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Aspects of Quantum Transport (I & II)

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Aspects of Quantum Transport

An updated review of a few famous and other less famous issues concerning low-dimensional quantum dynamics and the absence of "chaos" therein.

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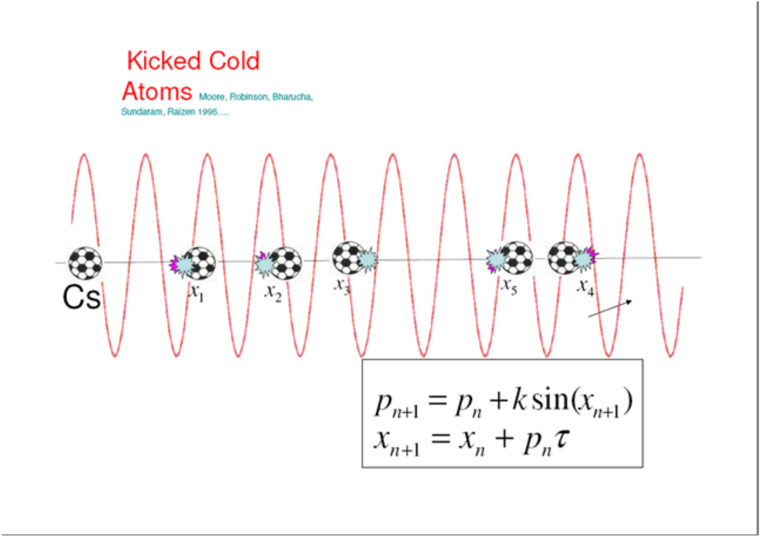
Kicked Cold Atoms



Kicked Hamiltonians

$$e^{-\frac{i}{\hbar}t(\hat{T}+\hat{V})} = \lim_{N\to\infty,\tau\to 0,N\tau=t} \left(e^{-\frac{i}{\hbar}\tau\hat{T}} e^{-\frac{i}{\hbar}\tau\hat{V}}\right)^N$$

For fixed N, τ , the rhs is the propagator from time $t = 0_-$ to time $t = N\tau_-$ of the **Kicked Hamiltonian** :

$$\hat{H}(t) = \hat{T} + \tau\hat{V} \sum_{n\in\mathbb{Z}} \delta(t - n\tau)$$

The kicked dynamics may be drastically different from the dynamics which is generated by $\hat{T} + \hat{V}$. In the 1-freedom case, the latter is classically integrable , but the former has , generically, a mixed phase space.
Path integrals for kicked dynamics are ordinary N-fold integrals.

Instances of Kicked Dynamics

($\hat{X}, \hat{P}$: canonical position & momentum operators for a point particle moving in a line)

Pendulum $\longrightarrow$ Kicked Rotor

$$\hat{T} = \frac{1}{2}\hat{P}^2 \ , \ \hat{V} = \mu \cos(\hat{X})$$

Harper $\longrightarrow$ "Kicked Harper"

$$\hat{T} = \lambda \cos(\hat{P}) \ , \ \hat{V} = \mu \cos(\hat{X})$$

Wannier-Stark $\longrightarrow$ Kicked Accelerator

$$\hat{T} = \frac{1}{2}\hat{P}^2 + \eta \hat{X} \ , \ \hat{V} = \mu \cos(\hat{X})$$

From Atoms to Rotors

In experiments, atoms move in (approximately) straight lines. However, the kicking potential is periodic in space. Quasi-momentum is then conserved. If the spatial period is 2π, then q.mom. = fractional part of momentum $:= \beta$ and the Brillouin zone is $\mathbb{B}^{(\mathrm{P})} = [0, 1[$. $\hbar = 1$.

