The Abdus Salam International Centre for Theoretical Physics

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2058-S-9

Pseudochaos and Stable-Chaos in Statistical Mechanics and Quantum Physics

21 - 25 September 2009

Pseudo-chaos in piecewise isometric systems

F. VIVALDI

Queen Mary, University of London, U.K.

Strada Costiera 11, 34151 Trieste, Italy - Tel.+39 040 2240 111; Fax +39 040 224 163 - sci_info@ictp.it

pseudo-chaos

in piecewise isometric systems

Franco Vivaldi
Queen Mary, University of London

with
J H Lowenstein (New York)

Mathematics:

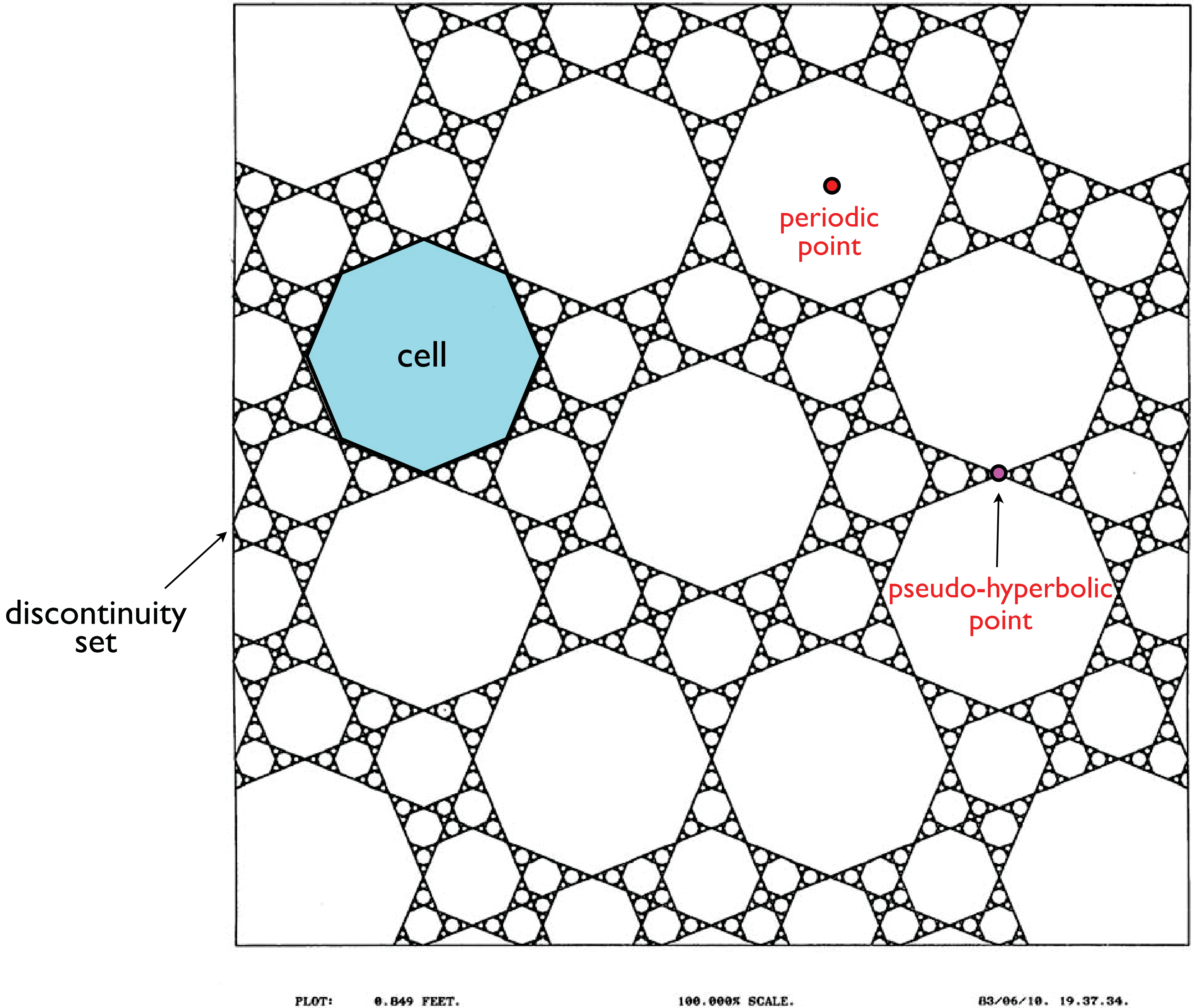
- polygonal outer billiards
- polygon-exchange maps
- non-ergodic rotations on tori
- piecewise isometries

Applications:

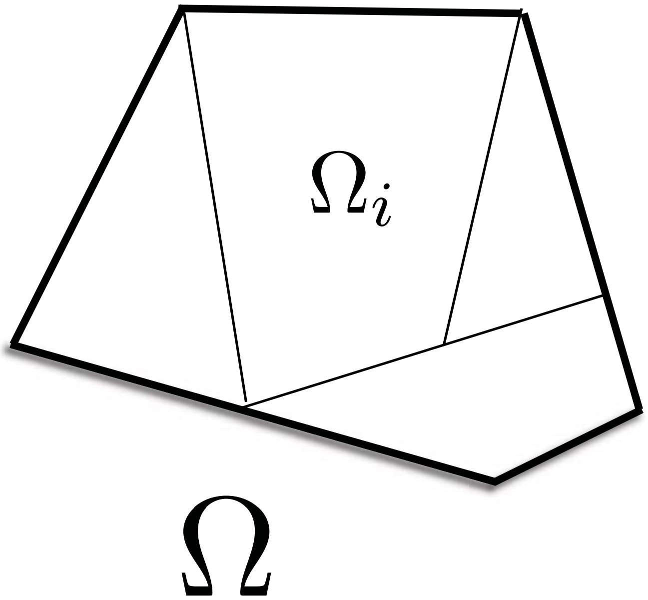
- digital filters
- sigma-delta modulators
- digital printing
- micro electromechanical systems
- voltage-controlled oscillators
- dynamics of round-off errors

pretty pictures

ugly proofs


periodic
point
cell
pseudo-hyperbolic
point
discontinuity
set
PLOT: 0.849 FEET.
100.000% SCALE.
83/06/10. 19.37.34.

Piecewise isometries



the space:

$$\Omega \subset \mathbb{R}^n$$

$$\Omega = \overline{\bigcup \Omega_i}$$

a finite collection of pairwise disjoint open polytopes (intersection of open half-spaces), called the atoms.

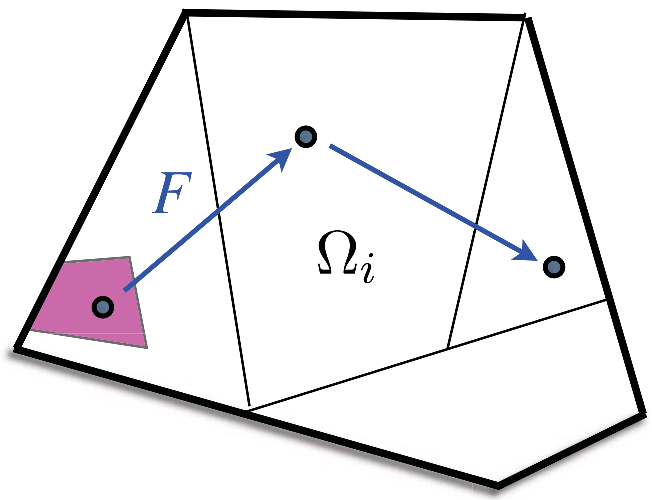
the dynamics: $F : \Omega \to \Omega$ $\quad F|_{\Omega_i}$ is an isometry

Theorem (Gutkin & Haydin 1997, Buzzi 2001)
The topological entropy of a piecewise isometry is zero.

