $\tau_{\mathfrak{X}}^{-1}(L_i) \subset X$. The union $C \cup \bigcup_{i=1}^n V_i$ is a compact neighborhood of x_0 in X of which $\mathcal{U} = \{C, V_1, \ldots V_n\}$ is a formal affinoid covering. The canonical reduction of each V_i is also compatible with $\mathrm{sp}_{\mathfrak{X}} : X \to \mathfrak{X}_s$. Moreover, each V_i has canonical reduction which is a semistable curve with a single double point. So, the minimum semistable model of V_i is associated to the trivial covering $\{V_i\}$ of V_i, and it is a semistable k°-formal scheme whose reduction has two components and a single (double) intersection point. It is therefore a basic formal annulus $\mathcal{V}_i$ embedded as an open formal subscheme of $\mathfrak{X}$. The Shilov boundary $\Gamma(V_i)$ consists of x_0 and of a point y_i of type (2), corresponding to $r_i \in L_i$. Similarly, the minimum formal model $\mathfrak{C}$ of C is an open subscheme of $\mathfrak{X}$ which is a basic formal disk and has Shilov boundary $\Gamma(C) = \{x_0\}$. Since the restriction of f to $S(X)$, hence to L, is continuous at x_0, for any given $\varepsilon > 0$, we may find $r_i \in |k^\times| \cap (0, 1)$ so close to 1 that $f(y_i) > f(x_0) - \varepsilon$. Property (2) then implies that $f(x) > f(x_0) - \varepsilon$ for all $x \in V_i$, and the theorem follows. $\square$

3 Normalized radius of convergence

3.1 Formal structures

Notation 3.1.1. In this section, k is supposed to be non trivially valued and *of characteristic zero.* Here X is the complement of a reduced positive divisor $\mathcal{Z} = \{z_1, \ldots, z_r\}$ (possibly empty) in a compact rig-smooth connected k-analytic curve $\overline{X}$. So, if $\overline{X} = \mathcal{M}(\mathscr{A})$, $z_1, \ldots, z_r$ correspond to r distinct maximal ideals of the k-affinoid algebra $\mathscr{A}$. We assume here that $\overline{X}$ is the generic fiber of a Q-semistable k°-formal scheme $\overline{\mathfrak{X}} = \overline{\mathfrak{X}}'/H$, represented over a finite galois extension k'/k of group H, by the semistable k'°-formal scheme $\overline{\mathfrak{X}}'$. It is well known that the inverse image $\mathcal{Z}'$ of $\mathcal{Z}$ in $\overline{X}' = \overline{X} \otimes k'$, determines a finite flat closed reduced subscheme $\mathfrak{Z}'$ of $\overline{\mathfrak{X}}'$. We denote by $\mathfrak{Z}$ the quotient $\mathfrak{Z}'/H$, a flat normal closed formal subscheme of $\overline{\mathfrak{X}}$ of relative dimension zero. We call $\mathfrak{Z}$ the *schematic closure* of $\mathcal{Z}$ in $\overline{X}$.

It is known [41] that, for a further finite galois extension k''/k', there is an admissible blow-up $\mathfrak{Y}'$ of $\overline{\mathfrak{X}}'\widehat{\otimes}k''^\circ$ such that the inverse image of $\mathfrak{Z}''$ of $\mathfrak{Z}'$ in $\mathfrak{Y}'$ is an effective relative Cartier divisor $\mathfrak{Z}''$ of $\mathfrak{Y}'$ finite étale over $\mathrm{Spf}\, k''^\circ$ which does not intersect the singular locus of $\mathfrak{Y}'$. In other words, upon replacement of $\overline{\mathfrak{X}}$ by its Q-blow-up $\mathfrak{Y} := \mathfrak{Y}'/\mathrm{Gal}\,(k''/k)$, we may assume that the strict inverse image of $\mathfrak{Z}$ in $\mathfrak{Y}$ is Q-étale over k°.

So, we will assume in the following that $\mathfrak{Y} \to \overline{\mathfrak{X}}$ is a Q-blow-up such that the schematic closure $\mathfrak{Z}$ of $\mathcal{Z}$ in $\mathfrak{Y}$ is Q-étale and is contained in the maximal open Q-smooth formal subscheme of $\mathfrak{Y}$. We point out that this fact is equivalent to the requirement that the disks $D_{\mathfrak{Y}}(z_1, 1^-), \ldots, D_{\mathfrak{Y}}(z_r, 1^-)$ are distinct and that their boundary points $\tau_{\mathfrak{Y}}(z_1), \ldots, \tau_{\mathfrak{Y}}(z_r)$ are vertices of $\mathbf{S}(\mathfrak{Y})$.

We keep the notational distinction between $\mathcal{Z}$ and $\mathfrak{Z}$ mainly for future use.

In any case,

$$X = \overline{X} \setminus \mathcal{Z}\ . \tag{3.1.1.1}$$

We generalize the functor $\mathcal{FS}^Q(X) \to \mathcal{GP}(X)$, $\mathfrak{X} \mapsto \mathbf{S}(\mathfrak{X})$, described in theorem 1.5.5 for compact X, to the present non compact case, as follows.

Definition 3.1.2. *A* Q-subpolygon *of X is a triple $\mathbf{S} = (S, \mathcal{V}, \mathcal{E})$ consisting of a connected closed subset S of X, of a finite set of points of type* (2) *$\mathcal{V} \subset X$ (the* vertices of $\mathbf{S}$*), and of a finite set $\mathcal{E}$ of* open edges *of* $\mathbf{S}$. *It is required that the elements of $\mathcal{E}$ are either open Q-segments (the* open edges of finite length of $\mathbf{S}$*) or open Q-half-lines (the* open edges of infinite length of $\mathbf{S}$*), and that S is the disjoint union of the elements of $\mathcal{E}$ and of $\mathcal{V}$. A Q-subpolygon $\mathbf{S} = (S, \mathcal{V}, \mathcal{E})$ of X is* complete *if $X \setminus S$ is a union of open disks.*

Definition 3.1.3. *Let $\mathfrak{Y} \to \overline{\mathfrak{X}}$ be a Q-semistable model of $\overline{X}$ above $\overline{\mathfrak{X}}$ with the property described above. We define the* $\mathfrak{Y}$-skeleton *of $X = \overline{X} \setminus \mathcal{Z}$ as the Q-subpolygon of X $\mathbf{S}_{\mathfrak{Z}}(\mathfrak{Y}) = (S_{\mathfrak{Z}}(\mathfrak{Y}), \mathcal{V}(\mathfrak{Y}), \mathcal{E}(\mathfrak{Y}) \cup \{\ell_1, \ldots, \ell_r\})$, where $S_{\mathfrak{Z}}(\mathfrak{Y}) = S(\mathfrak{Y}) \cup \bigcup_{i=1}^r \ell_i$, and ℓ_i is the half-line $S(D_{\mathfrak{Y}}(z_i, 1^-) \setminus \{z_i\})$, for $i = 1, \ldots, r$. A* $\mathfrak{Y}$-normalized coordinate *at z_i is an isomorphism $T_{z_i} : D_{\mathfrak{Y}}(z_i, 1^-) \xrightarrow{\sim} D(0, 1^-)$ with $T_{z_i}(z_i) = 0$. We will say that the open half-line ℓ_i* connects *z_i to $\tau_{\mathfrak{Y}}(z_i)$. For any rigid point $x \in X_0$, we define the* $\mathfrak{Y}$-maximal open disk $D_{\mathfrak{Y},\mathfrak{Z}}(x, 1^-)$ in X, centered at x *as the maximal open disk contained in $D_{\mathfrak{Y}}(x, 1^-) \cap X$.*

The boundary point $\tau_{\mathfrak{Y},\mathfrak{Z}}(x)$ of $D_{\mathfrak{Y},\mathfrak{Z}}(x, 1^-)$ in $\overline{X}$ belongs to $S_{\mathfrak{Z}}(\mathfrak{Y})$ and, as in (1.5.5.1) we have a canonical continuous retraction

$$\tau_{\mathfrak{Y},\mathfrak{Z}} : X \to S_{\mathfrak{Z}}(\mathfrak{Y})\ . \tag{3.1.3.1}$$

Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-Module of finite type equipped with an X/k-connection ∇. Notice that the abelian sheaf

$$\mathcal{E}^\nabla = \mathrm{Ker}\,(\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_{X/k})$$

for the G-topology of X, is not in general locally constant. We use a canonical coordinate $T : D_{\mathfrak{Y},\mathfrak{Z}}(x,1^-) \xrightarrow{\sim} D(0,1^-)$, with $T(x) = 0$, on $D_{\mathfrak{Y},\mathfrak{Z}}(x,1^-)$, and define, for $r \in (0,1)$, $D_{\mathfrak{Y},\mathfrak{Z}}(x,r^\pm)$ as $T^{-1}(D(0,r^\pm))$. For any $r \in (0,1) \cap |k|$, the restriction of $\mathcal{E}$ to $D = D_{\mathfrak{Y},\mathfrak{Z}}(x,r^+) \subset D_{\mathfrak{Y},\mathfrak{Z}}(x,1^-)$ is a free $\mathcal{O}_D$-Module of finite type. Let us choose a basis $\underline{e} : \mathcal{E}_{|D} \xrightarrow{\sim} \mathcal{O}_D^\mu$. The coordinate column vector $\vec{y}$ with respect to the basis $\underline{e}$ of a horizontal section of $(\mathcal{E}_{|D}, \nabla_{|D})$ satisfies a system of differential equations of the form (0.1.0.1) where $G \in M_{\mu\times\mu}(\mathcal{O}_X(D))$ depends on r and on the choice of the basis $\underline{e}$. By iteration of the system (0.1.0.1) we obtain, for any $i \in \mathbb{N}$, the equations

(3.1.3.2) $$(i!)^{-1}(\frac{d}{dT})^i\,\vec{y} = G_{[i]}\,\vec{y} \quad ,$$

with $G_{[i]} \in M_{\mu\times\mu}(\mathcal{O}_X(D))$.

The power series $Y(T) = \sum_i G_{[i]}(x)(T-x)^i$ converges in a neighborhood U of $x \in D$ to the unique solution matrix of the system (0.1.0.1) such that $Y(x)$ is the $\mu \times \mu$ identity matrix. So, $\underline{e}\,Y \in \mathcal{E}^\nabla(U)^\mu$ is a row of horizontal sections of $\mathcal{E}(U)$. For any $r \in (0,1) \cap |k|$, the value

(3.1.3.3) $$\mathcal{R}_r(x) := \min(r, \liminf_{i\to\infty} |G_{[i]}(x)|^{-1/i}) \in (0,r] \ ,$$

is the radius of the maximal open disk $D(0,\rho^-)$, contained in $D(0,r^+)$, such that the power series $Y(T)$ converges in $D(0,\rho^-)$. It is independent of the choice of T and of the basis $\underline{e}$ of $\mathcal{E}$ restricted to $D_{\mathfrak{Y},\mathfrak{Z}}(x,r^+)$. The matrix $Y(T)$ is in fact an *invertible* solution matrix of the system (0.1.0.1) holomorphic in $D(0,\mathcal{R}_r(x)^-)$, since the differential system for the wronskian $w := \det Y$, namely

(3.1.3.4) $$\frac{d\,w}{dT} = (\operatorname{Tr} G)\,w \ ,$$

has no singularity in $D(0,1^-)$. We conclude that $\mathcal{E}^\nabla_{|D_{\mathfrak{Y},\mathfrak{Z}}(x,\mathcal{R}_r(x)^-)}$ is the constant sheaf k^μ on $D_{\mathfrak{Y},\mathfrak{Z}}(x,\mathcal{R}_r(x)^-)$.

For $r_1 < r_2 < 1$ we have $\mathcal{R}_{r_1}(x) \leq \mathcal{R}_{r_2}(x) \leq 1$, and therefore the quantity

(3.1.3.5) $$\mathcal{R}(x) := \lim_{r\to 1} \mathcal{R}_r(x) \quad (\ r \in |k| \cap (0,1)\ ) \ ,$$

is well-defined and belongs to $(0,1]$. We conclude that $\mathcal{E}^\nabla_{|D_{\mathfrak{Y},\mathfrak{Z}}(x,\mathcal{R}(x)^-)}$ is a locally constant sheaf with fiber k^μ on $D_{\mathfrak{Y},\mathfrak{Z}}(x,\mathcal{R}(x)^-)$ (for both the natural and the G-topology). Since $D_{\mathfrak{Y},\mathfrak{Z}}(x,\mathcal{R}(x)^-)$ equipped with the natural topology is contractible, $\mathcal{E}^\nabla_{|D_{\mathfrak{Y},\mathfrak{Z}}(x,\mathcal{R}(x)^-)}$ is in fact the constant sheaf k^μ on $D_{\mathfrak{Y},\mathfrak{Z}}(x,\mathcal{R}(x)^-)$, and $\mathcal{E}_{|D_{\mathfrak{Y},\mathfrak{Z}}(x,\mathcal{R}(x)^-)}$ is free of rank μ.

