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Open billiards: Regular, chaotic and pseudo-chaotic

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Open billiards:
Regular, chaotic and pseudo-chaotic

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Open dynamics

Define a "hole" as a subset of phase space, at which trajectories escape. This has many applications:

- Illuminating structures and Poincare recurrence in the corresponding closed system, for example fractal measures in a Hamiltonian phase space for which the invariant measure is uniform.

- Describing metastability and rare events, for example chemical reaction rates, migration of asteroids.

- Relating dynamics to thermodynamics via the escape rate formalism of Gaspard et al

- Physical escape or scattering problems, eg microlasers, room acoustics

- Nondestructive investigation of internal dynamics by measurements of escaping particles.

Open billiards

An open billiard is a dynamical system consisting of point particle moving with constant velocity in a domain except for specular collisions with the boundary and absorption at a hole. We are typically interested in the survival probability given an initial equilibrium distribution as a function of time and hole shape and size. Almost all this work is for $t \to \infty$.

- Open billiards: Applications and theory
- Hyperbolic example: Diamond, small hole, expansion in correlation functions. Bunimovich and CD, EPL **80**, 40001 (2007).
- Integrable example: Circle, small hole, expansion involving the Riemann zeta function. Bunimovich and CD, Phys Rev Lett **94**, 100201 (2005).
- Intermittent example: Stadium, arbitrary hole on the straight section, leading coefficient. CD and Georgiou, Physica D (to appear); arxiv.org/0812.3095
- Pseudochaos example: Finite Ehrenfest gases. CD and Cohen, J Stat Phys **101** 775-817 (2000).

Motivation

Open billiards provide:

- Examples of open dynamical systems covering many cases from integrable to strongly hyperbolic.
- Connections with number theory.
- A description of many physical systems and experiments involving a particle or small(ish) wavelength wave in a cavity.
- Models for statistical mechanics and molecular dynamics.

Dynamical classification of billiards

Integrable Circle, ellipse, rectangle, three triangles.

Pseudo-integrable Polygons with rational angles.

Parabolic Polygons with irrational angles.

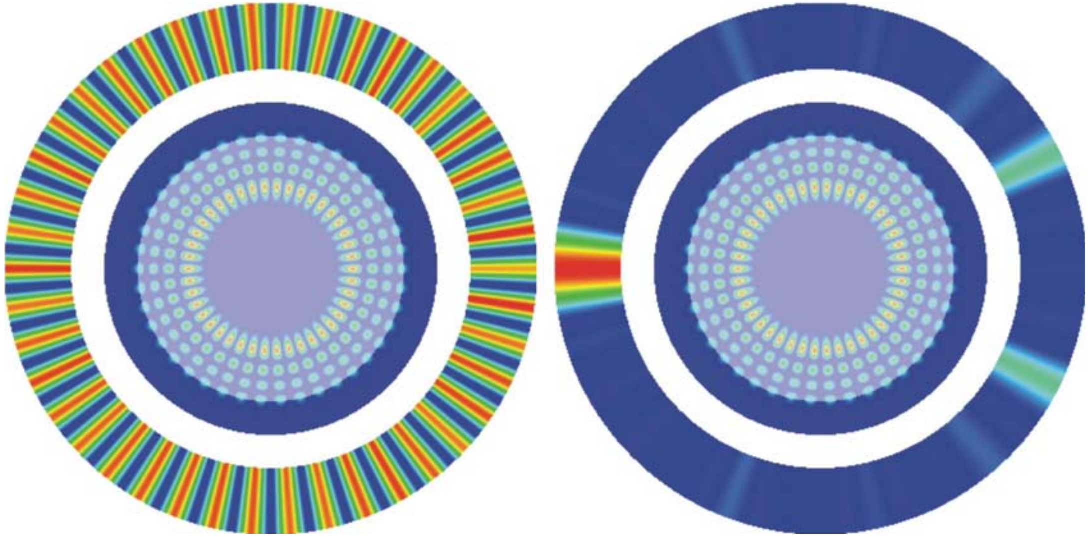
Mixed Mushroom, generic curved.

Hyperbolic Stadium, Sinai, Cardioid.

Exotic External field, Riemannian metric, global topology, thin barrier, unbounded domain, alternative reflection laws, moving boundary.

Open Trajectories absorbed at hole(s), subset of boundary, incidence angles, or the interior.

Application: Microlasers



Microlasers are cavities containing an active (lasing) medium that trap light due to total internal reflection. Thus the "hole" is the entire boundary, but only trajectories sufficiently close to the normal direction can escape. Placing a small scatterer (right) breaks the symmetry and allows strong directivity in conjunction with low losses. Here we have an internal wavelength about 1/6 the radius, ie not too small, yet geometric optics is still useful in determining the optimal position of the scatterer. [CD, Morozov, Sieber, Waalkens, 2008-9; numerous theoretical and experimental papers in the physics literature]

Open dynamical systems

Given a map Φ on a set M, of which part $H \subset M$ denotes the "hole", we can distinguish four types of points x in $M \backslash H$, depending on whether x has images and/or preimages in H. Typically we have

- The repeller, set of points with neither images nor preimages in H.

- The unstable manifold of the repeller (contains the set of points with no preimages in H).

- The stable manifold of the repeller (contains the set of points with no images in H).

- The transient set, set of points with both images and preimages in H.

In the case of a billiard, Φ is reversible, so there are obvious statements relating the forward and backward iterations of the map. Measures are then defined supported on one or more of these sets. Example: triadic Baker map, repeller dimension $2\ln 2/\ln 3$, manifolds dimension $\ln 2/\ln 3 + 1$, transient set dimension 2.