Cells

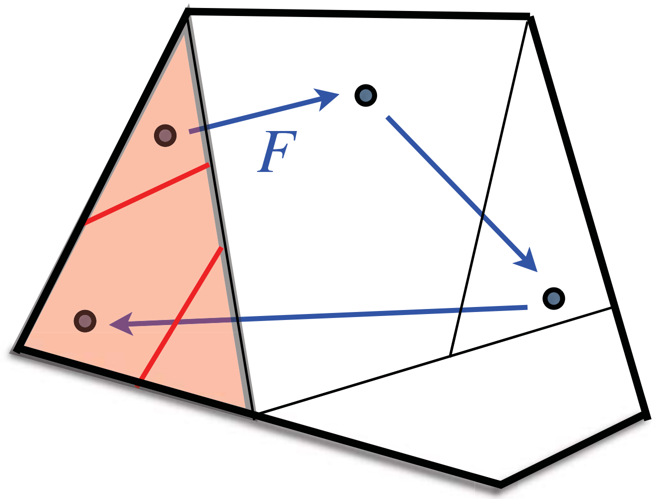
The points of an orbit visit atoms in succession, defining a symbolic dynamics.

A cell is a set of points with the same symbolic dynamics; cells are convex sets.



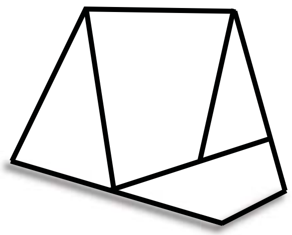
Induced maps

The first return map to an atom defines a new PWI on a smaller domain. This process may be continued recursively.



Topology

Iterating the boundary of the atoms:



$$\partial\Omega = \bigcup \partial\Omega_i$$

$$\mathcal{D} = \bigcup_{t\in\mathbb{Z}} F^t(\partial\Omega)$$ **discontinuity set**

$$\Pi = \Omega \setminus \overline{\mathcal{D}}$$ **periodic set** (union of cells of positive measure)

$$\mathcal{E} = \overline{\mathcal{D}} \setminus \mathcal{D}$$ **exceptional set** (asymptotic phenomena)

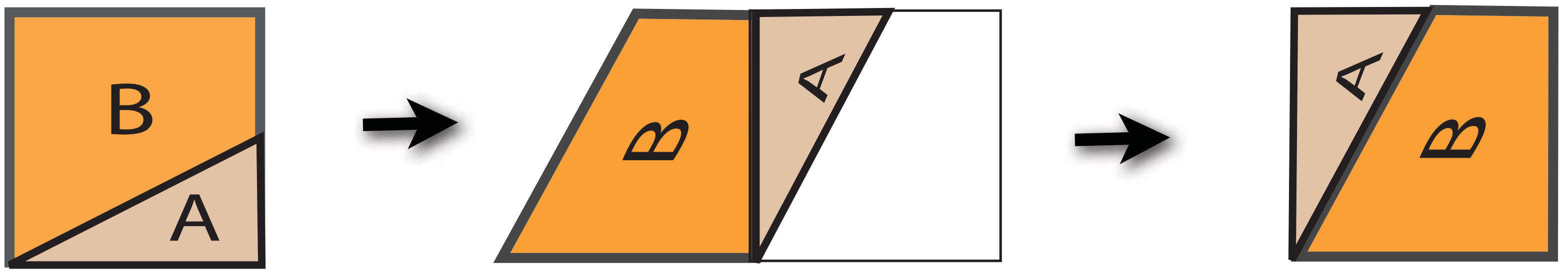
Conjecture (Goetz 1998). *If* $\overline{\mathcal{D}} \neq \Omega$, *then* $\overline{\mathcal{D}}$ *has empty interior.*

Rotations on tori: the standard model

$\Omega = \mathbb{T}^2 \simeq [0,1)^2$ $\qquad$ $F : (x, y) \mapsto (\lambda x - y, x) \pmod 1$

$\lambda = 2\cos(2\pi\nu)$

$\lambda = \frac{1}{2}$



Adler, Kitchens and Tresser, ETDS (2000):

"What surprised us most about these maps, is how quickly we ran out of cases which are amenable of any detailed analysis"

The relevant arithmetical environment is the field

$$\mathbb{Q}(\lambda) \qquad \lambda = 2\cos(2\pi\nu)$$

There are two basic cases:

$\nu \in \mathbb{Q}$

rational rotations

cells are polygons

field is algebraic

$\nu \notin \mathbb{Q}$

irrational rotations

cells are ellipses

field may be algebraic

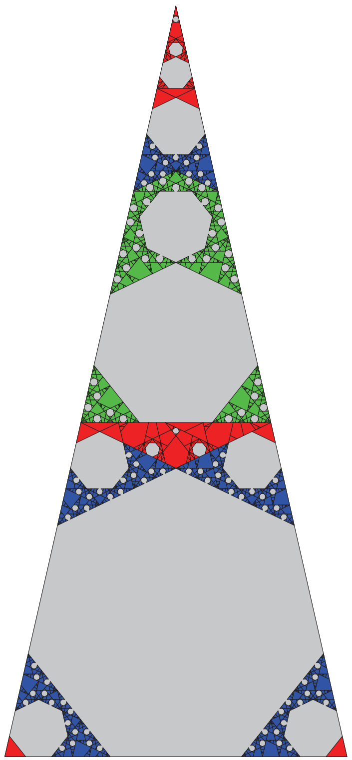
or transcendental

Rational rotations

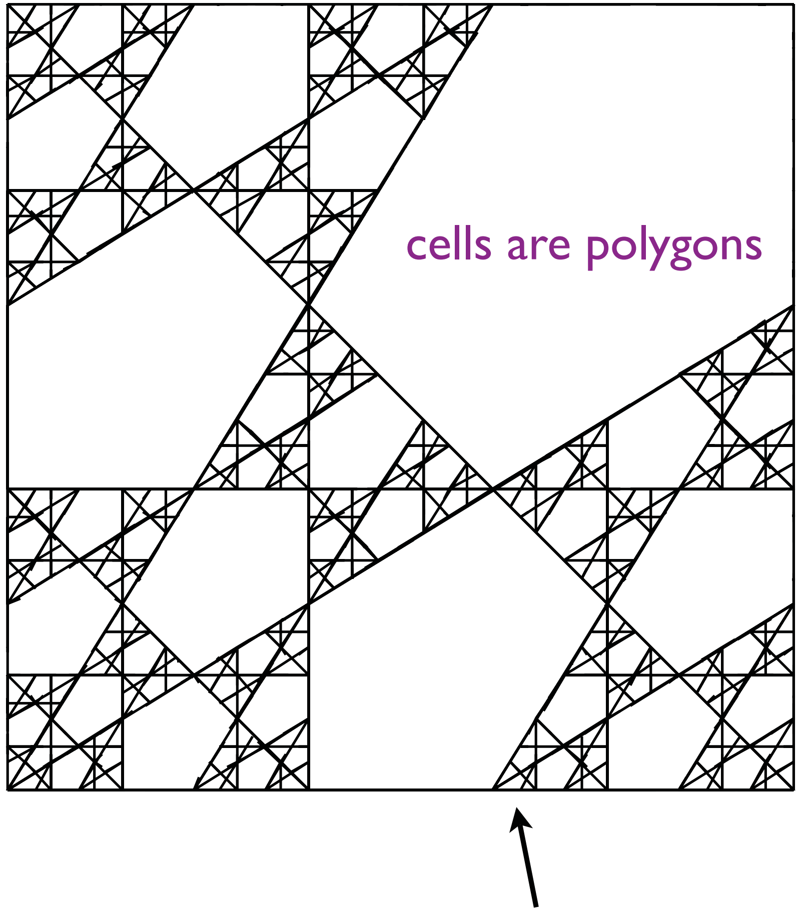
$$\lambda = 2\cos(2\pi\nu) \qquad \nu \in \mathbb{Q}$$