From the previous discussion, we conclude the following

Proposition 3.1.4. *If for some analytic domain $V \subset X$, $\mathcal{E}^\nabla_{|V}$ is locally constant, then it is necessarily a local system of k-vector spaces of rank μ on V and the canonical monomorphism*

(3.1.4.1) $$\mathcal{E}^\nabla_{|V} \otimes_k \mathcal{O}_V \hookrightarrow \mathcal{E}_{|V} \ ,$$

is an isomorphism. For any $x \in X(k)$, $D_{\mathfrak{Y},\mathfrak{Z}}(x,\mathcal{R}(x)^-)$ is the maximal open disk $\mathcal{D}$ centered at x, not intersecting $S_{\mathfrak{Z}}(\mathfrak{Y})$, and such that $\mathcal{E}^\nabla$ is a locally constant sheaf on $\mathcal{D}$, or, equivalently, such that the restriction of $(\mathcal{E},\nabla)$ to $\mathcal{D}$ is isomorphic to the trivial connection on $\mathcal{O}^\mu_{\mathcal{D}}$.

We may then give the

Definition 3.1.5. *Let $X = \overline{X} \setminus \mathfrak{Z}$ be a rig-smooth connected k-analytic curve as in 3.1.1. Let $(\mathcal{E}, \nabla)$ be an object of* $\mathbf{MIC}(X/k)$*, with $\mathcal{E}$ locally free of rank μ for the G-topology. We fix a semistable model $\mathfrak{Y}$ of $\overline{X}$ as in (3.1.1). For any k-rational point $x \in X$, we define the* $\mathfrak{Y}$-normalized radius of convergence $\mathcal{R}_{\mathfrak{Y},\mathfrak{Z}}(x, (\mathcal{E}, \nabla))$ (or $\mathcal{R}_{\mathfrak{Y},z_1,\dots,z_r}(x, (\mathcal{E}, \nabla))$) of $(\mathcal{E}, \nabla)$ at *x as the radius $\rho \in (0, 1]$ of the maximal open disk $\mathcal{D}$ centered at x, contained in X and not intersecting the closed subset $S_{\mathfrak{Z}}(\mathfrak{Y}) \subset X$, such that $(\mathcal{E}, \nabla)_{|\mathcal{D}}$ is isomorphic to the trivial connection $(\mathcal{O}_{\mathcal{D}}, d_{\mathcal{D}})^{\mu}$.*

Lemma 3.1.6. *Let X, $\overline{X}$, $\mathfrak{Y}$, $(\mathcal{E}, \nabla)$ be as in the previous definition, and let $\psi : \mathfrak{Y}' \to \mathfrak{Y}$ be a Q-étale morphism of Q-semistable k°-formal schemes. Let $\overline{X}' = \mathfrak{Y}'_{\eta}$, $\mathcal{Z}' = \psi_{\eta}^{-1}(\mathcal{Z})$, $\mathfrak{Z}' = \psi^*(\mathfrak{Z})$, $X' = \overline{X}' \setminus \mathcal{Z}'$, and $g : X' \to X$ be the morphism induced by ψ_{η}. Then $g^*(\mathcal{E}, \nabla)$ is an object of* $\mathbf{MIC}(X'/k)$ *and*

$$\mathcal{R}_{\mathfrak{Y}',\mathfrak{Z}'}(x, g^*(\mathcal{E}, \nabla)) = \mathcal{R}_{\mathfrak{Y},\mathfrak{Z}}(g(x), (\mathcal{E}, \nabla)) \ , \ \forall x \in X' \ . \tag{3.1.6.1}$$

Proof. This follows from the discussion in (1.7) and the definitions. □

Notice that if $\mathcal{E}$ is free over $\mathcal{D} = D_{\mathfrak{Y},\mathfrak{Z}}(x, 1^-)$, $x \in X(k)$, and we use a global basis $\underline{e}$ of $\mathcal{E}$ on $\mathcal{D}$ to transform ∇ into the system (0.1.0.1), where the coordinate T is normalized so that $\mathcal{D} \xrightarrow{\sim} D(0, 1^-)$, $x \mapsto 0$, formula 3.1.3.5 collapses to

$$\mathcal{R}(x, \Sigma) = \min(1, \liminf_{n\to\infty} |G_{[n]}(x)|^{-1/n}) \ . \tag{3.1.6.2}$$

Definition 3.1.7. *Let $\mathfrak{Y}$ be a Q-semistable model of $\overline{X}$ and $X = \overline{X} \setminus \mathcal{Z}$, as before. We will say that the $\mathcal{O}_{\overline{X}_G}$-Module $\overline{\mathcal{E}}$ is* Q-coherent *(resp* Q-locally free*) over $\mathfrak{Y}$ if it is of the form $\mathrm{sp}_{\mathfrak{Y}}^*(\overline{\mathfrak{E}})$, for a Q-coherent (resp. Q-locally free) $\mathcal{O}_{\mathfrak{Y}}$-Module $\overline{\mathfrak{E}}$ (1.1.16), where $\mathrm{sp}_{\mathfrak{Y}} : \overline{X}_G \to \mathfrak{Y}$ is viewed as a morphism of G-ringed spaces. We denote by* $\mathbf{MIC}_{\mathfrak{Y}}(X/k)$ *(resp.* $\mathbf{MIC}_{\mathfrak{Y}}(\overline{X}(*\mathcal{Z})/k)$*) the full subcategory of* $\mathbf{MIC}(X/k)$ *consisting of pairs $(\mathcal{E}, \nabla)$, where $\overline{\mathcal{E}}$ is an $\mathcal{O}_{\overline{X}_G}$-Module Q-coherent and Q-locally free over $\mathfrak{Y}$ (resp. and ∇ extends to a connection on $\overline{\mathcal{E}}$ with meromorphic singularities at $\mathcal{Z}$*

$$\overline{\nabla} : \overline{\mathcal{E}} \to \overline{\mathcal{E}} \otimes \Omega^1_{\overline{X}_G/k}(*\mathcal{Z}) \ \ ).$$

We will prove the following

Theorem 3.1.8. *Let $(\mathcal{E}, \nabla)$ be an object of* $\mathbf{MIC}_{\mathfrak{Y}}(X/k)$*. The function $X(k) \to \mathbb{R}_{>0}$, $x \mapsto \mathcal{R}_{\mathfrak{Y},\mathfrak{Z}}(x, (\mathcal{E}, \nabla))$, extends (uniquely) to a continuous function $X \to \mathbb{R}_{>0}$.*

The proof of theorem 3.1.8 will be based in fact on the construction of a function $x \mapsto \mathcal{R}_{\mathfrak{Y},\mathfrak{Z}}(x, (\mathcal{E}, \nabla))$, at all $x \in X$, extending the definition given in this section for k-rational points $x \in X(k)$. Namely, we set the following

Definition 3.1.9. *Let $(\mathcal{E}, \nabla)$ be an object of* $\mathbf{MIC}_{\mathfrak{Y}}(X/k)$*. For any $x \in X$, we consider the base-field extension*

$$\psi_{\overline{X},\mathscr{H}(x)/k} : \overline{X}' := \overline{X} \widehat{\otimes} \mathscr{H}(x) \to \overline{X} \ ,$$

inducing

$$\psi_{X,\mathscr{H}(x)/k} : X' := X \widehat{\otimes} \mathscr{H}(x) \to X \ ,$$

*and the canonical $\mathscr{H}(x)$-rational point x' of X' above x (*cf. *the comments following proposition 1.4.1 of [7]). For any Q-semistable model $\mathfrak{Y}$ of $\overline{X}$, $\mathfrak{Y}' = \mathfrak{Y} \widehat{\otimes} \mathscr{H}(x)^{\circ}$ is a Q-semistable model of $\overline{X}'$; let z_i' be the canonical inverse image of $z_i \in \overline{X}$ in $\overline{X}'$. We set*

$$(\mathcal{E}', \nabla') = \psi^*_{X,\mathscr{H}(x)/k}(\mathcal{E}, \nabla) \ ,$$

and

$$\mathcal{R}_{\mathfrak{Y},z_1,\dots,z_r}(x, (\mathcal{E}, \nabla)) := \mathcal{R}_{\mathfrak{Y}',z_1',\dots,z_r'}(x', (\mathcal{E}', \nabla')) \ . \tag{3.1.9.1}$$

Let $\varepsilon \in |k| \cap (0,1)$, and let

$$X^{(\varepsilon)} = \overline{X} \setminus \bigcup_{i=1}^{r} D_{\mathfrak{Y}}(z_i, \varepsilon^-) \ .$$

Recall that we have assumed that all the points $\tau_{\mathfrak{Y}}(z_i)$ are vertices of $\mathbf{S}(\mathfrak{Y})$. Therefore, if $\mathscr{P}_{\mathfrak{Y}} = \mathscr{P}(\mathcal{B}, \mathcal{C})$ is the semistable partition of $\overline{X}$ associated to $\mathfrak{Y}$, each of the open disks $D_{\mathfrak{Y}}(z_i, 1^-)$ is a maximal disk in an affinoid $C_i \in \mathcal{C}$, for $i = 1, \dots, r$. We consider the semistable partition $\mathscr{P}^{(\varepsilon)} = \mathscr{P}(\mathcal{B}^{(\varepsilon)}, \mathcal{C}^{(\varepsilon)})$, where, for each i, we have replaced the affinoid C_i by the two affinoids $C_i \setminus D_{\mathfrak{Y}}(z_i, 1^-)$ and $D_{\mathfrak{Y}}(z_i, \varepsilon^+)$, and have added the open annulus $D_{\mathfrak{Y}}(z_i, 1^-) \setminus D_{\mathfrak{Y}}(z_i, \varepsilon^+)$. So, $\mathscr{P}^{(\varepsilon)} \geq \mathscr{P}_{\mathfrak{Y}}$ corresponds to a morphism $\mathfrak{Y}^{(\varepsilon)} \to \mathfrak{Y}$ in $\mathcal{FS}^Q$, which is the admissible blow-up of the $\mathcal{O}_{\mathfrak{Y}}$-ideal of sections of $\mathcal{O}_{\mathfrak{Y}}$ whose pull-back to $\overline{X}$ vanishes on the affinoid disks $D_{\mathfrak{Y}}(z_i, \varepsilon^+) \setminus D_{\mathfrak{Y}}(z_i, \varepsilon^-)$, $i = 1, \dots, r$. The closed fiber $\mathfrak{Y}_s^{(\varepsilon)}$ is the union of $\mathfrak{X}_s$ and of r projective lines $\ell_i^{(\varepsilon)} \cong \mathbb{P}^1_{\tilde{k}}$. The line $\ell_i^{(\varepsilon)}$ is at the same time the canonical reduction of $D_{\mathfrak{Y}}(z_i, \varepsilon^+)$, and the image of $D_{\mathfrak{Y}}(z_i, \varepsilon^+)$ via $\mathrm{sp}_{\mathfrak{Y}^{(\varepsilon)}}$. Let $\mathfrak{X}^{(\varepsilon)} \subset \mathfrak{Y}^{(\varepsilon)}$ be the open formal subscheme $\mathrm{sp}^{-1}_{\mathfrak{Y}^{(\varepsilon)}}(\mathfrak{Y}_s^{(\varepsilon)} \setminus (\bigcup_{i=1}^r \ell_i^{(\varepsilon)})$.

We summarize the previous discussion:

Lemma 3.1.10. *There is a morphism of Q-semistable formal schemes $\mathfrak{X}^{(\varepsilon)} \to \mathfrak{Y}$, composite of an open immersion $\mathfrak{X}^{(\varepsilon)} \to \mathfrak{Y}^{(\varepsilon)}$ and of an admissible blow-up $\mathfrak{Y}^{(\varepsilon)} \to \mathfrak{Y}$, whose generic fiber identifies with the embedding $X^{(\varepsilon)} \subset \overline{X}$.*

The restriction $(\mathcal{E}, \nabla)_{|X^{(\varepsilon)}}$ of $(\mathcal{E}, \nabla)$ to $X^{(\varepsilon)}$, is an object of $\mathbf{MIC}_{\mathfrak{X}^{(\varepsilon)}}(X^{(\varepsilon)}/k)$, and, since $\forall x \in X^{(\varepsilon)}(k)$, $D_{\mathfrak{Y},\mathfrak{Z}}(x, 1^-) = D_{\mathfrak{X}^{(\varepsilon)}}(x, 1^-)$, and, according to our definition, we are allowed to replace the field k by its extension $\mathscr{H}(x)$, we conclude that $\mathcal{R}_{\mathfrak{X}^{(\varepsilon)}}(x, (\mathcal{E}, \nabla)_{|X^{(\varepsilon)}}) = \mathcal{R}_{\mathfrak{Y}, z_1, \dots, z_r}(x, (\mathcal{E}, \nabla))$, for any $x \in X^{(\varepsilon)}$. So, we may restrict our attention to the case where there are no z_i's and $\overline{X} = X = \mathfrak{Y}_\eta$, for a Q-semistable formal scheme $\mathfrak{Y}$. Let $(\mathfrak{B}, T)$ be an étale neighborhood of $\mathfrak{Y}$ which is a basic formal annulus or disk. Suppose that the image B of $(\mathcal{B}, T) = (\mathfrak{B}, T)_\eta$ in $\overline{X}$ is contained in X, and contains the point $x \in X$. Now, the canonical point $x' \in X'$ above x is an interior point of the image B' in $\overline{X}'$, of $\mathcal{B}' := \mathcal{B} \widehat{\otimes} \mathscr{H}(x) = \mathfrak{B}'_\eta$, where $\mathfrak{B}' := \mathfrak{B} \widehat{\otimes} \mathscr{H}(x)^\circ$. More precisely, $D_{\mathfrak{Y}', z'_1, \dots, z'_r}(x', 1^-)$ is contained in B', and, for any choice of an inverse image $y' \in \mathcal{B}'$ of x', it is canonically isomorphic to $D_{\mathfrak{B}'}(y', 1^-)$. If $\mathfrak{B}$ is a disk, the coordinate T of $\mathfrak{B}$ induces the normalized coordinate $T_x = T - T(x)$ on $D_{\mathfrak{Y}', z'_1, \dots, z'_r}(x', 1^-)$. If $\mathfrak{B}$ is an annulus, the étale coordinate T on $\mathfrak{B}$ (which is also an étale coordinate on $\mathfrak{B}'$) induces a normalized coordinate T_x on $D_{\mathfrak{Y}', z'_1, \dots, z'_r}(x', 1^-)$, with $T_x(x') = 0$, via (*cf.* (0.2.0.1))

$$T_x(y) = (T(y) - T(x))/T(x) \ , \ \forall y \in D_{\mathfrak{Y}', z'_1, \dots, z'_r}(x', 1^-) \ . \tag{3.1.10.1}$$