Bloch theory

$$L^2(\mathbb{R}) \simeq L^2(\mathbb{B}^{(\mathrm{P})}) \otimes L^2(\mathbb{T}) \quad , \quad \hat{U} = \int_{\mathbb{B}^{(\mathrm{P})}}^{\oplus} d\beta \; \hat{U}_\beta$$

Each $\hat{U}_\beta$ formally defines a rotor's dynamics. It is obtained by the replacement $X \to \theta := X\mathrm{mod}(2\pi)$, $\hat{P} \to -i\partial_\vartheta + \beta$

Example

Kicked Atom : $\hat{U} = e^{-i\mu\cos(\hat{X})} e^{-i\tau\hat{P}^2/2}$

Kicked Rotor $\hat{U}_\beta = e^{-i\mu\cos(\theta)} e^{-i\tau(-i\partial_\theta+\beta)^2/2}$

Spectral Bounds 1

$\psi(x,t)$: a wave packet propagating over the discrete lattice $\mathbb{Z}^n$ under any unitary dynamics, in discrete or continuous time. For $1 > \epsilon > 0$ and any time T let $R_\epsilon(T) > 0$ the minimal radius of a ball in $\mathbb{Z}^n$ centered at 0, such that the probability outside it (averaged in time from time 0 to time T) is less than ϵ.

Theorem

(IG 89, JM Combes 93, Y Last 96) There is $C > 0$ independent of t so that

$$R_\epsilon(T) \;>\; C\; T^{d_{\mathrm{H}}^{+}/n}\;.$$

where d_{H}^{+} is the (upper) Hausdorff dimension of the spectral measure of ψ.

Reminder: The spectral measure of a state ψ attaches to any Borel set $B \subseteq \mathbb{R}$ the probability that a measurement of energy in state ψ yields a result $E \in B$.

Corollary:

Growth of q-th moment :

$$M_q(T) = \frac{1}{T}\int_0^T dt \sum_{x\in\mathbb{Z}^n} |\psi(x,t)|^2 \|x\|^q$$
$$> C' T^{q d_{\mathrm{H}}^{+}/n} \tag{1}$$

$\Rightarrow$ In the 1-dim case sub-ballistic propagation possible only with a singular spectrum.
"Quantum Suppression of chaotic diffusion"

Dynamical Localization over the lattice $\mathbb{Z}^n$: $M_2(T)$ bounded in time.

dynamical localization $\Rightarrow$ pure point spectrum
$\Leftarrow$ false in general.

Semi-uniform exponential localization: Del Rio Jitomirskaya Last Simon

Spectral Bounds II

Information about decay (in space) of (generalized) eigenfunctions affords improved lower bounds. Pioneered by heuristic results by R Ketzmerick, K Kruse, S Kraut, T Geisel 97 .

Theorem

A. Kiselev, Y. Last 2000 Let the generalized eigenfunctions $u_E(x)$ satisfy

$$\sum_{\|x\|<R} |u_E|^2(x) \;\leq\; \nu(R)$$

for some strictly nondecreasing function ν and for all E in a set of positive spectral measure. Then

$$R_\epsilon(t) \;>\; C\,\nu^{-1}\left(t^{d_{\mathrm{H}}^{+}}\right)\,.$$

Warning: the above Thm. is somewhat loosely stated.

Smilansky's "irreversible" model.

A rotor (angular coordinate θ), coupled to a linear harmonic oscillator (coordinate q) by point interaction:

$$\hat{H} = -\frac{1}{2}\frac{\partial^2}{\partial\theta^2} - \frac{1}{2}\frac{\partial^2}{\partial q^2} + \frac{1}{2}q^2 + \alpha\, q\, \delta(\theta - \theta_0)\,. \tag{2}$$

$\alpha > 0$ a parameter, θ_0 a fixed point.

Theorem

(M Solomyak 04; SN Naboko, M Solomyak 06) If $\alpha < 1$, H has pure point spectrum in $[\frac{1}{2}\sqrt{1-\alpha^2}, +\infty)$. If $\alpha > 1$, the spectrum has a pure absolutely continuous component that coincides with $\mathbb{R}$.

WD Evans, M Solomyak 05 generalize the result to the case of $n > 1$ oscillators (interacting at different points).

Exponential instability.

Absolute Continuity of the spectrum + Solomyak's estimates on eigenfunctions + Kiselev & Last $\Rightarrow$

Proposition

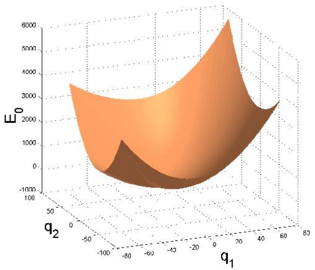
For $\alpha > 1$ the oscillator's energy grows exponentially fast in time.