Some rigorous results.

Self-similarity for quadratic fields (8 cases in all).

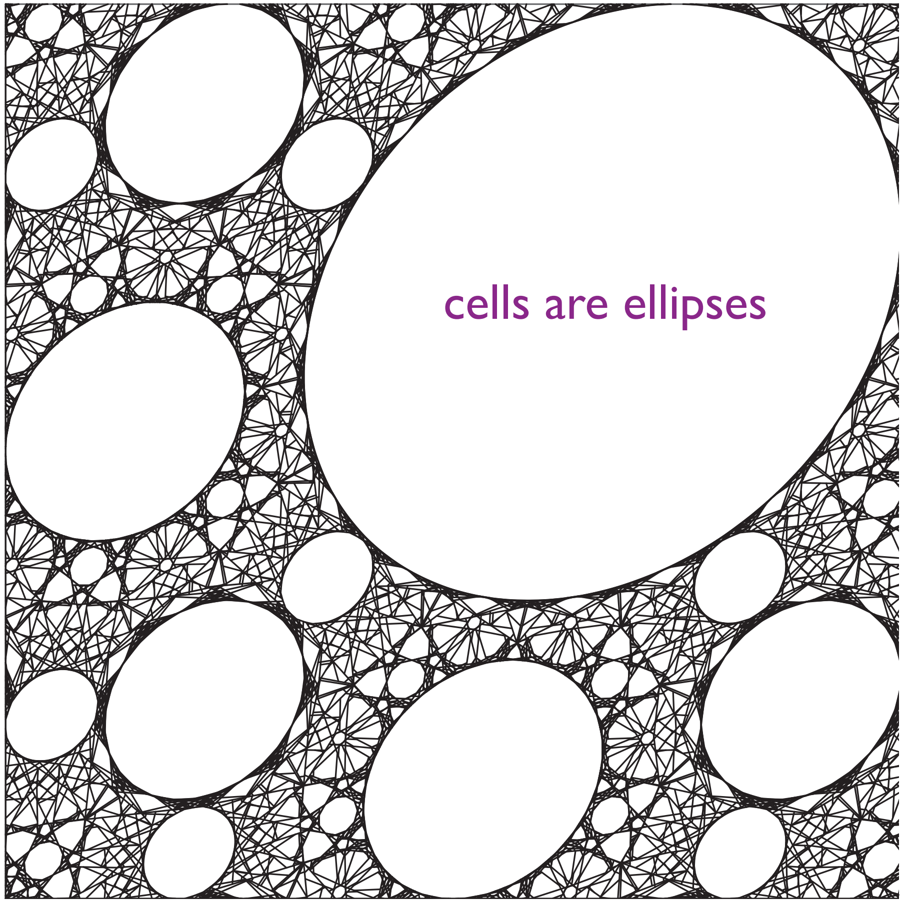


One known cubic case: finitely-generated renormalization structure



Irrational rotations

$\lambda = 2\cos(2\pi\nu) \qquad \nu \notin \mathbb{Q}$

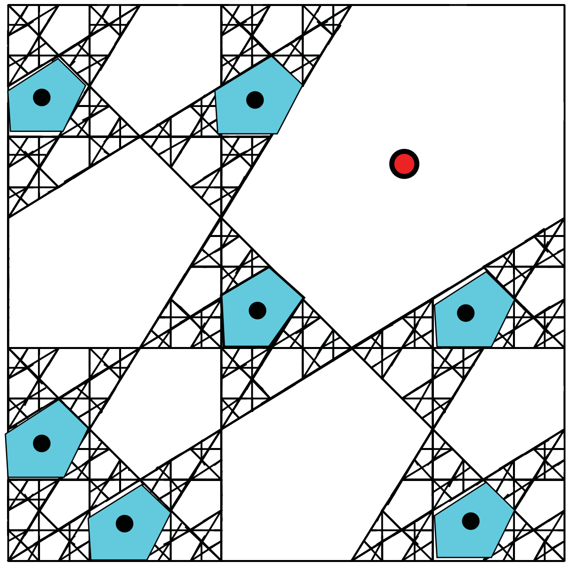


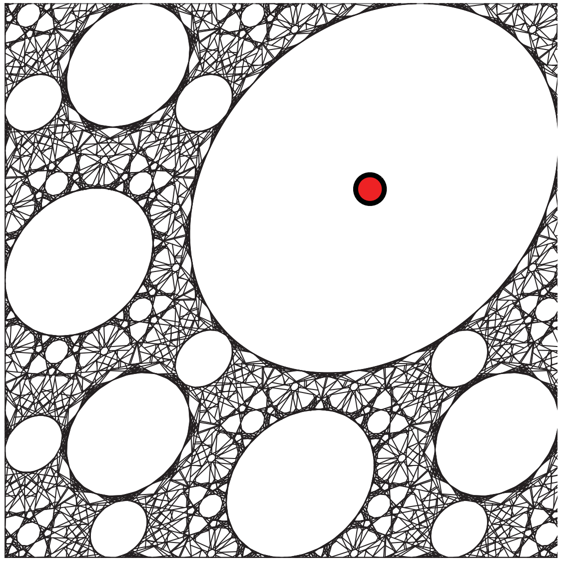
Very few rigorous results.

No known renormalization structure.