Conditionally invariant measures

For a review see Demers & Young, Nonlinearity 2006. A conditionally invariant measure is a Borel probability measure μ on $M\backslash H$ that satisfies

$$\frac{\Phi_*\mu(A)}{\Phi_*\mu(M\backslash H)} = \mu(A)$$

for all Borel subsets A of $M\backslash H$, where $\Phi_*\mu(A) = \mu(\Phi^{-1}(A))$. There are many such measures, obtained by giving an arbitrary measure on $\Phi^{-1}(H)$ and iterating backwards. The most common approach is to start with a uniform measure μ_0 and generate a "natural" conditionally invariant measure

$$\mu = \lim_{n\to\infty} \frac{\Phi_*^n\mu_0}{|\Phi_*^n\mu_0|} \qquad \Phi_*\mu = |\Phi_*\mu|\mu$$

if it can be shown that the limit exists and that Φ_* is continuous at μ in some topology. The survival probability given the initial uniform measure decays like $|\Phi_*\mu|^n$. The measure μ is supported on the repeller together with its unstable manifold. It is possible to give an invariant measure on the repeller alone by a suitable restriction of μ. The theory has many features in common with that of SRB measures, eg for uniformly hyperbolic systems μ is smooth in the unstable direction and the measure on the repeller satisfies generalised Pesin theorems.

Beyond uniform hyperbolicity

Most systems are *not* uniformly hyperbolic, and do *not* have an exponential escape rate. For example the survival probability $P(n) = |\Phi_*^n \mu_0|$ could have

- $P(n) = 0$ for finite n, eg if the hole is large, or some models with square scatterers.

- $P(n)$ decays superexponentially, eg the map $\Phi(x) = \sqrt{x} + x$ on a finite interval containing zero.

- $P(n) \sim 1/n$, eg a marginal family of orbits in a 2D billiard such as the stadium.

- $P(n) \to C$, a constant, eg an elliptical billiard with a small hole at one end; no orbits passing between the foci escape.

A variety of methods is needed.

Basic facts/notation for 2D billiards

The most convenient approach is usually the collision map $\Phi(x)$ where x denotes arc length l and angle ψ at the boundary. There is a function $T(x)$ giving the continuous time from x to $\Phi(x)$, from which continuous time properties may be calculated.

The equilibrium measure and mean collision time for a billiard with domain $D \subset R^2$ are

$$d\mu_0 = \frac{\cos\psi dl d\psi}{2|\partial D|} \qquad \int T(x) d\mu_0 = \frac{\pi |D|}{|\partial D|}$$

$\langle\rangle$ will indicate an average with respect to μ_0, so that correlation functions are written $\langle fg \circ \Phi^n \rangle - \langle f \rangle \langle g \rangle$ for functions $f, g : M \to R$.

Example 1: diamond billiard

Four circular arcs with centres at the vertices of a unit square, radius R.

$R = 1/2$ Tangential singular points, C/n correlations (Chernov, 2008).

$1/2 < R < 1/\sqrt{2}$ Exponential decay of correlations (Chernov, 1999).

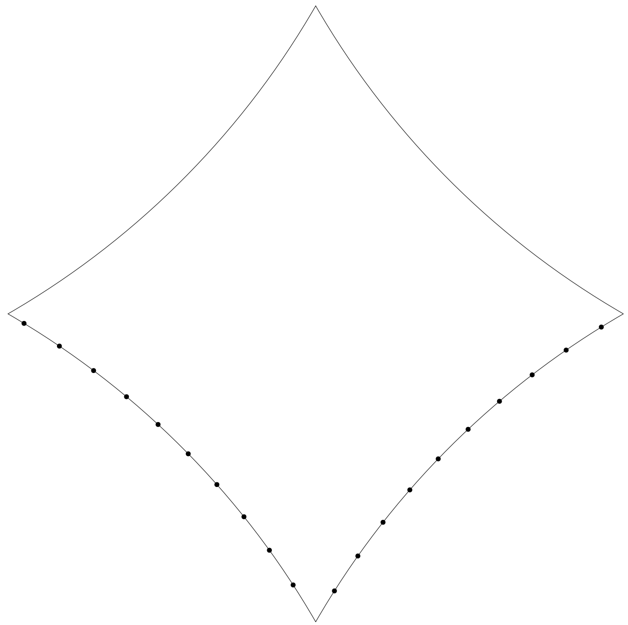
$R \to 1/\sqrt{2}$ Square, integrable and no decay of correlations.

For $R = 1/\sqrt{3}$ we have

$$|\partial D| = \frac{2\pi}{3\sqrt{3}}, \qquad |D| = 1 - \sqrt{\frac{4}{\sqrt{3}} - 1} - \frac{\pi}{9}, \qquad \langle T \rangle = \frac{\pi |D|}{|\partial D|}$$

We look for exponential escape in continuous time

$$\gamma = -\lim_{t\to\infty} \frac{1}{t} \ln P(t)$$



Phase functions

T Time to next collision

χ Characteristic function, 1 on boundary, 0 on hole

$\tau_k = \langle T^k \chi \rangle / \langle \chi \rangle$ Power of time, restricted to non-hole dynamics.