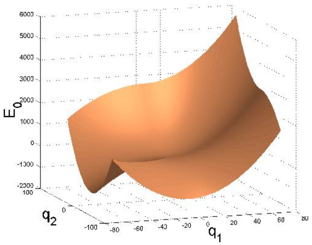
$$\text{total energy} \;=\; \frac{1}{2}\int_0^{2\pi} d\theta \int_{-\infty}^{\infty} dq \left|\frac{\partial\psi}{\partial\theta}\right|^2$$

$$+\frac{1}{2}\int_0^{2\pi} d\theta \int_{-\infty}^{\infty} dq \left\{ |\frac{\partial\psi}{\partial q}|^2 \;+\; q^2|\psi(\theta,q)|^2 \right\}$$

$$+\,\alpha \int_{-\infty}^{+\infty} dq\; q|\psi(0,q)|^2\;. \tag{3}$$

Below: Case of 2 oscillators. Lowest energy band for $\alpha = 0.7$ (left) and for $\alpha = 1.3$ (right).





Dynamical Localization: what is known

accumulated numerical evidence + assimilation to tight-binding models with disorder : whenever τ is sufficiently incommensurate to 2π, the kinetic energy of the KR stays bounded in time.

What is proven:

(Bourgain 02) For all sufficiently small μ:
1) U_β has p.p. spectrum, with exponentially localized eigenfunctions in momentum space. (If ϕ is an eigenfunction in momentum representation, and $|\phi(n_0)| > \epsilon$, then $|\phi(n)| < e^{-10^{-5}|n|}$ whenever $|n| > |n_0|^C + e^{(\log\log\frac{1}{\epsilon})^2}$.)
2) Dynamical Localization follows : if $\psi \in \mathcal{H}_1(\mathbb{T})$ then

$$\sup_{t\in\mathbb{Z}} ||U_\beta^t\psi||_{\mathcal{H}_1} < +\infty$$

for all (τ,β) in $[0,2\pi]\times[0,1]$ with the exception of a "small" set, the measure of which tends to 0 as $\mu\to 0$.

KR Resonances

A KR resonance is said to occur whenever $\hat{U}_\beta$ commutes with a momentum translation $\hat{T}^Q$ (Q a strictly positive integer), where $\hat{T} : \psi(\theta) \to e^{i\theta}\psi(\theta)$. This happens if β is rational and τ is commensurate to 2π.

Proposition

(Izrailev, Shepelyansky 1980; Dana, Dorofeev 06) $\hat{U}_\beta$ *commutes with* $\hat{T}^Q$ *if, and only if, (i)* $\tau = 2\pi P/Q$ *with* P *integer, (ii)* $\beta = \nu/P + Q/2 \mod(1)$, *with* ν *an arbitrary integer.*

If $P/Q = p/q$ with p, q coprime then q is the order of the resonance . Resonances with $Q = q$ are termed primitive.
At resonances, "Quasi-Position" ϑ is conserved: $\vartheta \equiv \theta \mod 2\pi/Q$ and $\vartheta \in \mathbb{B}_q^{(x)} \equiv [0, 2\pi/Q[$.

"θ changes by multiples of $2\pi/Q$"

Theorem

(Izrailev , Shepelyansky 1980) Identify $L^2(\mathbb{T})$ *and* $L^2(\mathbb{B}_q^{(\mathrm{x})}) \otimes \mathbb{C}^Q$ *through* $\psi(\theta) \rightleftarrows \{\psi(\vartheta + 2\pi(n-1)/Q)\}_{n=1,\ldots,q}$. *Then at a resonance with* $\tau = 2\pi P/Q$ *and* $\beta = \beta_r$,

$$\hat{U}_{\beta_r} \; = \; \int_{\mathbb{B}_q^{(\mathrm{x})}}^{\oplus} d\vartheta \; \mathfrak{X}(\beta_r, \mu, \vartheta) \; ,$$

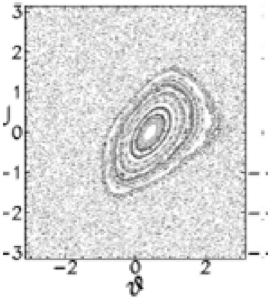
where $\mathfrak{X}(\beta_r, \mu) : [0, 2\pi] \to \mathbb{U}(Q)$ *is defined by :*

$$\mathfrak{X}_{jk}(\beta_r, \mu, \vartheta) = e^{-i\mu\cos(\vartheta + 2\pi(j-1)/Q)} G_{jk} \; , \tag{4}$$

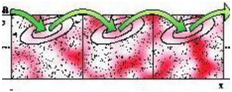
$$G_{jk} = \frac{1}{Q} \sum_{l=0}^{Q-1} e^{-\pi i p (l+\beta_r)^2/q} \; e^{2\pi i (j-k) l/Q} \; . \tag{5}$$

Transporting islands: Accelerator Modes.