Conjecture (Ashwin 1997) *The exceptional set of a 2-D irrational piecewise isometry has positive Lebesgue measure.*

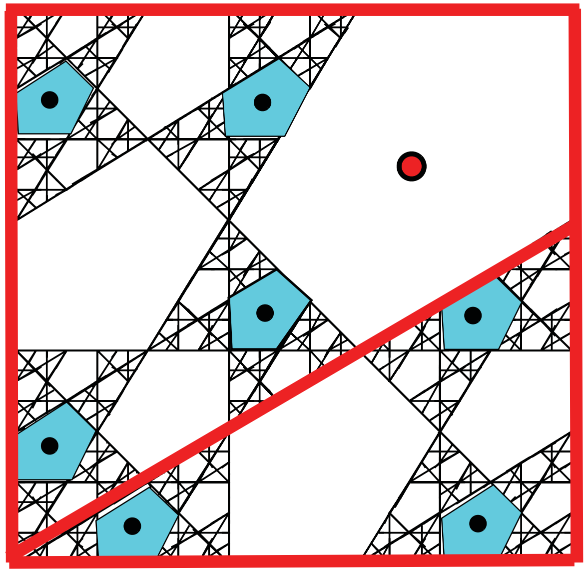
Periodic points

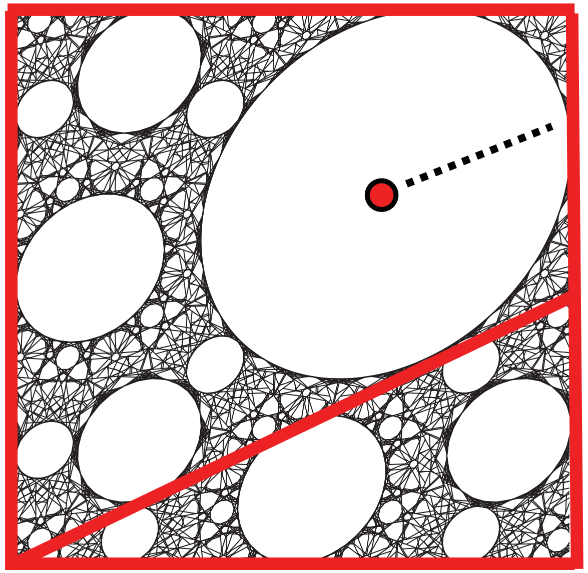




at the centre of each cell, there is an elliptic periodic point

Periodic points



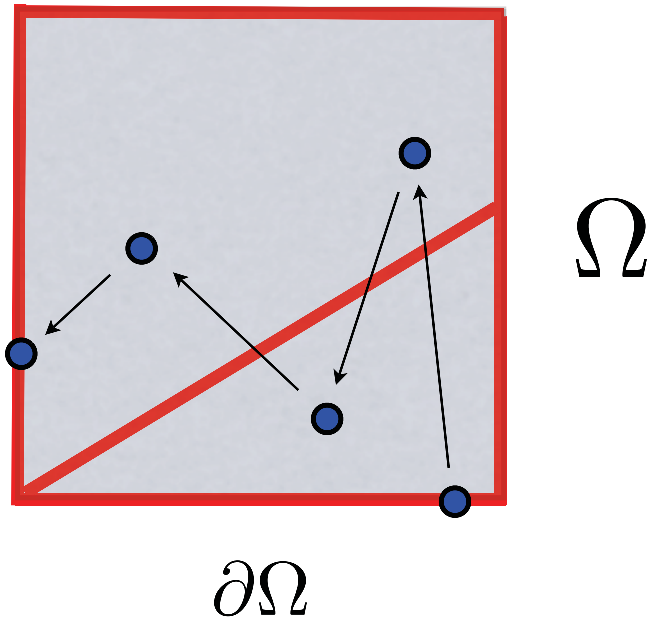


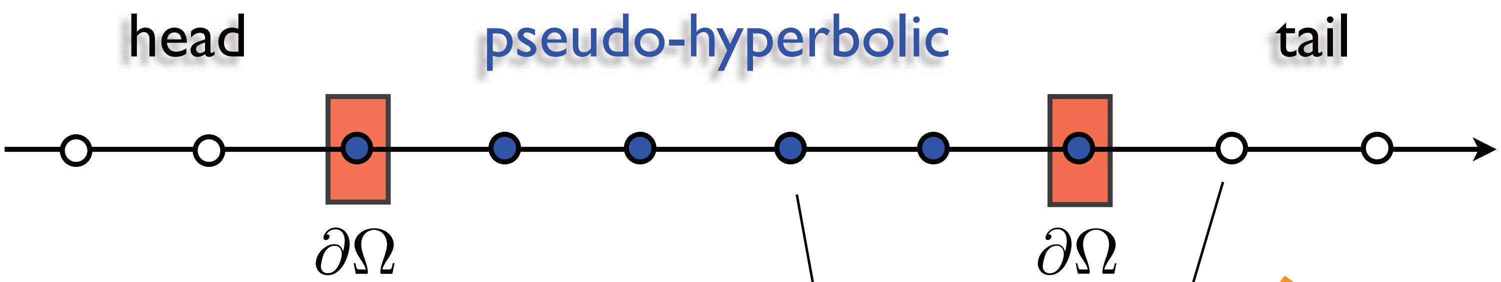
at the centre of each cell, there is an elliptic periodic point

the size of a cell is determined by the minimal distance of the periodic orbit from the boundary of the atoms

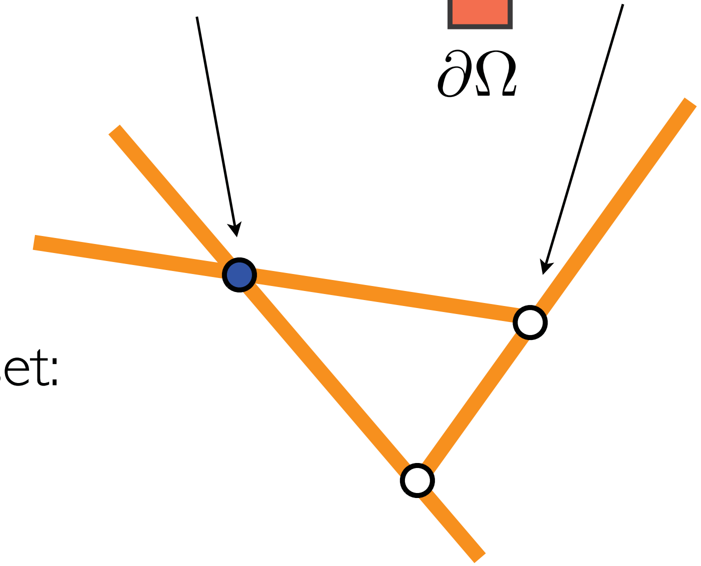
Pseudo-hyperbolic points

A point is pseudo-hyperbolic if its orbit reaches the boundary of the atoms in both time directions.





points on discontinuity set:



Non-archimedean absolute values provide a natural way of measuring the size of elliptic and pseudo-hyperbolic points.

Standard model: $$x_{t+1} \equiv \lambda x_t - x_{t-1} \pmod 1$$

λ **rational:** Under iteration, the primes dividing the denominator of λ will occur with increasing exponents in the denominator of x_t.

Need a concept of size, for which x_t becomes **bigger**.

- Fix a prime p. The *p-adic value* $\nu_p(m)$ of an integer m is defined to be the largest k such that p^k divides m, with $\nu_p(0) = \infty$.

- Letting $\nu_p(m/n) = \nu_p(m) - \nu_p(n)$, we extend the definition of ν_p to the rationals.

Define: $$|r|_p = e^{-\nu_p(r)} \qquad r \in \mathbb{Q}$$

λ **transcendental:** The point x_t depends polynomially on λ. Under iteration, the degree increases.

- For a polynomial $f(\lambda)$ with rational coefficients we let $\nu_\infty(f) = -\deg(f)$, with $\nu_\infty(0) = \infty$.