$\chi_s = e^{sT} \chi$ Weighted characteristic function

$\widehat{\chi}_s = \chi_s / \langle \chi_s \rangle - 1$ Zero average version of χ_s

$\widehat{T} = T - \tau_1$ Non-hole zero average version of T

$\widehat{\tau}_k = \langle \widehat{T}^k \chi \rangle / \langle \chi \rangle$

The generating function

The escape rate is the leading pole of

$$\int_0^\infty e^{st} P(t)dt = \int_0^\infty e^{st} \langle \chi^0 \chi^1 \ldots \chi^{N_t} \rangle dt$$

with superscripts indicating discrete time evolution. Up to a bounded factor this is

$$\int_0^\infty \langle \chi_s^0 \chi_s^1 \ldots \chi_s^{N_t} \rangle dt = \sum_{N=0}^{\infty} \langle \chi_s^0 \chi_s^1 \ldots \chi_s^N T^N \rangle$$

Assuming escape is not dominated by $T \to 0$ orbits we come to

$$G(s) = \sum_{N=0}^{\infty} G_N(s) = \sum_{N=0}^{\infty} \langle \chi_s^0 \chi_s^1 \ldots \chi_s^N \rangle$$

Extracting the leading behaviour,

$$G_N(s) = \langle \chi_s \rangle^{N+1} \left[1 + \sum_{0 \le j < k \le N} \langle \hat{\chi}_s^j \hat{\chi}_s^k \rangle + \sum_{0 \le j < k < l \le N} \langle \hat{\chi}_s^j \hat{\chi}_s^k \hat{\chi}_s^l \rangle + \ldots \right]$$

with problematic convergence...

The cumulant expansion

$G(s)$ diverges at the first zero of

$$g(s) = \lim_{N\to\infty} \ln \frac{G_{N+1}(s)}{G_N(s)}$$

so taking the difference of the logarithms of the previous series gives

$$\begin{aligned} g(s) &= \ln \langle \chi_s \rangle + \sum_{n=2}^{\infty} Q_n(s) \\ Q_2(s) &= \sum_{0<j} \langle \hat{\chi}_s^0 \hat{\chi}_s^j \rangle \\ Q_3(s) &= \sum_{0<j<k} \langle \hat{\chi}_s^0 \hat{\chi}_s^j \hat{\chi}_s^k \rangle \\ Q_4(s) &= \sum_{0<j<k<l} \left[\langle \hat{\chi}_s^0 \hat{\chi}_s^j \hat{\chi}_s^k \hat{\chi}_s^l \rangle - \langle \hat{\chi}_s^0 \hat{\chi}_s^j \rangle \langle \hat{\chi}_s^k \hat{\chi}_s^l \rangle - \ldots \right] - \frac{3}{2} Q_2(s)^2 \end{aligned}$$

which each term is now convergent, given sufficiently fast decay of multiple correlations.

Piecewise linear example

Consider $2x(mod 1)$ map, with hole at $[0, 1/4)$,

$$T = \begin{cases} T_1 & 1/4 \le x < 1/2 \\ T_2 & 1/2 \le x < 3/4 \\ T_3 & 3/4 \le x < 1 \end{cases}$$

Then we find

$$g(s) = \ln[(e^{sT_3} + \sqrt{e^{2sT_3} + 4e^{s(T_1+T_2)}})/4]$$

Expanding in s, the series converges at least to the escape rate

$$e^{\gamma(T_1+T_2)} + 2e^{\gamma T_3} = 4$$

Expanding in the order of correlation also appears to converge. This does not tell us about the small hole limit.

Small s expansion

The hole, and hence the escape rate s are expected to be small...

$$
\begin{aligned}
\widehat{\chi}_s &= \widehat{\chi} + s\widehat{T}(1+\widehat{\chi}) + \frac{s^2}{2}(\widehat{T}^2 - \widehat{\tau}_2)(1+\widehat{\chi}) \\
&\quad + \frac{s^3}{6}(\widehat{T}^3 - 3\widehat{T}\widehat{\tau}_2 - \widehat{\tau}_3)(1+\widehat{\chi}) + \ldots \\
\ln\langle\chi_s\rangle &= \ln\langle\chi\rangle + s\tau_1 + \frac{s^2}{2}\widehat{\tau}_2 + \frac{s^3}{6}\widehat{\tau}_3 \\
&\quad + \frac{s^4}{24}(\widehat{\tau}_4 - 3\widehat{\tau}_2^2) + \ldots \\
Q_2(s) &= \sum_{0<j}\langle\widehat{\chi}^0\widehat{\chi}^j\rangle + s\sum_{0<j}\left[\widehat{T}^0(1+\widehat{\chi}^0)\widehat{\chi}^j\rangle + \ldots\right] \\
&\quad + \ldots
\end{aligned}
$$

How big are the correlations?

Let h denote the size of the hole relative to the boundary.

$$\ln \langle \chi \rangle \sim h$$

$$\widehat{T} \sim 1$$

$$\widehat{\chi} \sim \begin{cases} h & \text{boundary} \\ 1 & \text{hole} \end{cases}$$

Thus $\langle \widehat{\chi}^0 \widehat{\chi}^j \rangle$ is of order h^2 unless the measure of phase space returning to the hole after j collisions is large. For chaotic billiards with small holes, the outgoing angle needs to be precisely specified, so the return probability is also of order h^2. Higher correlations will also be of order h^2 if the hole(s) lie on short periodic orbits, otherwise smaller.