If the pull-back of $\mathcal{E}$ on $\mathcal{B}$ is free, there exists a global basis $\underline{e}$ of $\mathcal{E}$ on the basic affinoid annulus $(\mathcal{B}, T)$, and we use such a global basis to transform ∇ into the system (0.1.0.1), formulas 3.1.3.5 and 3.1.6.2 become

$$\mathcal{R}(x, \Sigma) = \min(1, \liminf_{n \to \infty} |G_{[n]}(x)|^{-1/n}) \ , \tag{3.1.10.2}$$

for all $x \in B$, if B is a disk, and

$$\begin{aligned} \mathcal{R}(x, \Sigma) &= \min(1, |T(x)|^{-1} \liminf_{n \to \infty} |G_{[n]}(x)|^{-1/n}) \\ &= \min(|T(x)|, \liminf_{n \to \infty} |G_{[n]}(x)|^{-1/n}) |T(x)|^{-1} \ , \end{aligned} \tag{3.1.10.3}$$

for all $x \in B$, if B is an annulus. Notice that the previous formulas are independent of the choice of the global basis $\underline{e}$.

The previous formulas are especially useful if $(\mathcal{E},\nabla)$ is an object of $\mathbf{MIC}_{\mathfrak{Y}}(\overline{X}(*\mathcal{Z})/k)$ as in 3.1.7 and $(\mathcal{B},T)$ is a basic affinoid annulus or disk, generic fiber of a basic formal annulus or disk $(\mathfrak{B},T)$ varying in a covering of the type described in lemma 1.3.3. Then, $\mathcal{R}_{\mathfrak{Y},\mathfrak{Z}}(x,(\mathcal{E},\nabla))$ may be calculated locally on $\mathcal{B}$, that is

$$\mathcal{R}_{\mathfrak{Y},\mathfrak{Z}}(x,(\mathcal{E},\nabla)) = \mathcal{R}_{\mathfrak{B}}(x,(\mathcal{E},\nabla)_{|\mathcal{B}}) \ , \ \forall x \in \mathcal{B} \ , \tag{3.1.10.4}$$

and, if $\mathcal{E}_{|\mathcal{B}}$ is free of finite type and the connection is described via (0.1.0.1) in terms of an $\mathcal{O}(\mathcal{B})$-basis $\underline{e}$ of $\mathcal{E}(\mathcal{B})$, this in turn is computed as in (3.1.10.3).

3.2 Apparent singularities and change of formal model

We describe here how the function $x \mapsto \mathcal{R}_{\mathfrak{Y},\mathfrak{Z}}(x,(\mathcal{E},\nabla))$ gets modified, by a change of formal model $\mathfrak{Y}$ of $\overline{X}$ or by addition of an extra point $z = z_{r+1} \in X(k)$ to $\mathcal{Z} = \{z_1,\dots,z_r\}$. Notice that, if $\mathfrak{Y}_1 \to \mathfrak{Y}$ is a morphism of Q-semistable models of $\overline{X}$ inducing the identity on $\overline{X}$, there is a well-defined function $\rho_{\mathfrak{Y}_1/\mathfrak{Y}} : \overline{X}(k) \to (0,1) \cap |k|$, such that

$$D_{\mathfrak{Y}_1}(x,1^-) = D_{\mathfrak{Y}}(x,\rho_{\mathfrak{Y}_1/\mathfrak{Y}}(x)^-) \ , \ \forall \, x \in \overline{X}(k) \ . \tag{3.2.0.5}$$

The function $\rho_{\mathfrak{Y}_1/\mathfrak{Y}}$ is extended to all points $x \in \overline{X}$, by using the canonical $\mathscr{H}(x)$-rational point $x' \in \overline{X}' := \overline{X}\widehat{\otimes}\mathscr{H}(x)$ above x. Then $\mathfrak{Y}' := \mathfrak{Y}\widehat{\otimes}\mathscr{H}(x)^\circ$ (resp. $\mathfrak{Y}_1' := \mathfrak{Y}_1\widehat{\otimes}\mathscr{H}(x)^\circ$) is a Q-semistable model of $\overline{X}'$, and

$$\rho_{\mathfrak{Y}_1/\mathfrak{Y}}(x) := \rho_{\mathfrak{Y}_1'/\mathfrak{Y}'}(x') \ . \tag{3.2.0.6}$$

The following lemma follows easily from the description 1.5.5.2 of $\mathbb{H}(\overline{X}) \subset \overline{X}$.

Lemma 3.2.1. *Let $\varphi : \mathfrak{Y}_1 \to \mathfrak{Y}$ be a morphism in $\mathcal{FS}^Q(\overline{X})$ and $\tau_{\mathfrak{Y}} : \overline{X} \to \mathbf{S}(\mathfrak{Y})$ (resp. $\tau_{\mathfrak{Y}_1} : \overline{X} \to \mathbf{S}(\mathfrak{Y}_1)$, resp. $\tau_{\mathfrak{Y},\mathfrak{Y}_1} : \mathbf{S}(\mathfrak{Y}_1) \to \mathbf{S}(\mathfrak{Y})$) be the retraction described in (1.5.5.3) (resp. in (1.5.5.3), resp. in (1.5.5.1)). Then, for any $x \in \overline{X}$, $\tau_{\mathfrak{Y}_1}(x)$ and $\tau_{\mathfrak{Y}}(x)$ belong to the same fiber $\mathcal{C}$ of $\tau_{\mathfrak{Y},\mathfrak{Y}_1} : \mathbf{S}(\mathfrak{Y}_1) \to \mathbf{S}(\mathfrak{Y})$. We have*

$$\rho_{\mathfrak{Y}_1/\mathfrak{Y}}(x) = \ell_{\mathcal{C}}([\tau_{\mathfrak{Y}_1}(x),\tau_{\mathfrak{Y}}(x)])^{-1} \ , \tag{3.2.1.1}$$

where $\ell_{\mathcal{C}}([\tau_{\mathfrak{Y}_1}(x),\tau_{\mathfrak{Y}}(x)])$ denotes the length of the segment $[\tau_{\mathfrak{Y}_1}(x),\tau_{\mathfrak{Y}}(x)] \subset \mathcal{C} \subset \mathbb{H}(\overline{X})$. In particular, the function $\rho_{\mathfrak{Y}_1/\mathfrak{Y}}$ is continuous.

Proof. We may go to a finite galois extension of k, and consider only semistable formal schemes. If $\varphi_1 : \mathfrak{Y}_1 \to \mathfrak{Y}$ and $\varphi_2 : \mathfrak{Y}_2 \to \mathfrak{Y}_1$ are morphisms in $\mathcal{FS}(X)$, clearly $\rho_{\mathfrak{Y}_2/\mathfrak{Y}} = (\rho_{\mathfrak{Y}_2/\mathfrak{Y}_1} \circ (\varphi_1)_\eta) \cdot \rho_{\mathfrak{Y}_1/\mathfrak{Y}}$. So, we are reduced to proving the statement when either τ_φ is trivial, in which case it is trivial, or simple, in which case it is simple. ☐

Similarly, if $\overline{\mathfrak{X}}$ is as before, $\mathfrak{Y}_1 \to \mathfrak{Y} \to \overline{\mathfrak{X}}$ are morphisms in $\mathcal{FS}^Q(\overline{X})$, and $z_{r+1},\dots,z_{r+s} \in X(k)$ are such that $\{D_{\mathfrak{Y}_1}(z_1,1^-),\dots,D_{\mathfrak{Y}_1}(z_{r+s},1^-)\}$ consists of $r+s$ distinct disks, with $\tau_{\mathfrak{Y}_1}(z_1),\dots,\tau_{\mathfrak{Y}_1}(z_{r+s})$ vertices of $\mathbf{S}(\mathfrak{Y}_1)$, let $\mathcal{Z} = \{z_1,\dots,z_r\}$ (resp. $\mathcal{Z}_1 = \{z_1,\dots,z_{r+s}\}$). We may define a function $\rho_{(\mathfrak{Y},\mathfrak{Z})/(\mathfrak{Y}_1,\mathfrak{Z}_1)} : \overline{X} \setminus \mathcal{Z}_1 \to (0,1) \cap |k|$, such that

$$D_{\mathfrak{Y}_1,\mathfrak{Z}_1}(x,1^-) = D_{\mathfrak{Y},\mathfrak{Z}}(x,\rho_{(\mathfrak{Y},\mathfrak{Z})/(\mathfrak{Y}_1,\mathfrak{Z}_1)}(x)^-) \ , \ \forall \, x \in \overline{X}(k) \setminus \mathcal{Z}_1 \ , \tag{3.2.1.2}$$

extended as before to all points of $\overline{X} \setminus \mathcal{Z}_1$ and such that

$$\rho_{(\mathfrak{Y},\mathfrak{Z})/(\mathfrak{Y}_1,\mathfrak{Z}_1)}(x) = \ell_{\mathcal{C}}([\tau_{\mathfrak{Y}_1,\mathfrak{Z}_1}(x),\tau_{\mathfrak{Y},\mathfrak{Z}}(x)])^{-1} \ , \tag{3.2.1.3}$$

where $\tau_{\mathfrak{Y},\mathfrak{Z}}$ is the function of (3.1.3.1), $\mathcal{C}$ is the connected (and simply connected) component of $S_{\mathfrak{Z}_1}(\mathfrak{Y}_1)\setminus S_{\mathfrak{Z}}(\mathfrak{Y})$ such that both $\tau_{\mathfrak{Y}_1,\mathfrak{Z}_1}(x)$ and $\tau_{\mathfrak{Y},\mathfrak{Z}}(x)$ belong to $\mathcal{C}$, and $\ell_{\mathcal{C}}([\tau_{\mathfrak{Y}_1,\mathfrak{Z}_1}(x),\tau_{\mathfrak{Y},\mathfrak{Z}}(x)])$ denotes the length of the segment $[\tau_{\mathfrak{Y}_1,\mathfrak{Z}_1}(x),\tau_{\mathfrak{Y},\mathfrak{Z}}(x)] \subset \mathcal{C} \subset \overline{X} \setminus \mathcal{Z}_1$.

The general situation is as follows.

Lemma 3.2.2. *Let notation be as before. Let $(\mathcal{E},\nabla)$ is an object of $\mathbf{MIC}_{\mathfrak{Y}}(\overline{X}(*\mathcal{Z})/k)$, let $\mathfrak{Y}_1 \to \mathfrak{Y}$ be an admissible blowing-up, and $z_{r+1},\dots,z_{r+s} \in X(k)$ be such that $\{D_{\mathfrak{Y}_1}(z_1,1^-),\dots$ $\dots, D_{\mathfrak{Y}_1}(z_{r+s},1^-)\}$ consists of $r+s$ distinct disks. Let $\mathcal{Z}_1 = \{z_1,\dots,z_{r+s}\}$. Then $(\mathcal{E},\nabla)$ may be regarded as an object of $\mathbf{MIC}_{\mathfrak{Y}_1}(\overline{X}(*\mathcal{Z}_1\})/k)$ and*

$$\mathcal{R}_{\mathfrak{Y}_1,\mathfrak{Z}_1}(x,(\mathcal{E},\nabla)) = \min(1,\mathcal{R}_{\mathfrak{Y},\mathfrak{Z}}(x,(\mathcal{E},\nabla))/\rho_{(\mathfrak{Y},\mathfrak{Z})/(\mathfrak{Y}_1,\mathfrak{Z}_1)}(x)) \ \ , \forall\, x \in \overline{X}\setminus \mathcal{Z}_1 \ . \tag{3.2.2.1}$$

Remark 3.2.3. Let $(\mathcal{E},\nabla)$ be an object of $\mathbf{MIC}_{\mathfrak{Y}}(\overline{X}(*\mathcal{Z})/k)$ and let $z = z_{r+1} \in X(k) \cap (D_{\mathfrak{Y}}(z_1,1^-)\setminus\{z_1\})$. Let T_{z_1} be a $\mathfrak{Y}$-normalized coordinate at z_1 and $\mathfrak{Y}_1 \to \mathfrak{Y}$ be the blowing-up of the ideal $(T_{z_1}(z),T_{z_1})$ of $\mathcal{O}_{\mathfrak{Y}}$. Then, for $\mathcal{Z}_1 = \mathcal{Z}\cup\{z_{r+1}\}$, with schematic closure $\mathfrak{Z}_1$ in $\mathfrak{Y}$,
(3.2.3.1)

$$\mathcal{R}_{\mathfrak{Y}_1,\mathfrak{Z}_1}(x,(\mathcal{E},\nabla)) = \begin{cases} \mathcal{R}_{\mathfrak{Y},\mathfrak{Z}}(x,(\mathcal{E},\nabla)), & \text{if} \quad x \notin D_{\mathfrak{Y}}(z_1,T_{z_1}(z)^-), \\ \min(1,\mathcal{R}_{\mathfrak{Y},\mathfrak{Z}}(x,(\mathcal{E},\nabla))/|T_{z_1}(x)|), & \text{if} \quad x \in D_{\mathfrak{Y}}(z_1,1^-)\setminus\{z_1\}, \end{cases}$$

The previous formula shows that the unnecessary consideration of the apparent singularity z_{r+1} results in a loss of information.