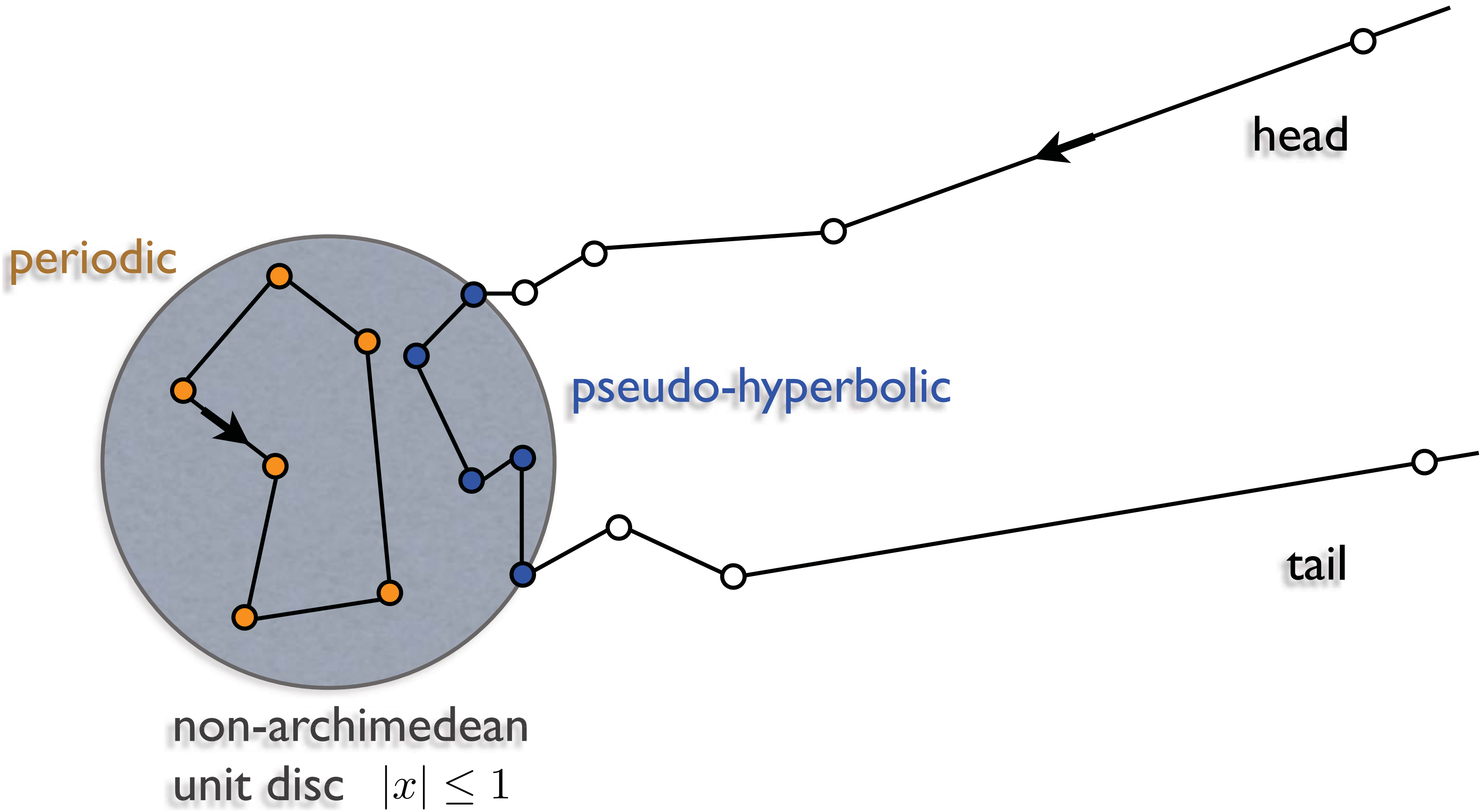
- We extend the definition of ν_∞ to rational functions $r(\lambda) = f(\lambda)/g(\lambda)$ via $\nu_\infty(r) = \nu_\infty(f) - \nu_\infty(g)$.

Define:

$$|r|_\infty = e^{-\nu_\infty(r)} \quad r \in \mathbb{Q}(\lambda)$$

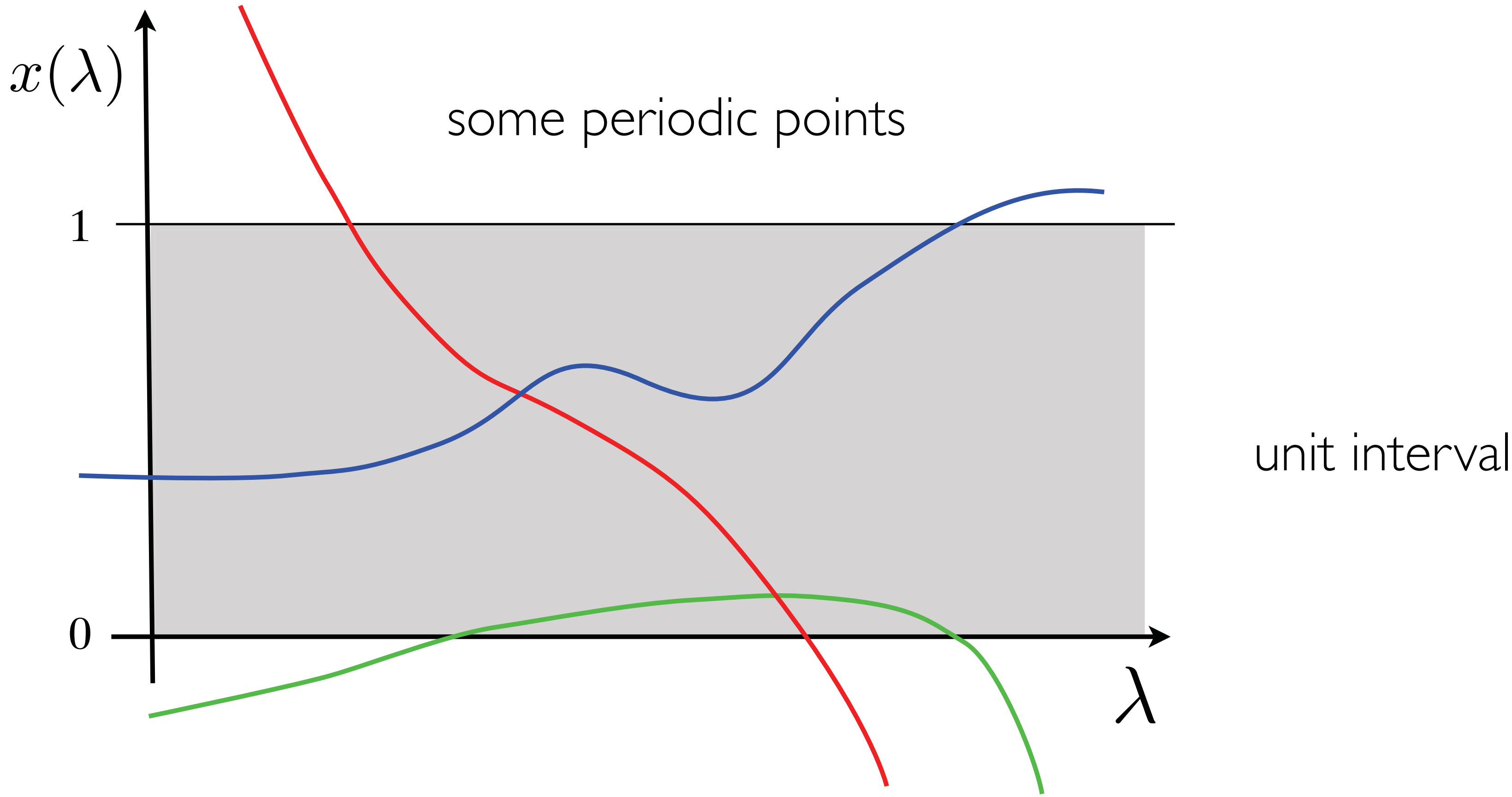
Main lemma:

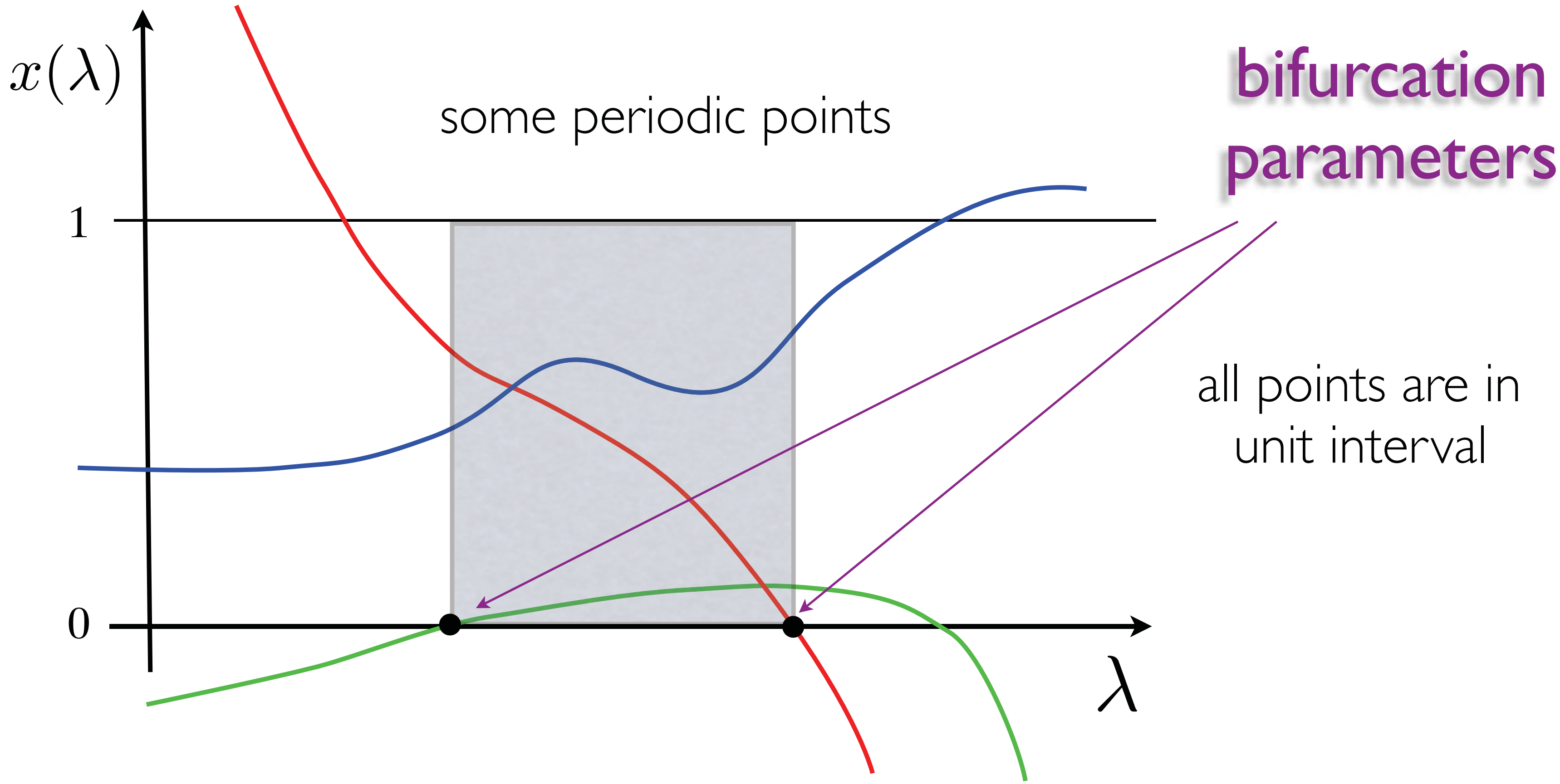
Standard model: the parameter is **rational** or **transcendental**



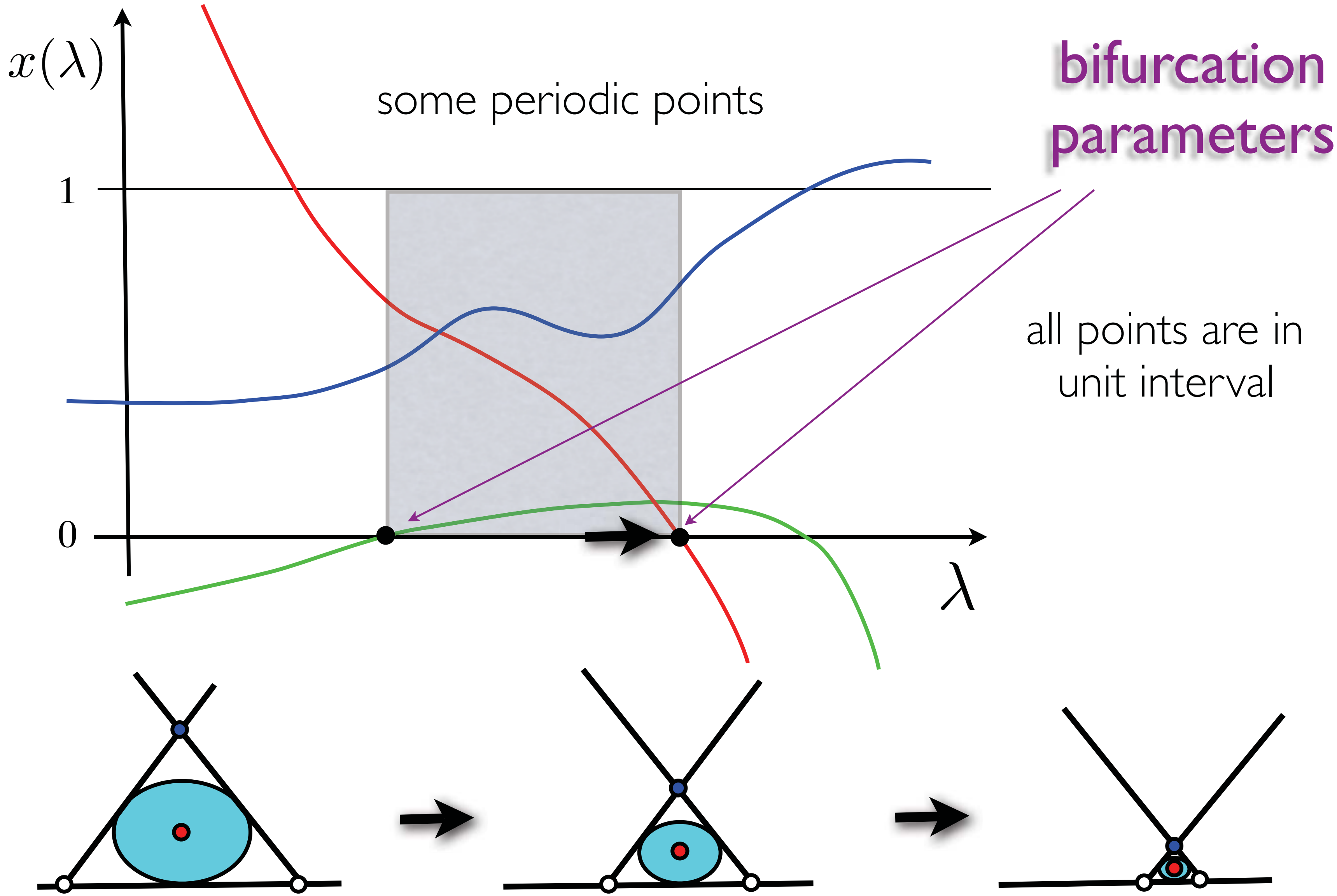
Theorem (Lowenstein & FV)
For rational or transcendental values of λ, *the standard model has no unstable periodic orbits.*