The expansion of $g(s)$

The leading behviour is

$$g^{(1)}(s) = \ln \langle \chi \rangle + s\tau_1$$

which has a zero at s proportional to h as expected. Continuing with $s \sim h$...

$$g^{(2)}(s) = \frac{s^2 \widehat{\tau}_2}{2} + \sum_{0<j} \langle \widehat{\chi}_s^{(1)0} \widehat{\chi}_s^{(1)j} \rangle + Q_3(0) + \ldots$$

where $\widehat{\chi}_s^{(1)} = \widehat{\chi} + s\widehat{T}(1+\widehat{\chi})$ is the first order part of $\widehat{\chi}_s$ and the higher cumulants only contribute for short periodic orbits. Finally we have the first main result:

$$\gamma^{(1)} = -\frac{\ln \langle \chi \rangle}{\tau_1}$$

$$\gamma^{(2)} = -\frac{g^{(2)}(\gamma^{(1)})}{\tau_1}$$

The correlation sum contains

$$\widehat{\chi}_{\gamma^{(1)}}^{(1)} \approx \widehat{\chi} + \frac{h}{\langle T \rangle} \widehat{T}(1 + \widehat{\chi}) \equiv u$$

One vs two holes

We can compare the escape rate of a system with holes A and B to the individual escape rates. We have

$$\chi_{AB} = \chi_A + \chi_B - 1$$

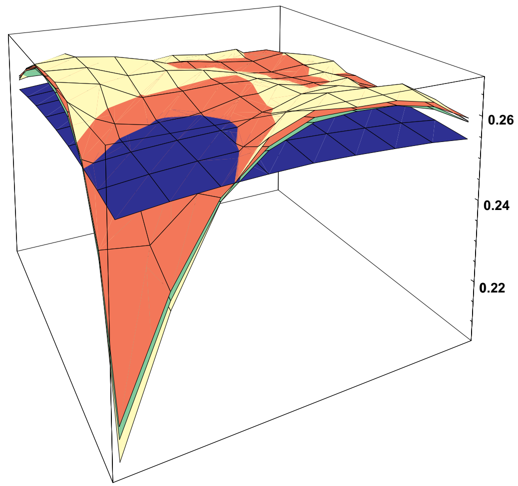
giving the second main result:

$$\begin{aligned}\gamma_{AB} &= \gamma_A + \gamma_B \\ &\quad - \frac{1}{\langle T \rangle} \left\{ \sum_{j=-\infty}^{\infty} \langle u_A^0 u_B^j \rangle \right. \\ &\quad \left. + \sum_{n=3}^{\infty} [Q_{nAB}(0) - Q_{nA}(0) - Q_{nB}(0)] \right\}\end{aligned}$$

The higher cumulants are small unless both holes are on the same periodic orbit.

Numerics

- Diamond billiard with $R = 1/\sqrt{3}$
- Trajectory of 10^8 collisions sampled
- Chosen hole size is $R/20$



Yellow Exact

Blue First order

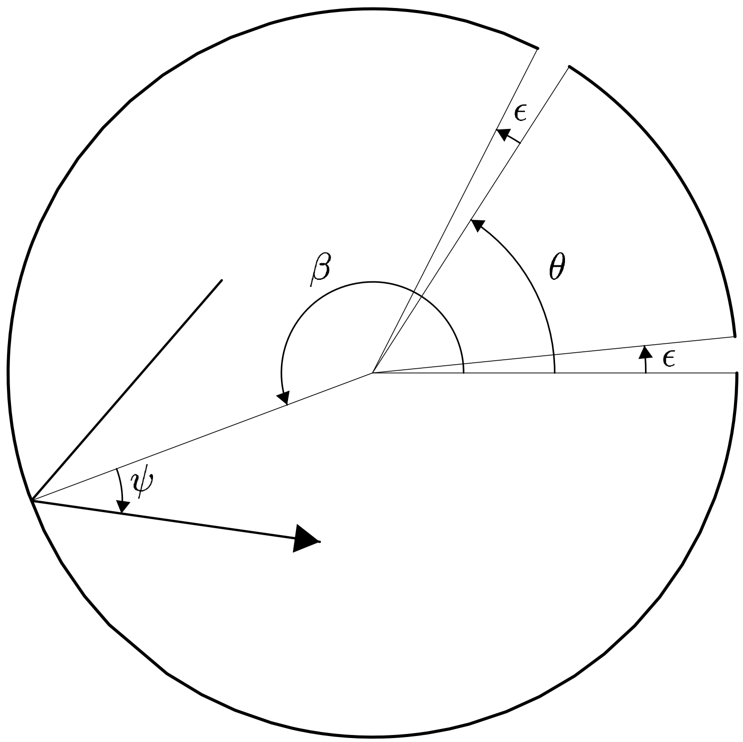
Green Second order expansion

Red Second order from 1-hole

Example 2: The circle

Dynamics $(\varphi, \psi) \to (\varphi + \pi - 2\psi, \psi)$ is just rotation on a circle, periodic for $\psi/\pi = 1/2 - m/n$, dense and uniform for irrational ψ/π. For the open problem, put holes at $\varphi \in [0, \epsilon] \cup [\theta, \theta + \epsilon]$ and find time to escape $t = 2\cos\psi_0 N(\varphi_0, \psi_0)$ where N counts collisions. Starting from the equilibrium measure at time zero, consider the probability $P(t)$ of remaining until time t, specifically

$$P_\infty = \lim_{t\to\infty} tP(t)$$



Long lived orbits are nearly periodic

- The $N + 1$ values $\varphi_0, \varphi_1, \ldots \varphi_N$ contain two at a distance less than ϵ if $N + 1 > 2\pi/\epsilon$.

- Define the period n to be the smallest positive integer so that $|\varphi_n - \varphi_0| < \epsilon$. Thus $n < 2\pi/\epsilon$.

- In units of n collisions, the orbit precesses slowly enough to be captured by the holes.

- All very long living orbits have small precession and are close to a periodic orbit.

Precisely: Let $t > 4[2\pi/\epsilon]$, then every connected component of the set of (φ, ψ) that survive to time t contains a unique interval of never escaping periodic orbits.