4 Differential systems on an annulus: review of classical results

In this section k is a non-archimedean field extension of $\mathbb{Q}_p$. We let k^{alg} be an algebraic closure of k.

We recall the k-Banach algebra $\mathscr{H}(r_1,r_2) \subset \mathcal{O}(B(r_1,r_2))$ of *analytic elements* on the open annulus (0.1.0.2), with $0 < r_1 \leq r_2$. Its elements are uniform limits on $B(r_1,r_2)$ of rational functions in $k(T)$ without poles in $B(r_1,r_2)$. The Banach norm of $\mathscr{H}(r_1,r_2)$ is of course the supnorm on $B(r_1,r_2)$. An analytic element $\overline{f} \in \mathscr{H}(r_1,r_2)$ defines an analytic function $f : B(r_1,r_2) \to \mathbb{A}^1$ and a continuous extension of f to the closure $B^+(r_1,r_2) = B(r_1,r_2)\cup\{t_{0,r_1},t_{0,r_2}\}$ of $B(r_1,r_2)$ in $\mathbb{A}^1$, still denoted by $f : B^+(r_1,r_2) \to \mathbb{A}^1$. We consider here a differential system Σ of the form (0.1.0.1) with G a $\mu\times\mu$ matrix with elements in $\mathscr{H}(r_1,r_2)$. The function radius of convergence of Σ according to Christol-Dwork [20] is defined for $\rho \in [r_1,r_2]$ as

$$R(\Sigma,\rho) := \min(\rho, \liminf_{k\to\infty} |G_{[k]}(t_{0,\rho})|^{-1/k}) \ , \tag{4.0.3.2}$$

where $G_{[k]}$ is the matrix describing the action of $(k!)^{-1}(\frac{d}{dT})^k$ on any formal column solution $\vec{y}$. Notice that $R(\Sigma,\rho)/\rho$ coincides with $\mathcal{R}(t_{0,\rho},\Sigma)$, as defined in (3.1.10.3), if the entries of G are analytic functions on some basic affinoid annulus $B = \mathfrak{B}_\eta$ with $B(r_1,r_2) \subset B \subset B[r_1,r_2]$, and $\mathcal{R}(t_{0,\rho},\Sigma)$ is taken with respect to $\mathfrak{B}$.

We follow the common practice of saying that a function F, defined on a subset of $\mathbb{R}_{>0}$ and taking values in $\mathbb{R}_{>0}$, has a certain property $\mathcal{P}$ *logarithmically* if the function $\log\circ F\circ\exp$ has the property $\mathcal{P}$. Christol-Dwork observe that the function $\rho \mapsto R(\Sigma,\rho)$ is logarithmically concave (*i.e.* logarithmically $\cap$-shaped) on the interval $[r_1,r_2]$.

This result (first part of [20, 2.5]) follows immediately from the well-known fact that, for any $f \in \mathscr{H}(r_1,r_2)$, the function $\rho \mapsto |f(t_{0,\rho})|$ is logarithmically convex and continuous on the interval $[r_1,r_2]$. Moreover, if for $a<b$, $\forall i \in \mathbb{N}$, $\varphi_i : [a,b] \to \mathbb{R}$ is a convex (resp. concave) function, then

$$\varphi = \limsup_{i\to\infty} \varphi_i \ \ (\text{resp. } \ \varphi = \liminf_{i\to\infty} \varphi_i \ )$$

is convex (resp. concave). So, $\rho \mapsto R(\Sigma,\rho)$ is continuous on (r_1,r_2), and lower semicontinuous in r_1 and r_2, by logarithmic concavity.

In the second part of *loc.cit.* , Christol-Dwork show that, as a consequence of their theory of the convergence polygon of a differential operator *cf.* §2.4 of *loc.cit.* , $\rho \mapsto R(\Sigma, \rho)$ is also USC at r_1 and r_2. They conclude that $\rho \mapsto R(\Sigma, \rho)$ is continuous on $[r_1, r_2]$.

We recall that the system Σ is said to be *solvable* at $t_{0,r}$, for $r \in [r_1, r_2]$, if $R(\Sigma, r) = r$.

Pons [38, 2.2] proves the following theorem.

Theorem 4.0.4. *The function $\rho \mapsto R(\Sigma, \rho)$ is logarithmically a concave polygon with a finite number of sides on $[r_1, r_2]$. The slopes of the sides are rational numbers with denominator at most μ.*

Remark 4.0.5. The simpler case of this theorem, for a system solvable at r_1 and r_2, appears in §4 of [19]. We cannot follow the topological arguments of Pons *loc.cit.* . We prefer to review entirely the proof of her theorem, based on [20] and [43], to make it completely clear. Actually, we need to combine the main theorem of [20, Thm. 5.4], with its variation proved by Kedlaya [35, Thm. 6.15].

Proof. Let $\pi \in k^{\mathrm{alg}}$ be such that $\pi^{p-1} = -p$. By logarithmic concavity, the continuous function $R(\Sigma, r)/r$ is constantly equal to its maximum value M on an interval $[r_1', r_2'] \subset [r_1, r_2]$, with $r_1' \leq r_2'$, and it is strictly increasing (resp. decreasing) for $r \leq r_1'$ (resp. $r \geq r_2'$). Then precisely one of the following holds:

1. $M = 1$,
2. $M = |\pi|^{1/p^h}$, for some $h \in \mathbb{Z}_{\geq 0}$,
3. $|\pi|^{1/p^{h-1}} < M < |\pi|^{1/p^h}$, for some $h \in \mathbb{Z}_{>0}$,
4. $M < |\pi|$.

If $M < 1$, then the interval $[r_1, r_2]$ can be subdivided in a finite number of intervals of the form $I = [a, b]$, where $R(\Sigma, r)/r$ is either strictly increasing or strictly decreasing, and for which precisely one of the following holds:

1. there exists $h = h(I) \in \mathbb{Z}_{>0}$ such that $|\pi|^{1/p^{h-1}} < R(\Sigma, r)/r < |\pi|^{1/p^h}$, for $r \in (a, b)$,
2. $R(\Sigma, r)/r < |\pi|$, for $r \in (a, b)$.

If $M = 1$, we can find an increasing (resp. decreasing) sequence of points $h \mapsto a_h \in [r_1, r_1')$ (resp. $h \mapsto b_h \in (r_2', r_2]$), defined for $h \in \mathbb{Z}_{\geq 0}$ as soon as $R(\Sigma, r_1)/r_1 \leq |\pi|^{1/p^h}$ (resp. $R(\Sigma, r_2)/r_2 \leq |\pi|^{1/p^h}$), converging to r_1' (resp. r_2'), and such that $R(\Sigma, a_h)/a_h = |\pi|^{1/p^h}$ (resp. $R(\Sigma, b_h)/b_h = |\pi|^{1/p^h}$). The function $R(\Sigma, r)/r$ satisfies $|\pi|^{1/p^{h-1}} < R(\Sigma, r)/r < |\pi|^{1/p^h}$ for $r \in (a_{h-1}, a_h)$ (resp. $r \in (b_h, b_{h-1})$), as soon as a_{h-1} (resp. b_{h-1}) is defined. If $R(\Sigma, r_1)/r_1 < |\pi|$ (resp. $R(\Sigma, r_2)/r_2 < |\pi|$), then $R(\Sigma, r)/r < |\pi|$ in $[r_1, a_0)$ (resp. $(b_0, r_2]$).

When $R(\Sigma, r)/r < |\pi|$, the precise value of $R(\Sigma, r)$ has been evaluated by Young [43]. We recall that the residue field $\mathscr{H}(t_{0,r})$ is the completion of the rational field $k(T)$ under the absolute value $f(T) \mapsto |f(t_{0,r})|$, where

$$|(a_n T^n + \cdots + a_1 T + a_0)(t_{0,\rho})| = \max_{i=0,1,\ldots,n} |a_i| r^i$$

for a polynomial $a_n T^n + \cdots + a_1 T + a_0 \in k[T]$. The derivation d/dT (resp. $\delta := T d/dT$) extends to a continuous derivation of $\mathscr{H}(t_{0,r})/k$, of operator norm $1/r$ (resp. 1). By the theorem of the cyclic vector, the system Σ is equivalent over the field $\mathscr{H}(t_{0,r})$, for any $r \in [r_1, r_2]$, to a single differential operator

$$L = \delta^\mu - C_1 \delta^{\mu-1} - \cdots - C_\mu \ , \quad \delta = T \frac{d}{dT} \ .$$

Young's theorem asserts that if $R(\Sigma, r)/r < |\pi|$, or if $\exists\, j_0 = 1, \dots, \mu$ such that $|C_{j_0}(t_{0,r})| > 1$, then

$$R(\Sigma, r)/r = |\pi| \min_j |C_j(t_{0,r})|^{-1/j} .$$

In particular, if $R(\Sigma, r)/r < |\pi|$, then $R(\Sigma, r)/r$ can only take values of the form $|\pi|(|a|r^s)^{1/j}$, where $a \in k$ and $s \in \mathbb{Z}$ and j is an integer between 1 and μ.

Let us consider a finite étale covering $\varphi : B(s_1, s_2) \to B(r_1, r_2)$ which is a composite of maps of the following forms: a *Kummer covering* $x \mapsto x^N$, for some $N = 1, 2, \dots$, where $r_i = s_i^N$, $i = 1, 2$, a *dilatation* $x \mapsto ax$, for $a \in k^\times$, in which case $r_i = |a|s_i$, for $i = 1, 2$, and an *inversion* $x \mapsto \gamma x^{-1}$, with $r_1 = |\gamma| s_2^{-1}$ and $r_2 = |\gamma| s_1^{-1}$, assuming such a $\gamma \in k^\times$ exists. Among Kummer coverings we have the *Frobenius covering*[13] *of degree h*: $\varphi_h : B(r_1, r_2) \to B(r_1^p, r_2^p)$, $x \mapsto x^{p^h}$. For any φ as above, we may pull-back the system to a similar system on $B(s_1, s_2)$, which we denote by $\varphi^*\Sigma$, in an obvious way.

Remark 4.0.6. The dilatation map $\varphi : x \mapsto ax$, for $a \in k^\times$, transforms isomorphically the disc $D(0, 1^\pm)$ into the disc $D(0, |a|^\pm)$, sending $t_{0,1}$ to $t_{0,|a|}$. If $r \in |k^\times|$, and $|a| = r$, the system $\varphi^*\Sigma$ is associated to the operator

$$\varphi^* L = \delta^\mu - C_1(ax)\delta^{\mu-1} - \dots - C_\mu(ax)$$

satisfies $R(\varphi^*\Sigma, 1) = R(\Sigma, r)/r$. If $r \notin |k^\times|$, one can reason in the same way after the base change to $k_r := k\{r^{-1}T_1, rT_2\}/(T_1T_2 - 1)$. In this form Young's result appears with full details in [26, VI, 2.1]. In the same vein, one can avoid consideration of $R(\Sigma, r)$ as $r \to r_1$, and only discuss the case $r \to r_2$, by using the inversion $x \mapsto x^{-1}$.