- Except for trivial cases, all parameters correspond to irrational rotations.
- Unstable periodic orbits do exist for algebraic parameter values, for both rational and irrational rotations.


$x(\lambda)$
some periodic points
1
unit interval
0
λ

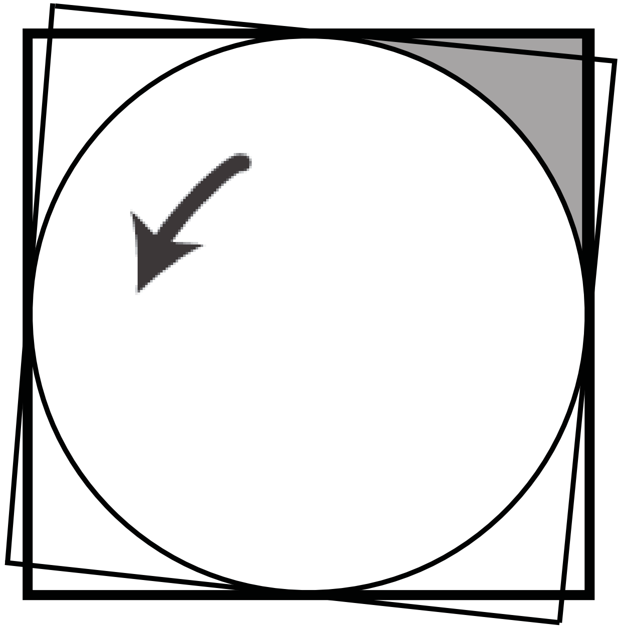


$x(\lambda)$
some periodic points
bifurcation parameters
1
all points are in unit interval
0
λ



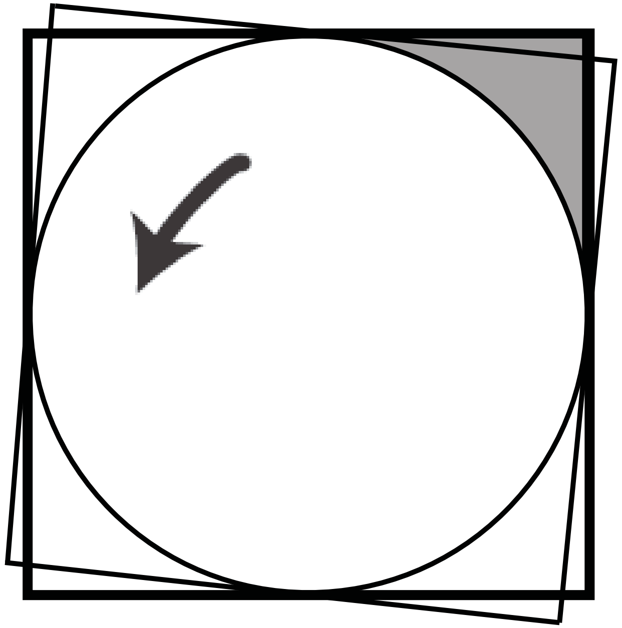
- Elliptic points collide with pseudo-hyperbolic points.
- A single polynomial characterizes each bifurcation.