Counting long lived orbits

$$\psi = \psi_{m,n} + \eta, \quad \eta \ll \epsilon$$

Orbit will survive for at least time t, $t/2\cos\psi_{m,n}$ collisions if

$$\varphi_0' = \left(\epsilon + \frac{\eta t}{\cos\psi_{m,n}}, \theta'\right) \cup \left(\theta' + \epsilon + \frac{\eta t}{\cos\psi_{m,n}}, \frac{2\pi}{n}\right)$$

if $\eta > 0$. Adding up these contributions:

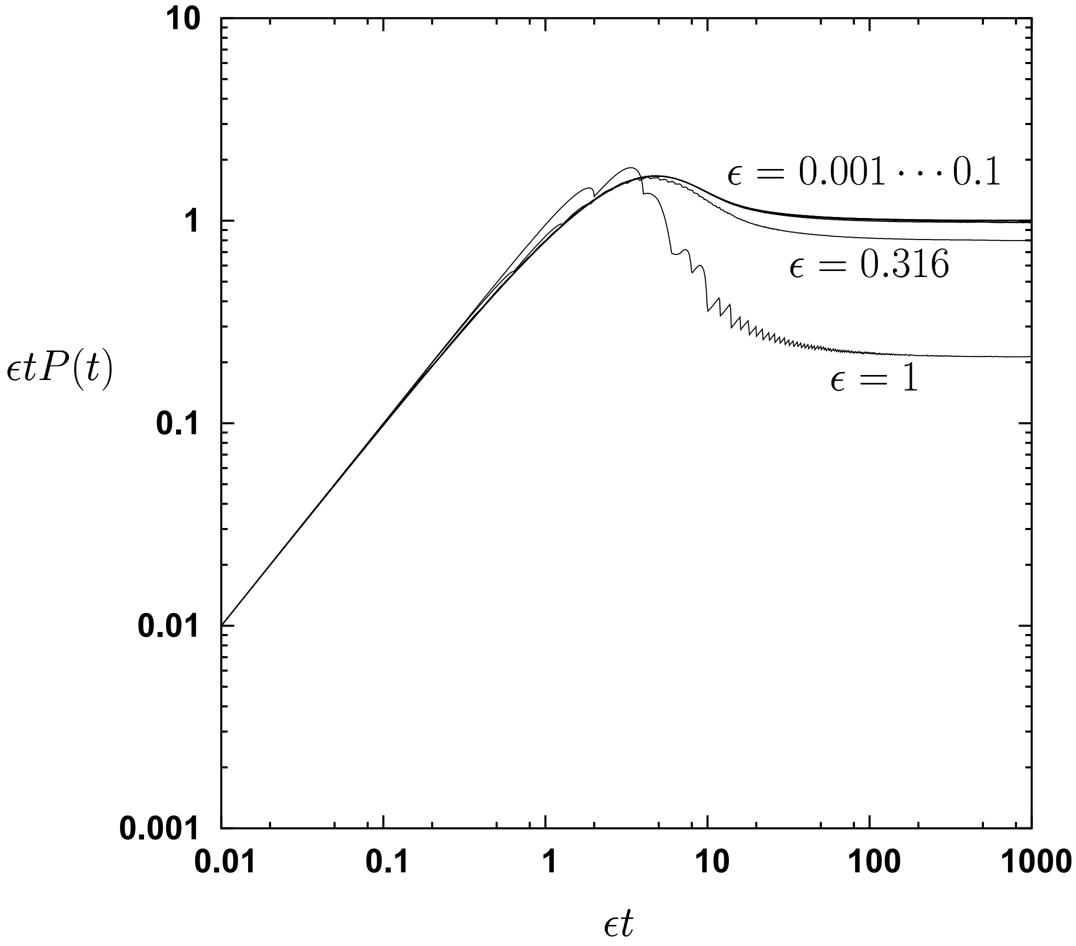
$$tP(t) \sim \frac{1}{4\pi} \sum_{m,n} n[g\left(\frac{2\pi}{n} - \theta' - \epsilon\right) + g(\theta' - \epsilon)] \sin^2 \frac{\pi m}{n}$$

where

$$g(x) = \begin{cases} x^2 & x > 0 \\ 0 & x \leq 0 \end{cases}$$

and the sum is over $1 \leq m < n < 2\pi/\epsilon$, $gcd(m,n) = 1$. The symbol $\sim$ means take $t \to \infty$, in which limit upper and lower bounds converge.

Finite time scaling



Summation over m

Use Ramanujan

$$\sum_{m=1}^{n-1} \exp(2\pi i m/n) = \mu(n)$$

where the sum is over $gcd(m,n) = 1$, μ is the Möbius function

$$\mu(n) = \begin{cases} 1 & n = 1 \\ -1 & n \quad \text{prime} \\ \mu(a)\mu(b) & n = ab, \quad gcd(a,b) = 1 \\ 0 & a^2|n, \quad a > 1 \end{cases}$$

and we find

$$P_\infty = \frac{1}{8\pi} \sum_{n=1}^{\infty} n[\phi(n) - \mu(n)][g\left(\frac{2\pi}{n} - \theta' - \epsilon\right) + g(\theta' - \epsilon)]$$

where

$$\phi(n) = n \prod_{p|n} (1 - p^{-1})$$

is the Euler totient function, the number of positive integers $\leq n$ which are coprime to n. Note $\phi(1) = 1$, so the $n = 1$ term vanishes.