Lemma 4.0.7. *Let $\gamma \in k^\times$ and $\varphi : B(|\gamma|, 1) \to B(|\gamma|, 1)$ be the inversion $x \mapsto \gamma/x$. Let Σ be a system on $B(|\gamma|, 1)$ as above. Then, for any $r \in [|\gamma|, 1]$,*

$$R(\varphi^*\Sigma, r)/r = R(\Sigma, |\gamma|/r)/(|\gamma|/r) .$$

Proof. An immediate calculation shows that for any k-rational point $a \in B(|\gamma|, 1)$ and any $R \in (|\gamma|, |a|)$, φ induces an isomorphism, of the disk $D(a, R^+)$ onto the disk $D(\gamma/a, (R\,|\gamma/a^2|)^+)$. Therefore, by the interpretation of $R(\Sigma, r)$ as the radius of the maximal disk centered at the canonical point a above $t_{0,r}$ in $\mathscr{H}(t_{0,r})\widehat{\otimes} B(|\gamma|, 1)$, for which $|x(a)| = r$, where a fundamental solution matrix of the base change of Σ at a converges, we deduce

$$R(\Sigma, |\gamma|/r) = (|\gamma|/r^2) R(\varphi^*\Sigma, r) ,$$

as promised. □

Of special interest is the pullback by Frobenius: the main theorem of [20] combined with [35, Thm. 6.15] asserts that if, for some fixed $h \in \mathbb{Z}_{>0}$, a system Σ of the form (0.1.0.1) satisfies

$$|\pi|^{1/p^{h-1}} < R(\Sigma, r)/r < |\pi|^{1/p^h}$$

for any $r \in (r_1, r_2)$, then there exists a system Σ_h with coefficients in $\mathscr{H}(r_1^{p^h}, r_2^{p^h})$, unique in the sense of $\mathscr{H}(r_1^{p^h}, r_2^{p^h})/k$-differential modules, such that $\Sigma \cong \varphi_h^*\Sigma_h$, where "$\cong$" means isomorphism of $\mathscr{H}(r_1, r_2)/k$-differential modules. Moreover

$$R(\Sigma_h, r^{p^h})/r^{p^h} = (R(\Sigma, r)/r)^{p^h} ,$$

[13]The semilinear version of the map φ_1 is often used instead. Namely, for any continuous lifting $\sigma \in \mathrm{Aut}(k)$ of the absolute Frobenius of $\widetilde{k}$, one considers $\phi^{(\sigma)} : B(r_1, r_2) \to B(r_1^p, r_2^p)\widehat{\otimes}_{k,\sigma} k$, which is $\sum_i a_i T^i \mapsto \sum_i a_i^\sigma T^{pi}$ at the ring level. See section 5.4 below.

so that $R(\Sigma_h, r)/r < |\pi|$ in $(r_1^{p^h}, r_2^{p^h})$, and Young's theorem can be applied. Notice that if the graph of $R(\Sigma_h, r)/r$ is logarithmically affine with slope α in the interval $[r_1^{p^h}, r_2^{p^h}]$, so is $R(\Sigma, r)/r$ in $[r_1, r_2]$:

$$R(\Sigma_h, r)/r = Cr^\alpha \Rightarrow R(\Sigma_h, r^{p^h})/r^{p^h} = Cr^{\alpha p^h} \Rightarrow R(\Sigma, r)/r = C^{1/p^h} r^\alpha .$$

So theorem 4.0.4 is proven in the case $R(\Sigma, r)/r < 1$ in $[r_1, r_2]$. If the function $R(\Sigma, r)/r$ reaches the maximum value 1 in $[r_1, r_2]$, we are in the case treated by Christol-Mebkhout, and can follow their argument [19, 4.2]. Namely, we operate as in case $M = 1$ above, and reduce, possibly by an inversion, to the case when $R(\Sigma, r)/r$ is strictly increasing to 1 in $[r_1, r_2]$. Then we consider the sequence $h \mapsto a_h \in [r_1, r_2]$, described above (where $r_1' = r_2$), defined for $h \in \mathbb{Z}_{\geq 0}$ as soon as $R(\Sigma, r_1)/r_1 \leq |\pi|^{1/p^h}$, converging to r_2, and such that $R(\Sigma, a_h)/a_h = |\pi|^{1/p^h}$. The function $R(\Sigma, r)/r$ satisfies $|\pi|^{1/p^{h-1}} < R(\Sigma, r)/r < |\pi|^{1/p^h}$ for $r \in (a_{h-1}, a_h)$, as soon as a_{h-1} (resp. b_{h-1}) is defined. If $R(\Sigma, r_1)/r_1 < |\pi|$, then $R(\Sigma, r)/r < |\pi|$ in $[r_1, a_0)$. So, by the previous argument, the function $R(\Sigma, r)/r$ is continuous, logarithmically concave and piecewise affine with rational slopes with denominator at most μ. But in this special case, those slopes are positive and must be decreasing as $r \to r_2$. The constraint on the denominator shows that there is a non negative rational number β with denominator bounded by μ, such that, for sufficiently big h, the function $R(\Sigma, r)/r$ on the interval $[a_{h-1}, a_h]$ is of the form $C_h r^\beta$. So, C_h is independent of h, and, since $R(\Sigma, r_2)/r_2 = 1$, $C_h = r_2^{-\beta}$. So, theorem 4.0.4 is proven in every case. □

Corollary 4.0.8. *Let X be a closed k-annulus in the analytic k-line $\mathbb{A}^1_k$, and (0.1.0.1) be a $\mu \times \mu$ system of linear differential equations on X. Let $r : S(X) \xrightarrow{\sim} [r_1, r_2]$ be the function "radius of a point" of [6, p. 78], restricted to $S(X)$. Then, the restriction of $x \mapsto R(\Sigma, r(x))/r(x)$ to $S(X)$ is the infimum of the constant 1 and a finite set of functions of the form*

$$|p|^{1/(p-1)p^h} |b|^{1/jp^h} r(x)^{s/j} , \tag{4.0.8.1}$$

where $j \in \{1, 2, \dots, \mu\}$, $s \in \mathbb{Z}$, $h \in \mathbb{Z} \cup \{\infty\}$, $b \in k^\times$.

Example 4.0.9. ([26, IV.7.3]) Let $\alpha \in k^\circ$ and regard

$$\Sigma_\alpha \ : \frac{dy}{dT} = \frac{\alpha}{1+T}\, y , \tag{4.0.9.1}$$

as an analytic differential equation on any strictly affinoid annulus of the form

$$X = B[-1; r_1, r_2] \subset \mathbb{A}^1_k \setminus \{-1\} ,$$

with its minimal semistable formal structure $\mathfrak{X}$. For any $x \in X$, the solution of Σ_α on $\mathbb{A}^1_{\mathscr{H}(x)}$ which takes the value 1 at (the canonical point over) x is

$$g_\alpha\left(\frac{T - T(x)}{1 + T(x)}\right) = \left(1 + \frac{T - T(x)}{1 + T(x)}\right)^\alpha \in 1 + (T - T(x))\mathscr{H}(x)[[T - T(x)]] .$$

Let

$$d(\alpha, \mathbb{Z}_p) = \inf\{|n - \alpha| \mid n \in \mathbb{Z}\} \in |k^\circ| ,$$

and assume $p^{-m-1} < d(\alpha, \mathbb{Z}_p) \leq p^{-m}$, with $m \in \mathbb{Z}_{\geq 0}$. The radius of convergence of the power series

$$g_\alpha(U) = 1 + \sum_{i=1}^{\infty} \binom{\alpha}{i} U^i ,$$

is then

$$r_\alpha := |\pi|^{\frac{1}{p^m}} (d(\alpha, \mathbb{Z}_p)p^m)^{-\frac{1}{p^{m+1}}} ,$$

so that

$$\mathcal{R}_{\mathfrak{X}}(x, \Sigma_\alpha) = R(\Sigma_\alpha, x)/|1 + T(x)| = r_\alpha$$

is of the form predicted by corollary 4.0.8 on $S(X) = S(\mathfrak{X})$ and is in fact constant all over X.

5 Dwork-Robba theory over basic annuli and disks

We keep the notation and assumptions of the previous section on the field k, but the system Σ of (0.1.0.1) is supposed to be defined on a basic affinoid annulus or disk (X, T), as in definition 1.3.2. Recall that (X, T) is the generic fiber of a formal coordinate neighborhood $(\mathfrak{X}, T)$ either in a standard formal annulus of height $|\gamma|$, $\mathrm{Spf}\, k^\circ\{S, T\}/(ST - \gamma)$, with $\gamma \in k^\circ$, or in the standard formal disk $\mathrm{Spf}\, k^\circ\{T\}$ over k°, and in particular that the coordinate T on X is the pull-back of the formal coordinate T on $\mathfrak{X}$. In this case, X has canonical strictly semistable reduction, and the corresponding strictly semistable model of X is the minimum one, and coincides with $\mathfrak{X}$. So, $D_{\mathfrak{X}}(x, 1^-) = D_X(x, 1^-)$, for any $x \in X \setminus S(X) = X \setminus S(\mathfrak{X})$. We denote by $|| - ||_X$ the supnorm on X. Let $\mathcal{E} \cong \mathcal{O}_X^\mu$ and let ∇ be the connection on $\mathcal{E}$, whose solutions are the column solutions of the system Σ. We consider the system of ordinary linear differential equations (0.1.0.1) on X. We define $G_{[i]}$, for $i \in \mathbb{N}$, as in (3.1.3.2), *for the global coordinate T on the formal annulus or disk $\mathfrak{X}$.*
If X is a disk, we set

(5.0.9.2) $$\mathcal{R}(x, \Sigma) = \min(1, \liminf_{n\to\infty} |G_{[n]}(x)|^{-1/n}) .$$

If X is an annulus, we set

(5.0.9.3) $$R(x, \Sigma) = \min(|T(x)|, \liminf_{n\to\infty} |G_{[n]}(x)|^{-1/n}) ,$$

and

(5.0.9.4) $$\mathcal{R}(x, \Sigma) = R(x, \Sigma)/|T(x)| .$$

Then, according to (3.1.10.2) and (3.1.10.3), we have in any case

$$\mathcal{R}_{\mathfrak{X}}(x, (\mathcal{E}, \nabla)) = \mathcal{R}(x, \Sigma) .$$

5.1 The Dwork-Robba theorem

We state here a useful form of the theorem of Dwork and Robba [26, Thm. 3.1, Chap. IV]. A multivariable version of it is given in [33] and [4, 4.2].

Theorem 5.1.1. *Let $x \in X$, Σ, $G_{[i]}$, for $i \in \mathbb{N}$, and $R = R(x, \Sigma)$ be as above. Then, for any $n \in \mathbb{N}$ we have the following estimate*

(5.1.1.1) $$|G_{[n]}(x)| \le C n^{\mu-1} R^{-n} ,$$

where

$$C = \max_{i\le\mu-1} \left(R^i |i!| \, ||G_{[i]}||_X\right) .$$

Proof. We may assume that $(\mathfrak{X}, T)$ *is* either a standard formal annulus of height $|\gamma|$, with $\gamma \in k^\circ$, or the standard formal disk over k°, *cf.* (1.3.2), and that $X = \mathfrak{X}_\eta$. We can also assume that k is algebraically closed. We first explain the notation and the result of remark 3.2 in [26, Chap. IV]. So, let $t = t_{a,\rho} \in X$, for $a \in k$ and $0 < \rho \leq 1$, and let $\mathcal{A}_{a,\rho} = \mathcal{O}_X(D_X(a, \rho^-))$ be the ring of analytic functions on $D_X(a, \rho^-$. Let $\mathcal{A}'_{a,\rho}$ be the quotient field of the ring $\mathcal{A}_{a,\rho}$, that is the field of *meromorphic functions* on $D_X(a, \rho^-)$. Notice that the k-linear derivation $\partial : f \mapsto \frac{df}{dT}$ of $\mathcal{A}_{a,\rho}$ extends uniquely to a k-linear derivation ∂ of $\mathcal{A}'_{a,\rho}$.

The *boundary seminorm*

$$|| \, ||_{a,\rho} : \mathcal{A}'_{a,\rho} \to \mathbb{R}_{\geq 0} \cup \{\infty\} ,$$

is defined, for any $f \in \mathcal{A}'_{a,\rho}$ as

$$||f||_{a,\rho} = \limsup_{R \to \rho^-} |f(t_{a,R})| .$$

We refer to *loc.cit.* for the properties of this seminorm, and in particular for the fact that, for any $f \in \mathcal{A}'_{a,\rho}$,

$$\text{(5.1.1.2)} \qquad || \frac{\partial^s f}{s!\, f} ||_{a,\rho} \leq \rho^{-s} \quad , \quad \forall s \geq 0 \ .$$

Now, notice that the k-linear continuous So, let us assume that we are given a $\mu \times \mu$ matrix of meromorhic functions on $D_X(a, \rho^-)$, such that

$$\text{(5.1.1.3)} \qquad \Sigma : \partial Y = G Y \ .$$

Then, for any $s \geq 0$,

$$\frac{1}{s!} \, (\partial^s Y) \, Y^{-1} = G_{[s]} \ ,$$

and theorem 3.1 and remark 3.2 of [26, Chap. IV] assert that

$$\text{(5.1.1.4)} \qquad ||G_{[s]}||_{a,\rho} \leq \rho^{-s} s^{\mu - 1} \sup_{0 \leq i \leq \mu - 1} (\rho^i ||G_{[i]}||_{a,\rho}) \ .$$

Now, consider a k-rational point x of X, and notice that there exists a solution matrix Y of (5.1.1.3) analytic in $D_X(x, R^-)$, with $R = R(x, \Sigma)$. If we insist that $Y(x)$ be the identity matrix, this solution is the sum of the convergent series of polynomial functions

$$Y = \sum_{i=0}^{\infty} G_{[s]}(x)(T - T(x))^i \ .$$

In this case, the matrices $G_{[s]}$ are analytic on X, and in particular on $D_X(x, R^-)$, so that

$$||G_{[s]}||_{a,\rho} = \sup_{y \in D_X(x, R^-)} |G_{[s]}(y)| \leq ||G_{[s]}||_X \quad ,$$

and the previous estimates for $\rho = R$ imply

$$\text{(5.1.1.5)} \qquad |G_{[s]}(x)| \leq R^{-s} s^{\mu - 1} \sup_{0 \leq i \leq \mu - 1} (R^i ||G_{[i]}||_X) \ .$$

So, our result is proven for x a k-rational point of X. For the general case, we extend the base field from k to any algebraically closed non-archimedean field k' over $\mathscr{H}(x)$, and consider the canonical k'-rational point $x' \in X \widehat{\otimes}_k k' =: X'$ above x. The system Σ is to be interpreted on X', and we write $\mathfrak{X}' = \mathfrak{X} \widehat{\otimes}_{k^\circ} k'^\circ$ and $(\mathcal{E}', \nabla')$ for the pull-back of $(\mathcal{E}, \nabla)$. The matrices $G_{[s]}$, however, do not change and $||G_{[i]}||_X = ||G_{[i]}||_{X'}$. Moreover, $R = R(x, \Sigma) = R(x', \Sigma)$ (while $\mathcal{R}_{\mathfrak{X}}(x, (\mathcal{E}, \nabla)) = \mathcal{R}_{\mathfrak{X}'}(x, (\mathcal{E}', \nabla'))$ by definition!). So, we obtain

$$\text{(5.1.1.6)} \qquad |G_{[s]}(x')| \leq R^{-s} s^{\mu - 1} \sup_{0 \leq i \leq \mu - 1} (R^i ||G_{[i]}||_X) \ ,$$

and the theorem is proven. □

The next corollary is the prototype of a *transfer theorem to an ordinary contiguous disk* [26, V.5].