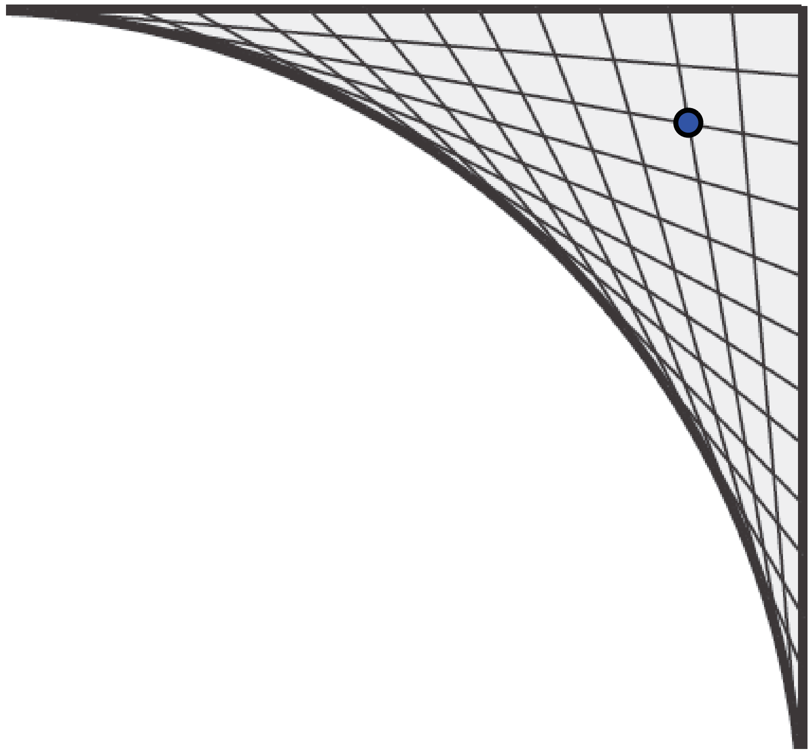
Near-rational behaviour



as λ approaches zero, the map approaches a rotation by $\pi/2$

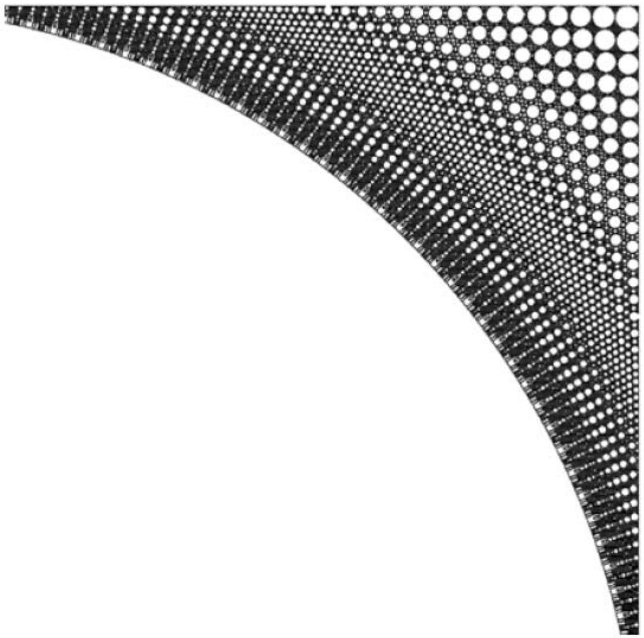
Near-rational behaviour

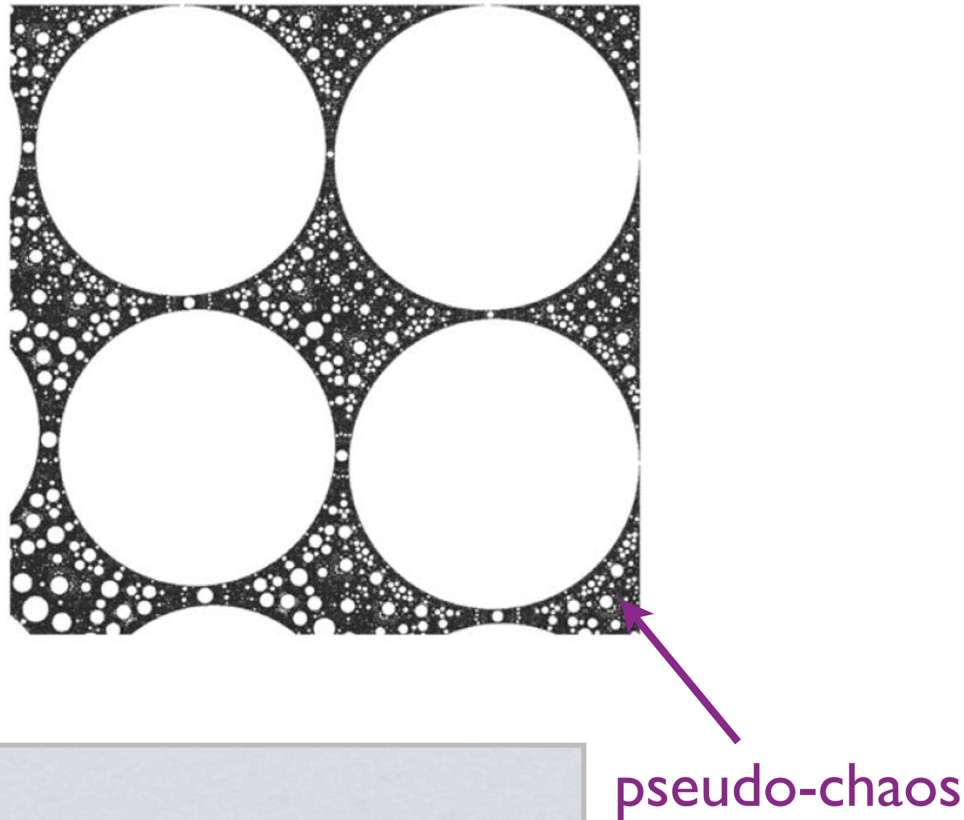




as λ approaches zero, the map approaches a rotation by $\pi/2$

pseudo-hyperbolic points become dense



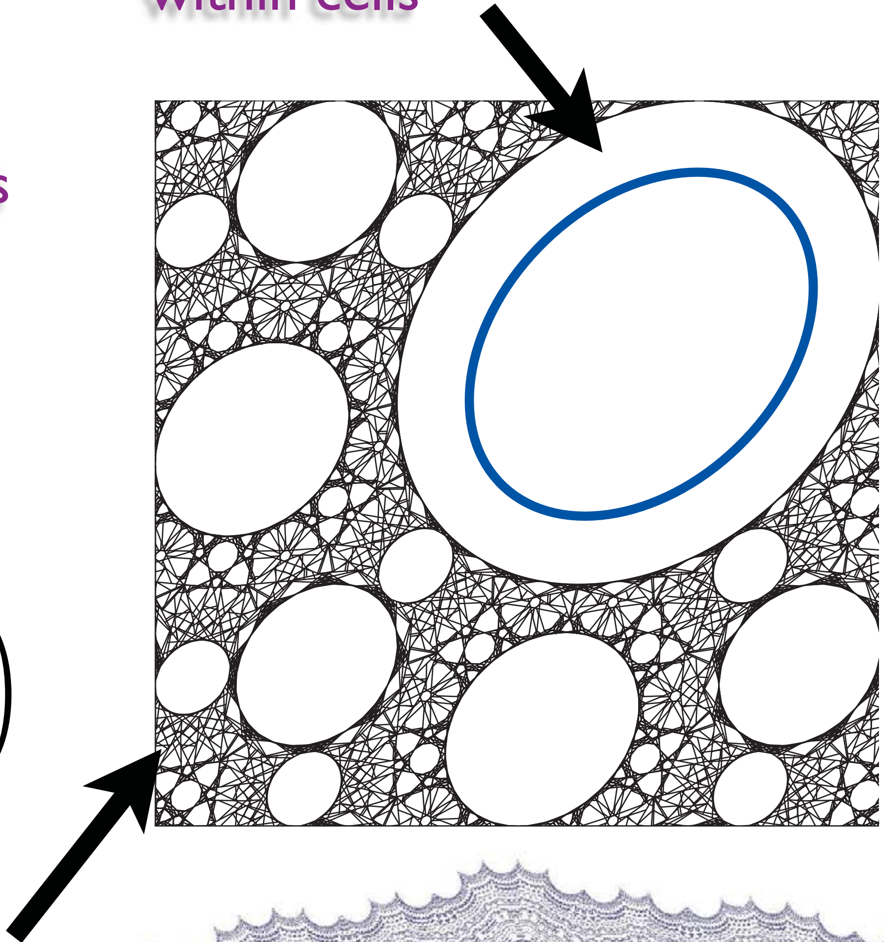


Theorem **(Lowenstein & FV)**
As λ tends to zero, the measure of the regular islands approaches a positive limit.

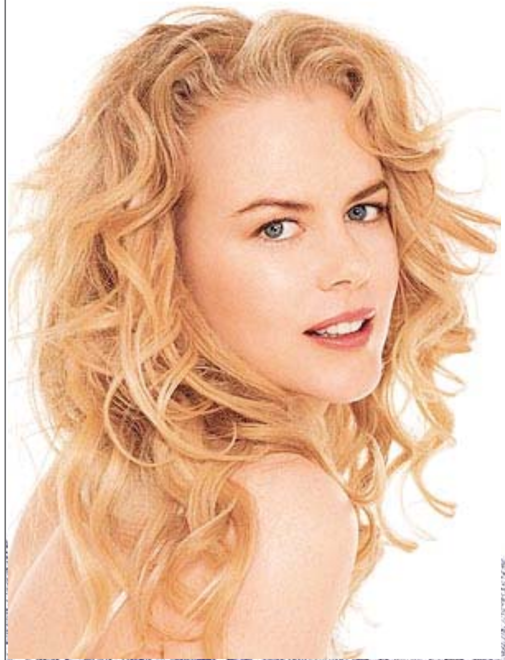
- The measure is computed explicitly
- Long, laborious proof, requiring computer assistance

Invariant curves

Smooth invariant curves exist within cells



Do non-smooth invariant curves exist in irrational PWIs?



$$\begin{pmatrix} 1/2 & -1 \\ 1 & 0 \end{pmatrix}$$