Small hole limit

Mellin transforms:

$$\tilde{P}(s) = \int_0^\infty P_\infty \epsilon^{s-1} d\epsilon$$

$$P_\infty = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \epsilon^{-s} \tilde{P}(s) ds$$

leads to

$$\begin{aligned} P_\infty &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{ds\, \epsilon^{-s} (2\pi)^{s+1}}{2s(s+1)(s+2)} \sum_{n=1}^{\infty} \frac{\phi(n) - \mu(n)}{n^{s+1}} \\ & \times \left\{ \left[1 - f\left(\frac{n\theta}{2\pi} \right) \right]^{s+2} + f\left(\frac{n\theta}{2\pi} \right)^{s+2} \right\} \end{aligned}$$

where f denotes fractional part. The small ϵ expansion is obtained by summing residues.

Final expression

For two holes separated by angle $\theta = 2\pi r/q$, we find

$$P_\infty = \sum_j \mathrm{Res}_{s=s_j} \tilde{P}(s)\epsilon^{-s}$$

$$\begin{aligned}
\tilde{P}(s) \quad = \quad & \frac{(2\pi)^{s+1}}{2s(s+1)(s+2)} \\
& \times \sum_{a=1}^{q} \frac{[1 - f(\frac{ap}{q})]^{s+2} + f(\frac{ap}{q})^{s+2}}{b^{s+1}\phi(q')} \\
& \times \sum_{\chi} \frac{\bar{\chi}(a')[\phi(b)L(s,\chi) - \mu(b)]}{L(s+1,\chi)\prod_{p|b}[1 - \chi(p)p^{-s-1}]}
\end{aligned}$$

with $b = gcd(a, q)$, $a' = a/b$, $q' = q/b$, characters χ modulo q', f is the fractional part, L is a Dirichlet L-function.

Special cases

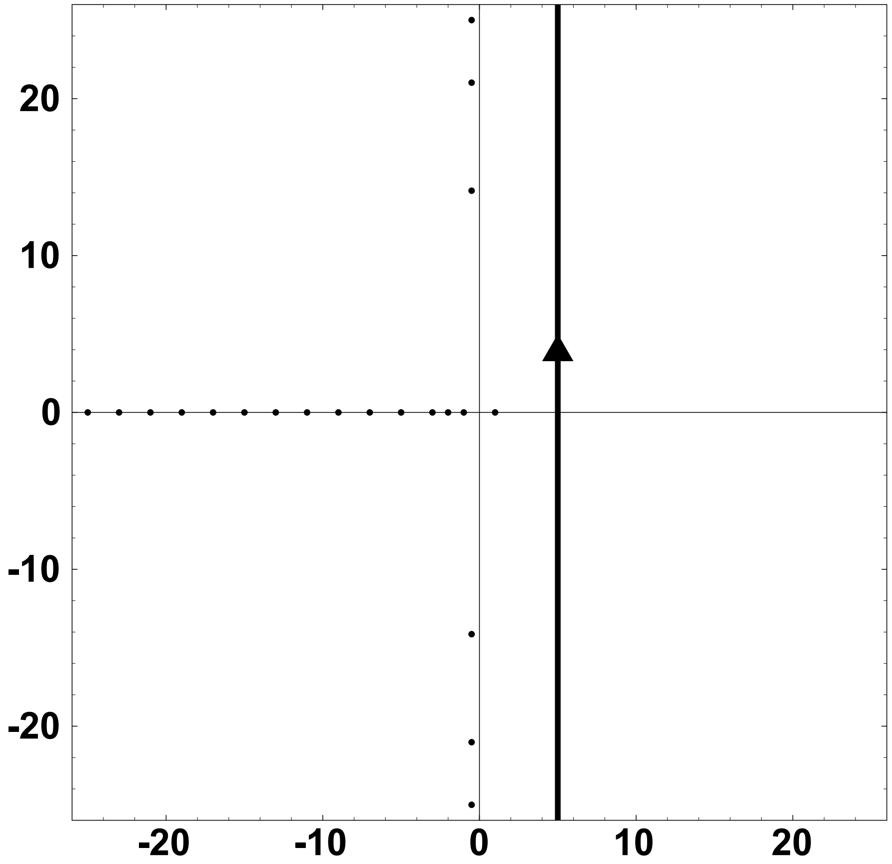
Odd characters cancel, so when $q = 1, 2, 3, 4, 6$ only Riemann zeta functions appear. The one and symmetric two hole cases are