Mixed Phase Space



Transporting Islands



Accelerator modes quantally decay due to tunneling but their presence results in much larger localization length.

Hanson, Ott, Antonsen 1984; Iomin, Fishman, Zaslavsky 2002,...

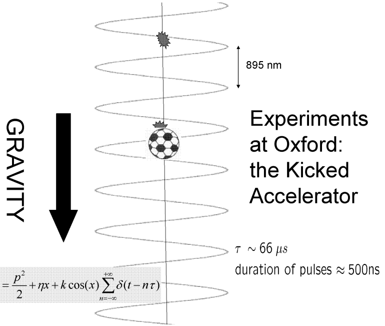
Eigenstates may ignore classical islands

Hufnagel,Ketzmerick, Otto, Schanz 02; Bäcker, Ketzmerick, Monastra 06;...

The chaotic sea recoils

Schanz, Dittrich, Ketzmerick 05;....

Kicked Accelerator.



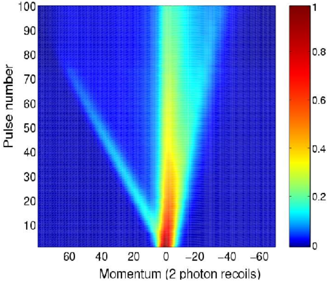
$$H = \frac{p^2}{2} + \eta x + k\cos(x)\sum_{n=-\infty}^{+\infty}\delta(t - n\tau)$$

Experiments at Oxford: the Kicked Accelerator

Quantum Accelerator Modes

QAMs were first discovered in experiments at Oxford.

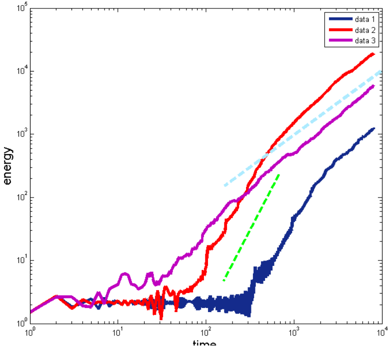
M.K. Oberthaler, R.M. Godun, M.B. d'Arcy, G.S. Summy and K. Burnett, PRL **83**, 4447, (1999)



Destruction of localization

($\eta\tau/2\pi$ irrational)

η increases through ■ ■ ■. Dashed lines: linear and quadratic growth.



Ballistic growth at intermediate times is due to Quantum Accelerator modes

Pseudoclassical Action

Hamiltonian "in the falling frame"

$$\frac{1}{2}(\hat{P} + \frac{\eta}{\tau}t)^2 + k\cos(\hat{X})\sum_{-\infty}^{\infty}\delta(t - n\tau)\ .$$

From the t-th kick to the $(t+1)$-th kick:
Unitary Propagator:

$$\hat{U}_t = e^{-ik\cos(\theta)}e^{-i\frac{\tau}{2}(-i\partial_\theta + \phi_t)^2}\ ,$$

$$\phi_t = \beta + \eta/2 + \eta t\ .$$

Near Resonance: $\tau = 2\pi\frac{p}{q} + \epsilon$

$$\hat{U}_t\psi(\theta) = e^{-ik\cos(\theta)}\sum_{s=0}^{q-1} G_s\, e^{-i\frac{\epsilon}{2}(-i\partial_\theta+\beta)^2}\psi(\theta - 2\pi s/q - \tau\phi_t) =$$

$$\frac{1}{\sqrt{2\pi i\epsilon}}\sum_{m\in\mathbb{Z}}\sum_{s=0}^{q-1} G_s \int_0^{2\pi} d\theta'\; e^{-\frac{i}{\epsilon}S(\theta,\theta',s,m,t)}\psi(\theta')$$

Action ($\tilde{k} := \epsilon k$):

$$S(\theta,\theta',s,m,t) = -\tilde{k}\cos(\theta) + \frac{1}{2}(\theta - \theta' - 2\pi s/q - 2\pi m - \tau\phi_t)^2$$

Propagation over t kicks: sum over paths. Each path is specified by $(\theta_0, \theta_1, \ldots, \theta_t)$, $(m_0, \ldots, m_t)$, $(s_0, \ldots, s_t)$.