Corollary 5.1.2. *Let D be an open Q-disk in X, and let t be its limit point in $X \setminus D$. Then $R(t,\Sigma) = \inf_{x\in D} R(x,\Sigma)$.*

Proof. Let $R = \inf_{x\in D} R(x,\Sigma)$, let r be the radius of D and let

$$C = \max_{i\le\mu-1} \left(r^i|i!|\,\left|\left|G_{[i]}\right|\right|_D\right) = \max_{i\le\mu-1} \left(r^i|i!|\,\left|G_{[i]}(t)\right|\right) \ .$$

If $R(x,\Sigma) > r$, for some $x \in D$, then $R(y,\Sigma) = R(x,\Sigma)$, for all y in an open Q-disk D' with $D \subsetneqq D' \subset X$. The statement is obvious in this case. So, we may assume that $R \le r$. We have, for any $x \in D$ and $n \ge 0$,

$$|G_{[n]}(x)| \le Cn^{\mu-1}R(x,\Sigma)^{-n} \le Cn^{\mu-1}R^{-n} \ . \tag{5.1.2.1}$$

So, we also have

$$|G_{[n]}(t)| = ||G_{[n]}||_D \le Cn^{\mu-1}R^{-n} \ , \tag{5.1.2.2}$$

which shows that $R \le R(t,\Sigma)$. On the other hand

$$|G_{[n]}(x)| \le ||G_{[n]}||_D = |G_{[n]}(t)| \le Cn^{\mu-1}R(t,\Sigma)^{-n} \ , \tag{5.1.2.3}$$

shows that $R(x,\Sigma) \ge R(t,\Sigma)$, for any $x \in D$. □

5.2 Upper semicontinuity of $x \mapsto R(x,\Sigma)$

We proceed in our discussion of the system (0.1.0.1) on the basic affinoid annulus or disk (X,T) to show that the function $x \mapsto R(x,\Sigma)$ is USC on X. Again, the multivariable version of this discussion appears in [4, 4.3].

We define

$$\widetilde{R}(x,\Sigma) = \liminf_{n\to\infty} |G_{[n]}(x)|^{-1/n} \ , \tag{5.2.0.4}$$

so that

$$R(x,\Sigma) = \begin{cases} \min(1,\widetilde{R}(x,\Sigma)) & \text{if} \quad X \text{ is a } Q\text{-disk} \\ \min(|T(x)|,\widetilde{R}(x,\Sigma)) & \text{if} \quad X \text{ is a } Q\text{-annulus,} \end{cases} \tag{5.2.0.5}$$

and will equivalently prove that the function $x \mapsto \widetilde{R}(x,\Sigma)$ is USC. We exclude the case when there exist $x \in X$ and s such that $G_{[n]}(x) = 0$, for any $n > s$, since $\widetilde{R}(y,\Sigma) = \infty$ and, for any $y \in X$,

$$R(y,\Sigma) = \begin{cases} 1 & \text{if} \quad X \text{ is a } Q\text{-disk} \\ |T(y)| & \text{if} \quad X \text{ is a } Q\text{-annulus,} \end{cases} \tag{5.2.0.6}$$

in that case. For $s = 1,2,\dots$ and for $x \in X$, let

$$\varphi_s(x) = \inf_{n\ge s} |G_{[n]}(\xi)|^{-1/n} \ . \tag{5.2.0.7}$$

So, $x \mapsto \varphi_s(x)$ is USC on X, and

$$\widetilde{R}(x) = \widetilde{R}(x,\Sigma) = \lim_{s\to\infty} \varphi_s(x) \ , \tag{5.2.0.8}$$

is the function introduced in (5.2.0.4). The Dwork-Robba theorem 5.1.1 implies that, $\forall\, \varepsilon > 0$, $\exists\, s_\varepsilon$ such that $\forall\, n$ with $n \geq s_\varepsilon$

$$|G_{[n]}(x)|^{1/n} \leq \frac{1+\varepsilon}{\widetilde{R}(x)} \ , \ \ \forall\, x \in X \ , \tag{5.2.0.9}$$

that is

$$|G_{[n]}(x)|^{-1/n} \geq \frac{\widetilde{R}(x)}{1+\varepsilon} \ , \ \ \forall\, x \in X \ . \tag{5.2.0.10}$$

Hence

$$\begin{array}{ccc} \forall\, \varepsilon > 0 \quad \exists\, s_\varepsilon & \text{such that} & \forall\, s \geq s_\varepsilon \\ \varphi_s(x) \leq & \widetilde{R}(x) & \leq (1+\varepsilon)\varphi_s(x) \quad \forall\, x \in X \ , \end{array} \tag{5.2.0.11}$$

because the sequence $s \mapsto \varphi_s$ is an increasing sequence of functions on U. Then, $\forall\, \varepsilon > 0$, $\exists\, s_\varepsilon$ such that

$$0 \leq \widetilde{R} - \varphi_s \leq \varepsilon \ \text{ on } X \ , \ \ \forall\, s \geq s_\varepsilon \ . \tag{5.2.0.12}$$

So, $\widetilde{R}$ is a uniform limit of USC functions on X, and is therefore USC.

5.3 The generalized theorem of Christol-Dwork

Theorem 5.3.1. *Let Σ be a system of linear differential equations on the basic affinoid annulus*[14] *(X, T), as above. The function $x \mapsto \widetilde{R}(x, \Sigma)$ restricts to a continuous function on $S(X)$.*

Proof. There is a natural homeomorphism $\eta : [r_0, 1] \xrightarrow{\sim} S(X)$, $r \mapsto \eta_r$, with $r_0 \in (0,1] \cap |k|$. We follow the method of [20], recalled in §4: the function (5.2.0.4)

$$\widetilde{R}(\eta_r, \Sigma) = \liminf_{n \to \infty} |G_{[n]}(\eta_r)|^{-1/n} \ ,$$

is logarithmically concave on $[r_0, 1]$, hence continuous on $(r_0, 1)$ and LSC at r_0 and 1. On the other hand, we showed above that $x \mapsto \widetilde{R}(x, \Sigma)$ is USC all over X. This proves the theorem. $\square$

5.4 Descent by Frobenius

We recall that, for any morphism of schemes $\pi : Y \to S$ where $p = 0$ on S, we have a canonical commutative diagram

$$\begin{array}{ccccc} Y & \xrightarrow{F_{\rm rel}} & Y^{(p)} & \xrightarrow{\Psi} & Y \\ & \pi \searrow & \downarrow \pi^{(p)} & \square & \downarrow \pi \\ & & S & \xrightarrow{F_{\rm abs}} & S \end{array} \tag{5.4.0.1}$$

where $F_{\rm abs}$ (resp. $F_{\rm rel}$) denotes absolute (resp. relative) Frobenius, and $\Psi \circ F_{\rm rel} = F_{\rm abs}$.

[14]the case of a disk is here trivial.

Let $(\mathfrak{X}, T)$ be a basic formal annulus over k° of radii $r_1, r_2 \in |k| \cap (0,1]$, as in (1.3.3.1). We denote $\pi : \mathfrak{X} \to \mathrm{Spf}\, k^\circ$ the structural morphism. In our case diagram 5.4.0.1 for $\mathfrak{X}_s$ becomes

(5.4.0.2)
$$\begin{array}{ccccc} \mathfrak{X}_s & \xrightarrow{F_{\mathrm{rel}}} & (\mathfrak{X}_s)^{(p)} & \xrightarrow{\Psi_{\mathfrak{X}_s}} & \mathfrak{X}_s \\ & \pi_s \searrow & \pi_s^{(p)} \Big\downarrow & \square & \pi_s \Big\downarrow \\ & & \mathrm{Spec}\, \widetilde{k} & \xrightarrow{F_{\mathrm{abs}}} & \mathrm{Spec}\, \widetilde{k} \end{array}$$

Let us choose a continuous ring automorphism $\sigma : k^\circ \to k^\circ$ lifting the absolute Frobenius $F_{\mathrm{abs}}^* : x \mapsto x^p$ of $\widetilde{k}$. We want to show

Lemma 5.4.1. *The left hand triangle in diagram 5.4.0.2 lifts to a commutative diagram of k°-formal schemes*

(5.4.1.1)
$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{F_{\mathrm{rel}}(\sigma)} & \mathfrak{X}^{(\sigma)} \\ & \pi \searrow & \Big\downarrow \pi^{(\sigma)} \\ & & \mathrm{Spf}\, k^\circ \end{array}$$

where $\pi_{\mathfrak{X}^{(\sigma)}} := \pi^{(\sigma)} : \mathfrak{X}^{(\sigma)} \to \mathrm{Spf}\, k^\circ$ is a formal annulus with special fiber $(\mathfrak{X}_s)^{(p)} \to \mathrm{Spec}\, \widetilde{k}$.

Proof. Let us start with the special case of $\mathfrak{X} = \mathfrak{B}(r_1, r_2)$. In this case we know that there exists a commutative diagram

(5.4.1.2)
$$\begin{array}{ccc} \mathfrak{B}(r_1, r_2) & \xrightarrow{\Phi} & \mathfrak{B}(r_1^p, r_2^p) \\ \pi \searrow & & \swarrow \pi^{(p)} \\ & \mathrm{Spf}\, k^\circ & \end{array}$$

where
$$\mathfrak{B}(r_1^p, r_2^p) = \mathrm{Spf}\, k^\circ\{S, T, U\}/(a_2^p S - T, TU - a_1^p) \ ,$$
$\pi^{(p)} : \mathfrak{B}(r_1^p, r_2^p) \to \mathrm{Spf}\, k^\circ$ is the natural structural morphism, and the map Φ is defined at the ring level by

(5.4.1.3)
$$\Phi^* : \sum_{h,i,j} a_{h,i,j} S^h T^i U^j \mapsto \sum_{h,i,j} a_{h,i,j} S^{ph} T^{pi} U^{pj} \ .$$

We need to use a semilinear version of this map, as follows. We consider the diagram

(5.4.1.4)
$$\begin{array}{ccc} \mathfrak{B}(r_1, r_2) & \xrightarrow{\Phi^{(\sigma)}} & \mathfrak{B}(r_1^p, r_2^p) \widehat{\otimes}_{k^\circ, \sigma} k^\circ \\ \pi \searrow & & \swarrow \pi^{(\sigma)} \\ & \mathrm{Spf}\, k^\circ & \end{array}$$

where the map $\Phi^{(\sigma)}$ is defined at the ring level by

(5.4.1.5)
$$(\Phi^{(\sigma)})^* : \sum_{h,i,j} a_{h,i,j} S^h T^i U^j \mapsto \sum_{h,i,j} a_{h,i,j}^\sigma S^{ph} T^{pi} U^{pj} \ .$$

We observe that the previous diagram 5.4.1.4 satisfies the requirements of the statements 5.4.1.2 for $\mathfrak{X} = \mathfrak{B}(r_1, r_2) =: \mathfrak{B}$, $\mathfrak{X}^{(\sigma)} = \mathfrak{B}(r_1^p, r_2^p) \widehat{\otimes}_{k^\circ, \sigma} k^\circ =: \mathfrak{B}^{(\sigma)}$, and in particular that $\Phi^{(\sigma)}$

lifts $F_{\mathrm{rel}} : \mathfrak{B}_s \to \mathfrak{B}_s^{(p)}$. We call $\pi_{\mathfrak{B}}$, $\pi_{\mathfrak{B}}^{(\sigma)}$ instead of simply π, $\pi^{(\sigma)}$, the structural morphisms of $\mathfrak{B}$ and $\mathfrak{B}^{(\sigma)}$, respectively. Consider the commutative diagram of cartesian squares

$$\begin{array}{ccccc} \mathfrak{X}_s & \xrightarrow{F_{\mathrm{rel}}} & (\mathfrak{X}_s)^{(p)} & \xrightarrow{\Psi_{\mathfrak{X}_s}} & \mathfrak{X}_s \\ \alpha_s\Big\downarrow & \square & \alpha_s^{(p)}\Big\downarrow & \square & \alpha_s\Big\downarrow \\ \mathfrak{B}_s & \xrightarrow{F_{\mathrm{rel}}} & (\mathfrak{B}_s)^{(p)} & \xrightarrow{\Psi_{\mathfrak{X}_s}} & \mathfrak{B}_s \end{array} \tag{5.4.1.6}$$