$$\tilde{P}_1(s) = \frac{(2\pi)^{s+1}[\zeta(s) - 1]}{2s(s+1)(s+2)\zeta(s+1)}$$

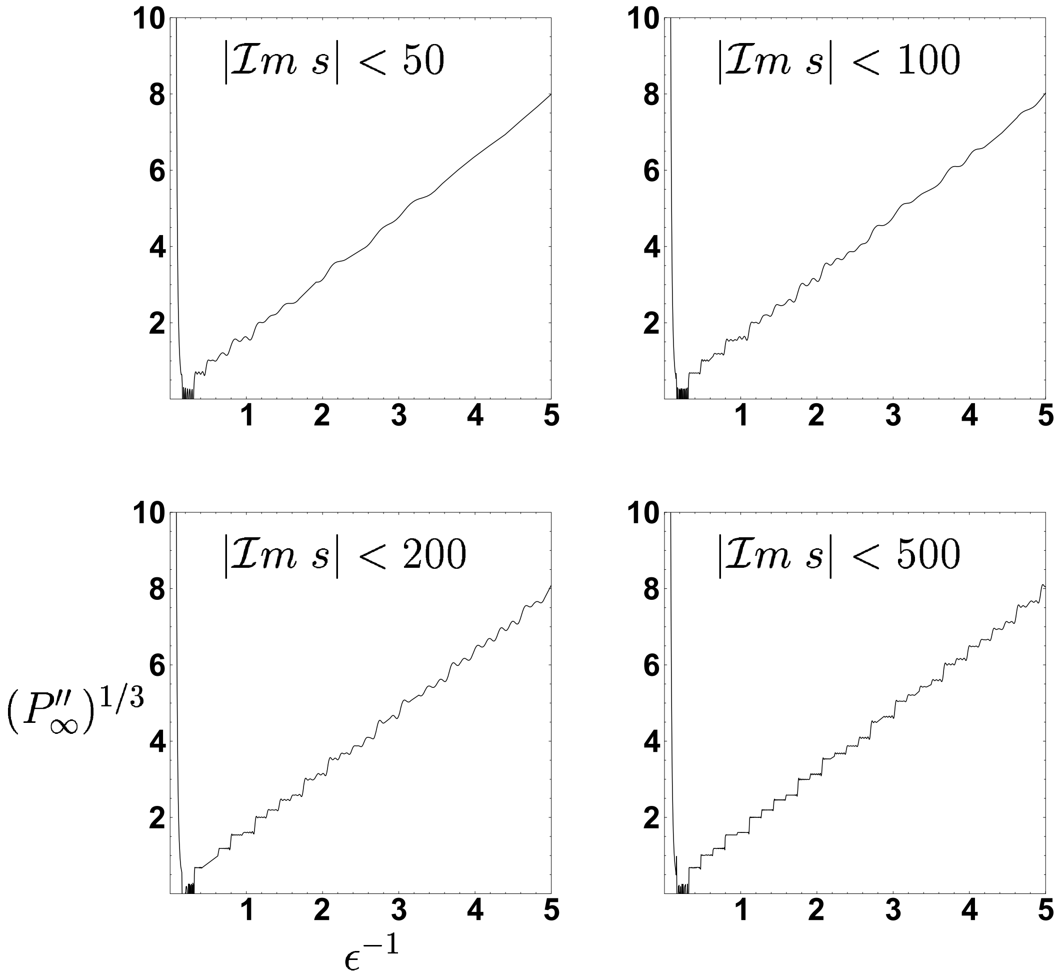
$$\tilde{P}_2(s) = \frac{(\pi)^{s+1}\zeta(s)}{s(s+1)(s+2)\zeta(s+1)}$$

with poles at odd $s \leq 1$ and at $\mathcal{R}e\, s = -1/2$ assuming the Riemann Hypothesis, and for $q = 1$ also $s = -2$.

The contour



Steps as sums over zeta zeros



Riemann reformulated

The Riemann hypothesis is thus

$$\lim_{\epsilon\to 0}\lim_{t\to\infty}\epsilon^{\delta}(tP_1(t)-2/\epsilon)=0$$

for every $\delta > -1/2$ where $P_1(t)$ is the probability of remaining after time t from an initial equilibrium distribution in the one hole problem.

An equivalent formulation is

$$\lim_{\epsilon\to 0}\lim_{t\to\infty}\epsilon^{\delta}t[P_1(t)-2P_2(t)]=0$$

with $P_2(t)$ the symmetric 2-hole probability.

The generalised Riemann hypothesis implies that the statement is also true for two holes with rational θ, but the converse statement is open, as is the case of irrational θ.

Irrational θ in 2-hole

The fractional parts are uniformly distributed, so make a "mean field" approximation, replacing the sum by an integral:

$$\begin{aligned}
\left\langle g\left(\frac{2\pi}{n}-\theta'-\epsilon\right)+g(\theta'-\epsilon)\right\rangle &= \frac{n}{3\pi}\left(\frac{2\pi}{n}-\epsilon\right)^3 \\
\langle\phi(n)\rangle &= \frac{6n}{\pi^2} \\
\langle\mu(n)\rangle &= 0
\end{aligned}$$

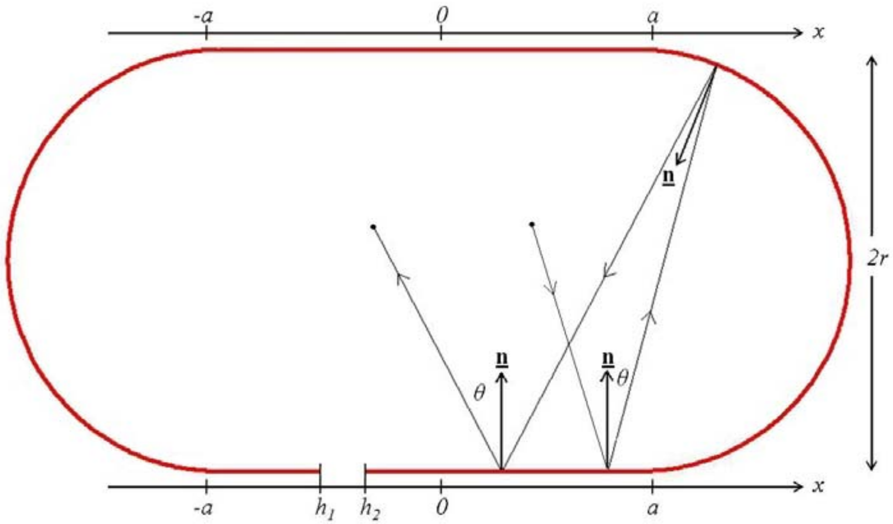
so that

$$tP(t)\approx\frac{1}{24\pi^2}\int_0^{2\pi/\epsilon} n^2\frac{6n}{\pi^2}\left(\frac{2\pi}{n}-\epsilon\right)^3 dn=\frac{1}{\epsilon}$$