Pseudoclassical Asymptotics

$$\epsilon \to 0 \;;\; k \to \infty \;;\; \tilde{k} = k\epsilon = \mathit{const.}$$

Stationary Phase selects paths with $(m_0, \ldots, m_t)$ and $(s_0, \ldots, s_t)$ arbitrary, and rays $(\theta_0, \theta_1, \ldots, \theta_t)$ that obey:

$$\theta_{t+1} = \theta_t + I_t + \tau\phi_t + 2\pi s_t/q \mod 2\pi \;,$$

$$I_{t+1} = I_t + \tilde{k}\, \sin(\theta_{t+1}) \;.$$

$q = 1$: the Pseudoclassical Limit

S Fishman , IG, L Rebuzzini PRL 89 (2002) 0841011; J Stat Phys 110 (2003) 911; A Buchleitner, MB d'Arcy, S Fishman, SA Gardiner, IG, ZY Ma, L Rebuzzini and GS Summy, PRL 96 (2006) 164101; IG, S Fishman, L Rebuzzini Nonlinearity 19 (2006); RHihinashvili, TOliker, YS Avizrats, A Iomin, S Fishman, IG Physica D 226 (2007)

Multiples of $2\pi/q$ drop out. Time dependence is removed by changing variable to:

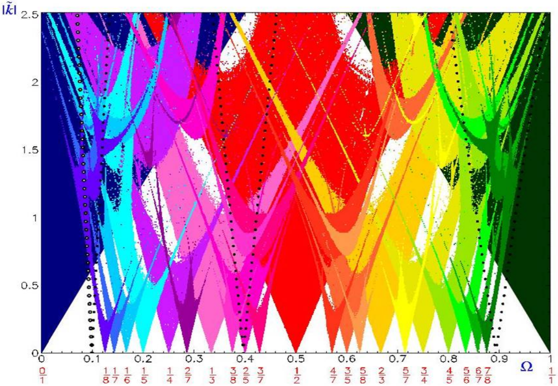
$$J_t = I_t + \frac{\eta}{2} + \delta\beta + \tau\eta t$$

(Difference linearly grows with time)

$$J_{t+1} = J_t + \tau\eta + \tilde{k}\sin(\theta_{t+1}),$$
$$\theta_{t+1} = \theta_t + J_t .$$

Rays are trajectories of a classical dynamical system on $\mathbb{T} \times \mathbb{R}$. Stable Periodic Orbits of the map on $\mathbb{T} \times \mathbb{T}$ $\rightsquigarrow$ Stable Accelerating Rays $\rightsquigarrow$ Quantum Accelerator Modes.

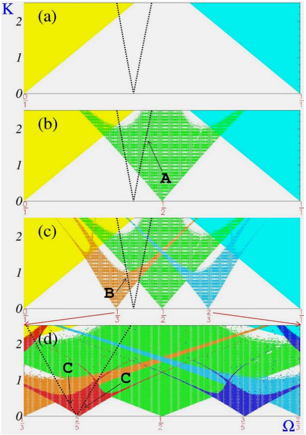
Phase Diagram of QAMs: Arnol'd Tongues



Farey approximation of Gravity

The observed modes along the "experimental path" are the rational (Farey) approximants to

$$\omega^* = \frac{2\pi M^2 g}{\hbar^2 G^3} .$$



$q > 1$: Near Higher-Order Resonances

Partial Similarity to quasiclassics with multi-component wavefunctions (e.g. spinors):

Littlejohn, Flynn 1991

IG, L Rebuzzini PRL 100 (2008) 234103; L.Rebuzzini, IG, R Artuso PR-A 79 (2009)

$$J_{t+1} = J_t + \tau\eta + \tilde{k}\,\sin(\theta_{t+1}) + \delta_t\ ,$$
$$\theta_{t+1} = \theta_t + J_t\ ,$$
$$\delta_t = \frac{2\pi}{q}(s_{t+1} - s_t)\ .$$

Rays are not trajectories of a unique classical system anymore. There is a ray for each choice of an integer string $\mathbf{s} := (s_0, \ldots, s_t)$: so rays exponentially proliferate with the number t of kicks. Each ray contributes an amplitude:

$$\frac{1}{\sqrt{q^t \epsilon |\det(\mathfrak{M}_t)|}}\ e^{\frac{i}{\epsilon}S_{\mathbf{s},\mathbf{m}} + i\Phi_{\mathbf{s},\mathbf{m}}}.$$

Stable Rays?