We now define $\alpha^{(\sigma)} : \mathfrak{X}^{(\sigma)} \to \mathfrak{B}^{(\sigma)}$ as the unique lifting of $\alpha_s^{(p)} : \mathfrak{X}_s^{(p)} \to \mathfrak{B}_s^{(p)}$, according to [8, Lemma 2.1]. We obtain a unique map $F_{\mathrm{rel}}^{(\sigma)} : \mathfrak{X} \to \mathfrak{X}^{(\sigma)}$ lifting $F_{\mathrm{rel}} : \mathfrak{X}_s \to \mathfrak{X}_s^{(p)}$, and a cartesian square

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{F_{\mathrm{rel}}^{(\sigma)}} & \mathfrak{X}^{(\sigma)} \\ \alpha\Big\downarrow & \square & \Big\downarrow\alpha^{(\sigma)} \\ \mathfrak{B} & \xrightarrow{\Phi^{(\sigma)}} & \mathfrak{B}^{(\sigma)} \end{array} \tag{5.4.1.7}$$

We now define $\pi^{(\sigma)} : \mathfrak{X}^{(\sigma)} \to \mathrm{Spf}\, k$ as $\pi_{\mathfrak{B}}^{(\sigma)} \circ \alpha^{(\sigma)}$, and obtain diagram 5.4.1.1. $\square$

We recall that there exists a continuous map of k°-modules

$$\Psi^* : \sum_{h,i,j} a_{h,i,j} S^h T^i U^j \mapsto \sum_{h,i,j} a_{ph,pi,pj} S^h T^i U^j \ , \tag{5.4.1.8}$$

such that $\Psi^* \circ \Phi^* = \mathrm{id}_{A(r_1^p, r_2^p)}$.

The generic fibers $\varphi = \Phi_\eta : B[r_1, r_2] \to B[r_1^p, r_2^p]$ and $\phi^{(\sigma)} = (\Phi^{(\sigma)})_\eta : B[r_1, r_2] \to B[r_1^p, r_2^p]\widehat{\otimes}_{k,\sigma} k$ of the previous maps are both classically used in the theory of descent of differential modules by Frobenius, as we have illustrated (using the linear map) in the previous section (where $\varphi = \varphi_1$). We have the following generalization of the theorem of descent by Frobenius of Dwork and Christol.

Theorem 5.4.2. (Descent by Frobenius) *Let $\mathfrak{X}$ be a basic formal annulus over k°, and let $F_{\mathrm{rel}}^{(\sigma)} : \mathfrak{X} \to \mathfrak{X}^{(\sigma)}$ be as in lemma 5.4.1. We denote by $\phi^{(\sigma)} : X := \mathfrak{X}_\eta \to X^{(\sigma)} := \mathfrak{X}_\eta^{(\sigma)}$ the generic fiber of $F_{\mathrm{rel}}^{(\sigma)}$. Let $(\mathcal{E}, \nabla)$ be an object of $\mathbf{MIC}_{\mathfrak{X}}(X/k)$ and assume that $\mathcal{R}_{\mathfrak{X}}(x, (\mathcal{E}, \nabla)) > p^{-\frac{1}{p-1}}$, $\forall x \in X$. Then, there exists a unique object $(\mathcal{F}, \nabla)$ of $\mathbf{MIC}_{\mathfrak{X}^{(\sigma)}}(X^{(\sigma)}/k)$, such that $(\mathcal{E}, \nabla) = (\phi^{(\sigma)})^*(\mathcal{F}, \nabla)$. For any $x \in X$, we have*

$$\mathcal{R}_{\mathfrak{X}^{(\sigma)}}(\phi^{(\sigma)}(x), (\mathcal{F}, \nabla)) = \mathcal{R}_{\mathfrak{X}}(x, (\mathcal{E}, \nabla))^p \ . \tag{5.4.2.1}$$

Proof. The construction of $(\mathcal{F}, \nabla)$ is the same, *mutatis mutandis* as the one of Christol-Dwork [20, 4.3] and of [26, V.7]. It is important however to use the coordinate-free presentation of Kedlaya [35, 6.3], that eliminates the problem of apparent singularities, which causes so much technical complication in [20] and [26]. We do not give details here. Notice however that one may assume, without loss of generality, that the field k contains the p-th roots of unity. The group μ_p acts on $\mathfrak{B}(r_1, r_2)$ via $\zeta \mapsto \tau_\zeta$, where $\tau_\zeta : (S, T, U) \mapsto (\zeta S, \zeta T, \zeta^{-1} U)$ and the map Ψ^* of formula 5.4.1.8 is

$$\Psi^* = p^{-1} \sum_{\zeta \in \mu_p} \tau_\zeta^* \ .$$

The quotient map $\mathfrak{B}(r_1, r_2) \to \mathfrak{B}(r_1, r_2)/\mu_p$ identifies with Φ in (5.4.1.1). The action of μ_p on $\mathfrak{B}(r_1, r_2)$ uniquely lifts to an action on $\mathfrak{X}$ and on $\mathfrak{X}^{(\sigma)}$. The quotient map $\mathfrak{X} \to \mathfrak{X}^{(\sigma)}/\mu_p$ identifies then with $F_{\mathrm{rel}}^{(\sigma)}$ in (5.4.1.2).

We may now follow the method of Kedlaya *loc.cit.* to conclude. □

6 Continuity and piecewise linearity of $x \mapsto \mathcal{R}_{\mathfrak{Y},\mathfrak{Z}}(x, (\mathcal{E}, \nabla))$

We use here the notation of definition 3.1.7. In particular k is a complete extension of $\mathbb{Q}_p$, $\overline{X} = \mathfrak{Y}_\eta$, for a semistable formal scheme $\mathfrak{Y}$, $\mathfrak{Z} = \{z_1, \ldots, z_r\} \subset \overline{X}(k)$ and $X = \overline{X} \setminus \mathfrak{Z}$.

Theorem 6.0.3. *Let $(\mathcal{E}, \nabla)$ be an object of $\mathbf{MIC}_{\mathfrak{Y}}(X/k)$. The function $X \to \mathbb{R}_{>0}$, $x \mapsto \mathcal{R}_{\mathfrak{Y},\mathfrak{Z}}(x, (\mathcal{E}, \nabla))$ is continuous.*

Proof. Let $\varepsilon \in |k| \cap (0, 1)$, and let $X^{(\varepsilon)} = \overline{X} \setminus \cup_{i=1}^r D_{\mathfrak{Y}}(z_i, \varepsilon^-)$. By lemma 3.1.10, there is a morphism of semistable formal schemes $\mathfrak{X}^{(\varepsilon)} \to \mathfrak{Y}$, composite of an open immersion $\mathfrak{X}^{(\varepsilon)} \to \mathfrak{Y}^{(\varepsilon)}$ and of an admissible blowing-up $\mathfrak{Y}^{(\varepsilon)} \to \mathfrak{Y}$, whose generic fiber identifies with the embedding $X^{(\varepsilon)} \subset \overline{X}$. The restriction $(\mathcal{E}, \nabla)_{|X^{(\varepsilon)}}$ of $(\mathcal{E}, \nabla)$ to $X^{(\varepsilon)}$, is an object of $\mathbf{MIC}_{\mathfrak{X}^{(\varepsilon)}}(X^{(\varepsilon)}/k)$, and $\mathcal{R}_{\mathfrak{X}^{(\varepsilon)}}(x, (\mathcal{E}, \nabla)_{|X^{(\varepsilon)}}) = \mathcal{R}_{\mathfrak{Y},z_1,\ldots,z_r}(x, (\mathcal{E}, \nabla))$, for any $x \in X^{(\varepsilon)}$. So, we are reduced to the case where there are no z_i's and $\overline{X} = X = \mathfrak{X}_\eta$, for a Q-semistable formal scheme $\mathfrak{X}$. For any non-archimedean field extension k'/k and Q-étale morphism of Q-semistable k'°-formal schemes $\psi : \mathfrak{Y} \to \mathfrak{X}\widehat{\otimes}k'^\circ$, the assignment

$$f_\psi(y) = \mathcal{R}_{\mathfrak{Y}}(y, \psi^*(\mathcal{E}, \nabla)) \ , \ \ \forall y \in \mathfrak{Y}_\eta \ , \tag{6.0.3.1}$$

is an étale-local system of functions $(f_\psi)_\psi$ on $\mathfrak{X}$ with values in $(0, 1]$, as defined in (2.0.2), by lemma 3.1.6 and the definitions.

In the end, by lemmas 1.3.3 and 2.0.3, we are reduced to the situation of §5, and in particular to proving the following

Lemma 6.0.4. *Let $(X, T) = (\mathfrak{X}, T)_\eta$ be a basic affinoid annulus, and let $x \in X$, Σ, $G_{[i]}$, for $i \in \mathbb{N}$, $R(x) := R(x, \Sigma)$ be as in theorem 5.1.1. Then the function $R : X \to \mathbb{R}_{>0}$ is continuous.*

Proof. Let $\widetilde{R}(x) := \widetilde{R}(x, \Sigma)$ be as in (5.2.0.4). We check conditions (1) to (5) in theorem 2.0.5 for the étale-local system of functions associated to R, as in definition 2.0.2. Notice that $R(x, \Sigma)$ is given by formula 3.1.6.2 in terms of any formal étale coordinate on $\mathfrak{X}$, and over any non-archimedean extension field k'/k. Condition (1) is obvious: if $x_0 \in X(k)$, $R(x, \Sigma) = R(x_0, \Sigma)$, $\forall x \in D_{\mathfrak{X}}(x_0, R(x_0, \Sigma)^-)$. Condition (2) is clear since, for any affinoid $V \subset X$,

$$|G_{[n]}(x)| \leq \max_{y \in \Gamma(V)} |G_{[n]}(y)| \ .$$

and therefore

$$\widetilde{R}(x, \Sigma) = \liminf_{n \to \infty} |G_{[n]}(x)|^{-1/n} \geq \min_{y \in \Gamma(V)} \widetilde{R}(y, \Sigma) \ .$$

So, the same holds true for $R(x, \Sigma)$. Conditions (3) and (4) have been proved together in section 5.3. Condition (5) is proven in section 5.2. □

This proves the theorem, too. □

Remark 6.0.5. In [4, 5.3] we gave a direct proof of this statement, in the case of a system (*i.e.* for $\mathcal{E}$ free) over an affinoid domain of $\mathbb{A}^1$.

Corollary 6.0.6. *Let μ be the rank of $\mathcal{E}$ on $X = \overline{X} \setminus \mathfrak{Z}$ and $\overline{X} = \mathfrak{Y}_\eta$, as before. Let E be an open edge of $\mathbf{S}_{\mathfrak{Z}}(\mathfrak{Y})$ and let $\overline{E}$ be the closure of E in $\overline{X}$. We have the following possibilities:*

1. *E is of finite length $r_E^{-1} \in [1,\infty)$ and split. Then, for any choice of one of the two canonical homeomorphism $\rho : E \xrightarrow{\sim} (r_E, 1)$, the restriction of $x \mapsto \mathcal{R}(x) := \mathcal{R}_{\mathfrak{Y},\mathfrak{Z}}(x,(\mathcal{E},\nabla))$ to E is the infimum of the constant 1 and a finite set of functions of the form*

$$|p|^{1/(p-1)p^h}|b|^{1/jp^h}\rho(x)^{s/j}\ , \tag{6.0.6.1}$$

where $j \in \{1,2,\dots,\mu\}$, $s \in \mathbb{Z}$, $h \in \mathbb{Z}\cup\{\infty\}$, $b \in k^\times$. If $\overline{E}\setminus E$ consists of 2 distinct points, ρ extends to $\rho : \overline{E} \xrightarrow{\sim} [r_E, 1]$, and the expression of $\mathcal{R}(x)$ holds true at the ends. If $\overline{E}\setminus E$ consists of a single point v, the previous expression for $\mathcal{R}(x)$, for $\rho(x) = \rho$, tends to the same value $\mathcal{R}(v)$, as $\rho \to r_E^+$ and as $\rho \to 1^-$.