$\mathfrak{M}_t$ is the stability matrix :

$$\mathfrak{M}_t = \begin{vmatrix} 2+\tilde{k}\cos(\theta_0) & -1 & 0 & \dots & \dots & 0 \\ -1 & 2+\tilde{k}\cos(\theta_1) & -1 & 0 & \dots & 0 \\ 0 & -1 & 2+\tilde{k}\cos(\theta_2) & -1 & \dots & 0 \\ 0 & 0 & \dots & \dots & \dots & 0 \\ 0 & 0 & \dots & -1 & 2+\tilde{k}\cos(\theta_{t-1}) & -1 \\ 0 & 0 & \dots & \dots & -1 & 2+\tilde{k}\cos(\theta_t) \end{vmatrix}$$

Herbert-Jones-Thouless formula:

$$\log(|\det(\mathfrak{M})|) = t\int dn(E)\log(|E|) = t \times \text{Lyapunov exponent}$$

As t increases, most sequences δ_t are random and so are θ_t : $\Rightarrow$ LE positive $\Rightarrow$ Each such ray yields an exponentially small contribution.

Stable Rays

Distinguished individual contributions expected of rays, whose matrices $\mathfrak{M}$ have extended states.

How to find stable rays

IG, L Rebuzzini PRL 100 (2008) 234103

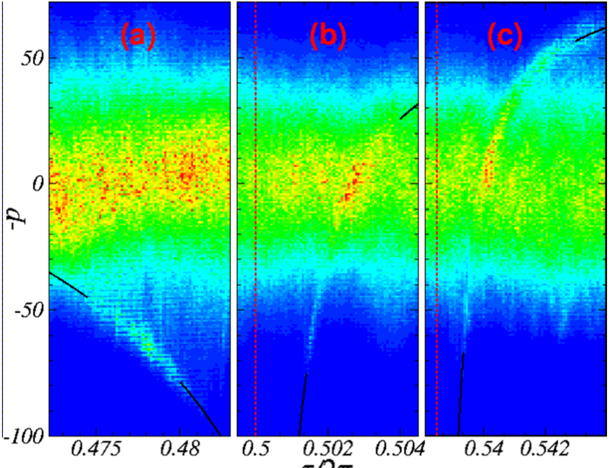
$$J_{t+1} = J_t + \tau\eta + \tilde{k}\sin(\theta_{t+1}) + \delta_t \,,$$
$$\theta_{t+1} = \theta_t + J_t \,,$$
$$\delta_t = \frac{2\pi}{q}(s_{t+1} - s_t) \,.$$

whenever δ_t is a periodic sequence of period T, T-fold iteration of the above equations defines a dynamical system on the 2-torus. Each stable periodic orbit of that system defines a stable ray that gives rise to an accelerator mode.

Acceleration

$$\frac{1}{\epsilon}\left\{\frac{2\pi}{T}\frac{\mathfrak{j}}{\mathfrak{p}} - \tau\eta - \frac{1}{T}\sum_{t=0}^{T-1}\delta_t\right\} \,.$$

(a) $q = 2$, $p = 1$, $T = 2$. Orbit with period 3 index 1.
(b) $q = 2$, $p = 1$, $T = 1$. Orbit with period 5 index 1.
(c) $q = 13$,$p = 10$, $T = 1$. Orbit with period 1 index 1.
$k = 0.8\pi$, $\eta = 0.126\tau$.



More Kicks Ahead

Experiments with Kicked Ultra-Cold Atoms still being done.
Relevant Theoretical Issues:

- Directed Transport (Quantum "Ratchet Effect")
- Terra Incognita: Multiple Kicks

  connection between Kicked-Harper and certain doubly-kicked rotors: J.Wang, J.Gong, T.Monteiro...
- Many-Body Kicked Systems: Hubbard Model ...