2. *E is of finite length $r_E^{-1} \in [1,\infty)$ and non split. Then, there is a canonical homeomorphism $\rho : E \xrightarrow{\sim} [r_E, 1)$, which extends to a homeomorphism $\rho : \overline{E} \xrightarrow{\sim} [r_E, 1]$, and the restriction of $x \mapsto \mathcal{R}(x)$ to $\overline{E}$ is the infimum of a finite set of functions as in 6.0.6.1.*

3. *if E is of infinite length, and z_i is a boundary point of E in $\overline{X}$, and $\rho : \overline{E} \xrightarrow{\sim} [0,1]$ is the canonical homeomorphism with $\rho(z_i) = 0$, then, $\forall\, \varepsilon \in |k| \cap (0,1)$ the restriction of $x \mapsto \mathcal{R}_{\mathfrak{Y},\mathfrak{Z}}(x,(\mathcal{E},\nabla))$ to $\rho^{-1}([\varepsilon,1])$ is an infimum of a finite set of functions as in 6.0.6.1.*

Proof. We may restrict to case 1. We know the result in case $\tau_{\mathfrak{Y}}^{-1}(\overline{E})$ is an annulus in $\mathbb{A}^1$. So, the restriction of $x \mapsto \mathcal{R}_{\mathfrak{Y},z_1,\dots,z_r}(x,(\mathcal{E},\nabla))$ to $\rho^{-1}([r_E/\varepsilon,\varepsilon])$ has the required properties for any $\varepsilon \in |k| \cap (r_E^{1/2}, 1)$. Let $R(r) = \mathcal{R}_{\mathfrak{Y},\mathfrak{Z}}(\rho^{-1}(r),(\mathcal{E},\nabla))$, and let $S = \log \circ R \circ \exp$. So, S is concave, piecewise linear and continuous in $[\log r_E, 0]$. We have to show that is a polygon with a finite number of sides. It is enough to show that the (right or left) slope of the function $S(t)$ is bounded as $t \to 0$. We may follow the strategy of proof of theorem 4.0.4. If we have $R(0) = 1$, the result is clear because in this case S is never decreasing and its slopes are non-negative. On the other hand, the slopes are rational numbers with denominator bounded by μ. So, there can only be a finite number of sides. If $R(0) < 1$, we apply we apply our result (5.4.2) of descent by Frobenius instead, and reduce eventually to the case explicitly computed by Young, precisely as in the proof of theorem 4.0.4. □

The last results show that the function $\log\rho \mapsto \log \mathcal{R}_{\mathfrak{Y},\mathfrak{Z}}(\rho^{-1}(-),(\mathcal{E},\nabla))$ on $[\log r_E, 0]$ as in (6.0.6.1) is a finite concave polygon ≤ 0. In particular, [26, Appendix I, Corollary],

Corollary 6.0.7. *Let E be an edge of finite length of $S_{\mathfrak{Z}}(\mathfrak{Y})$, parametrized as in corollary 6.0.6, and let $x, y \in \overline{E}$ be such that $\rho(x) \leq \rho(y)$. Then, if $\mathcal{R}_{\mathfrak{Y},\mathfrak{Z}}(x,(\mathcal{E},\nabla)) = \mathcal{R}_{\mathfrak{Y},\mathfrak{Z}}(y,(\mathcal{E},\nabla)) = 1$, one has $\mathcal{R}_{\mathfrak{Y},\mathfrak{Z}}(z,(\mathcal{E},\nabla)) = 1$, $\forall\, z \in \mathrm{sp}_{\mathfrak{Y}}^{-1}([x,y])$, where $[x,y] = \rho^{-1}([\rho(x),\rho(y)])$.*

Our last result makes use of the p-adic Turrittin theory of [3]. A crucial point needed in that paper in order to analytically reduce the system to its formal Turrittin canonical form, was the non-appearance of exponent differences of p-adic Liouville type. We recall that, for any differential system Σ as in (0.1.0.1), with G a matrix with entries in $k((T))$, the *exponents at 0* of Σ are formal invariants of the (fuchsian part of the) corresponding differential module $(\mathcal{E},\nabla)$, and the *exponent-differences at 0* are the exponents at 0 of the system corresponding to the differential module of endomorphisms $(\mathcal{E}nd_{k((T))}(\mathcal{E}),\nabla)$. So, the difference of two exponents at 0 is an exponent-difference at 0, but the converse is false in general.

Remark 6.0.8. We recall that, by the p-adic Roth theorem, algebraic numbers are p-adically non-Liouville.

Proposition 6.0.9. *In the previous notation, assume* $(\mathcal{E},\nabla)$ *is an object of* $\mathbf{MIC}_{\mathfrak{Y}}(\overline{X}(*\mathfrak{Z})/k)$ *and let* E *be an open edge of infinite length of* $S_{\mathfrak{Z}}(\mathfrak{Y})$*, with closure* $\overline{E}$ *containing the* k*-rational point* z_i*. Let* $\rho_{z_i}(\mathcal{E},\nabla)$ *be the Poincaré-Katz rank [3] of* $(\mathcal{E},\nabla)$ *at* z_i*, and assume that the exponent-differences of* $(\mathcal{E},\nabla)$ *at* z_i *are* p*-adically non-Liouville numbers. Then, the restriction of* $x \mapsto \mathcal{R}_{\mathfrak{Y},\mathfrak{Z}}(x,(\mathcal{E},\nabla))$ *to* $\overline{E}$ *is the infimum of the constant 1 and a finite set of functions of the form*

$$|p|^{1/(p-1)p^h}|b|^{1/jp^h}\rho(x)^{s/j}\ , \qquad (6.0.9.1)$$

where $j \in \{1,2,\dots,\mu\}$*,* $s \in \mathbb{Z}$*,* $h \in \mathbb{Z}\cup\{\infty\}$*,* $b \in k^\times$ *and* $s/j \le \rho_{z_i}(\mathcal{E},\nabla)$*. We have*

$$\lim_{x\to z_i} \mathcal{R}_{\overline{\mathfrak{X}},\mathfrak{Z}}(x,(\mathcal{F},\nabla))/|T_{z_i}(x)|^{\rho_{z_i}(\mathcal{F},\nabla)} = C\ , \qquad (6.0.9.2)$$

for a non-zero constant $C \in |(k^{\mathrm{alg}})^\times|$*.*

Proof. This is a combination of the previous discussion with [3, prop. 4]. □

We have a variant of corollary 6.0.6:

Corollary 6.0.10. *Let* v *be the second end of* $\overline{E}$ *as in the proposition. Then, if* $(\mathcal{E},\nabla)$ *is regular at* z_i *and* $\mathcal{R}_{\mathfrak{Y},\mathfrak{Z}}(v,(\mathcal{E},\nabla)) = 1$*, we have* $\mathcal{R}_{\mathfrak{Y},\mathfrak{Z}}(z,(\mathcal{E},\nabla)) = 1$*,* $\forall\, z \in D_{\mathfrak{Y}}(z_i,1^-)\setminus\{z_i\}$*.*

We conclude

Proposition 6.0.11. *Let* $(\mathcal{E},\nabla)$ *be an object of* $\mathbf{MIC}_{\mathfrak{Y}}(\overline{X}(*\mathfrak{Z})/k)$*, let* $\mathfrak{Y}_1$*,* $z_{r+1},\dots,z_{r+s}$ *be as in equation 3.2.2.1 and* $\mathfrak{Z}_1 = \{z_1,\dots,z_{r+s}\}$*,* $X_1 = \overline{X}\setminus\mathfrak{Z}_1$*. Then* $z \mapsto \mathcal{R}_{\mathfrak{Y},\mathfrak{Z}}(z,(\mathcal{E},\nabla))$ *is identically 1 on* X *if and only if* $z \mapsto \mathcal{R}_{\mathfrak{Y}_1,\mathfrak{Z}_1}(z,(\mathcal{E},\nabla))$ *is identically 1 on* X_1*.*

Proof. By (3.2.2.1), $\mathcal{R}_{\mathfrak{Y}_1,\mathfrak{Z}_1}(z,(\mathcal{E},\nabla)) \ge \mathcal{R}_{\mathfrak{Y},\mathfrak{Z}}(z,(\mathcal{E},\nabla))$, $\forall\, z \in X_1$. So, let us assume that $\mathcal{R}_{\mathfrak{Y}_1,\mathfrak{Z}_1}(z,(\mathcal{E},\nabla)) = 1$, $\forall\, z \in X_1$. Suppose first that $\mathfrak{Y} = \mathfrak{Y}_1$, and $s = 1$. So, the disk $D_{\mathfrak{Y}}(z_{r+1},1^-)$ contains no singularity of $(\mathcal{E},\nabla)$, and, by the transfer theorem 5.1.2, for any $z \in D_{\mathfrak{Y}}(z_{r+1},1^-)$, $1 \ge \mathcal{R}_{\mathfrak{Y},\mathfrak{Z}}(z,(\mathcal{E},\nabla)) \ge \mathcal{R}_{\mathfrak{Y},\mathfrak{Z}}(v,(\mathcal{E},\nabla)) = \mathcal{R}_{\mathfrak{Y},\mathfrak{Z}_1}(v,(\mathcal{E},\nabla)) = 1$, where v is the boundary point of $D_{\mathfrak{Y}}(z_{r+1},1^-)$ in $\mathbf{S}(\mathfrak{Y})$. Let us assume instead that $s = 0$ and $\mathbf{S}(\mathfrak{Y}_1)$ is obtained from $\mathbf{S}(\mathfrak{Y})$ by addition of one open edge E and one vertex v. Let w be the second end of $\overline{E}$. Then, the edge E corresponds to a k-rational open annulus $B(r_E,1)$ in an open disk $D(0,1^-) \cong D_{\mathfrak{Y}}(a,1^-)$, via a $\mathfrak{Y}$-normalized coordinate at a, where the vertex v (resp. w) corresponds to t_{0,r_E^+} (resp. to $\tau_{\mathfrak{Y}}(a) = t_{0,1}$). Now, again by (5.1.2), $\mathcal{R}_{\mathfrak{Y},\mathfrak{Z}}(w,(\mathcal{E},\nabla)) \ge \mathcal{R}_{\mathfrak{Y},\mathfrak{Z}}(v,(\mathcal{E},\nabla)) = \mathcal{R}_{\mathfrak{Y}_1,\mathfrak{Z}}(v,(\mathcal{E},\nabla)) = 1$, We conclude from proposition 6.0.10 that $\mathcal{R}_{\mathfrak{Y},\mathfrak{Z}}(z,(\mathcal{E},\nabla))$, $\forall\, z \in X$. Suppose now $s = 0$ and $\mathbf{S}(\mathfrak{Y}_1)$ is obtained from $\mathbf{S}(\mathfrak{Y})$ by subdividing one open edge E of $\mathbf{S}(\mathfrak{Y})$ as $E = E_1 \cup \{v\} \cup E_2$, where v (resp. E_1, E_2) is the only new vertex (resp. are the only new edges) of $\mathbf{S}(\mathfrak{Y}_1)$. Then, $D_{\mathfrak{Y}_1,\mathfrak{Z}}(z,1^-) = D_{\mathfrak{Y}_1,\mathfrak{Z}}(z,1^-)$, and therefpre, for any object $(\mathcal{F},\nabla)$ of $\mathbf{MIC}_{\mathfrak{Y}}(\overline{X}(*\mathfrak{Z})/k)$, $\mathcal{R}_{\mathfrak{Y}_1,\mathfrak{Z}}(z,(\mathcal{E},\nabla)) = \mathcal{R}_{\mathfrak{Y},\mathfrak{Z}}(z,(\mathcal{E},\nabla))$, for any $z \in X$; so the result follows also in this case. The general case is an iteration of the steps we have described. □

Corollary 6.0.12. *Theorem 0.2.3 holds.*

Proof. Let $(\mathcal{F},\nabla)$ be an object of $\mathbf{MIC}_{\overline{\mathfrak{X}}}(\overline{X}(*\mathfrak{Z})/k)$ satisfying the assumptions of Theorem 0.2.3. From the corollaries 6.0.7 and 6.0.10 we deduce that the function $x \mapsto \mathcal{R}_{\overline{\mathfrak{X}},\mathfrak{Z}}(z,(\mathcal{F},\nabla))$ is identically 1 on $X = \overline{X}\setminus\mathfrak{Z}$. Let D be an open disk in X. We may assume that D is isomorphic to $D(0,1^-)$, and must show that the restriction of $\mathcal{F}^\nabla$ to D is locally constant. If D is contained in one of the disks $D_{\overline{\mathfrak{X}}}(z_i,1^-)$, the restriction of $\mathcal{F}^\nabla$ to D is in fact constant. Similarly if $D \subset D_{\overline{\mathfrak{X}}}(x,1^-)$, for some $x \in X(k)$, $x \notin \cup_{i=1}^r D_{\overline{\mathfrak{X}}}(z_i,1^-)$. Otherwise, we may replace D by any smaller closed k-rational disk, and assume that the maximal point v of D is an interior point of an open edge E of $S_{\mathcal{Z}}(\mathfrak{Y})$. The closed subset $D \cap |S_{\mathcal{Z}}(\mathfrak{Y})|$ of

D, has a canonical graph structure, whose open edges (resp. vertices) are all the open edges (resp. vertices) of $S_{\mathcal{Z}}(\mathfrak{Y})$ contained in D and the extra open edge $E \cap D$ (resp. v). We call Γ this graph structure. So, Γ corresponds to a formal model $\mathfrak{D}$ of D, and clearly $z \mapsto \mathcal{R}_{\mathfrak{D}}(z, (\mathcal{F}, \nabla)_{|D})$ is identically 1 on D. But $\mathcal{F}_{|D}$ is free, so that $(\mathcal{F}, \nabla)_{|D}$ corresponds to a system Σ as in 0.1.0.1, over the closed disk $D(0, 1^+)$, with $R(t_{0,1}, \Sigma) = 1$. The transfer theorem shows that $\mathcal{R}(x, \Sigma) = 1$, $\forall x \in D(0, 1^+)$. □

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