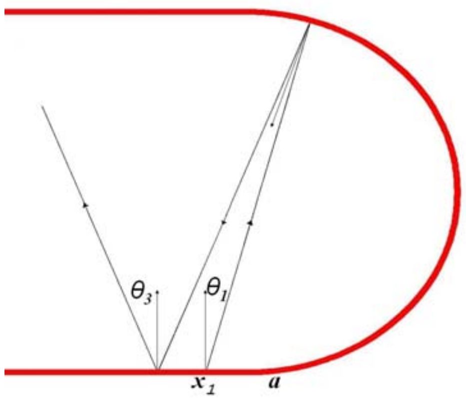
which is the same as all tested values of rational $\theta\neq 0$. Taking an irrational θ as a limit of rational θ suggests that $\tilde{P}(s)$ cannot be continued past a line of singularities at $\mathcal{R}e\, s=-1/2$.

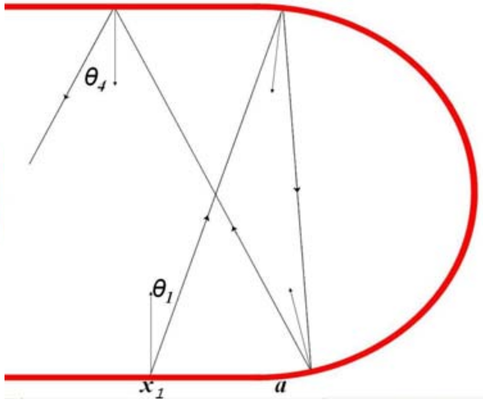
Example 3: The stadium

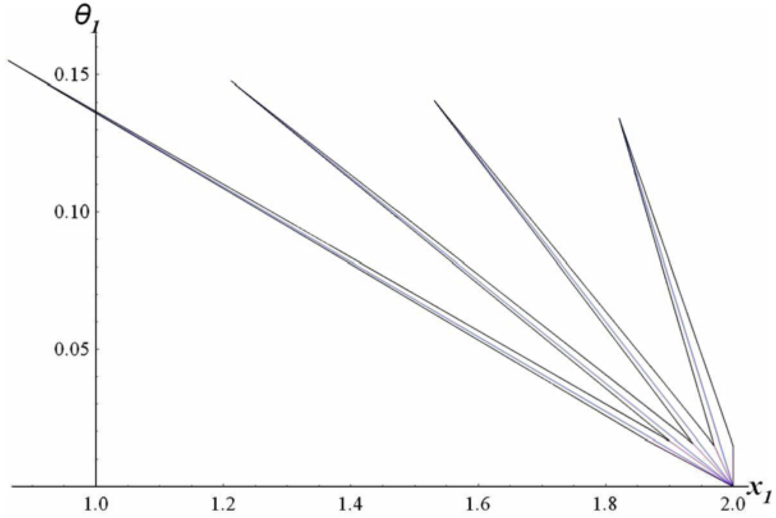
The stadium is a defocusing billiard, with strong ergodic properties (Bunimovich 1979; Chernov and Haskell 1996) but "bouncing ball" orbits that lead to $1/n$ decay of correlations (Vivaldi, Casati and Guarneri 1983, Markarian 2004, Chernov 2008). A major interest (and difficulty) of the stadium is its strong reinjection into the bouncing ball region; an orbit with small angle θ will be reinjected with angle in $[\theta/3, 3\theta]$.



Reflections from the circular arcs







Contributions to the survival probability

In the case that the hole is on the straight segment, there are two contributions for $t > 32ar/\epsilon$

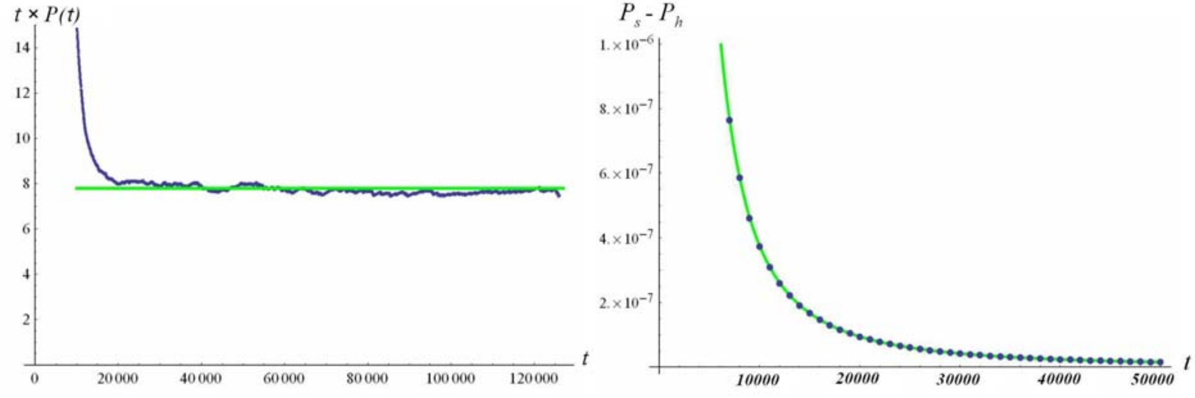
1. Orbits that move directly toward the hole:

$$P_1(t) = \frac{(a+h_1)^2 + (a-h_2)^2}{(8a+4\pi r)t} + O(1/t^2)$$

2. Orbits that move away from the hole, reflect from a circular arc and return to the hole, result obtained by summing a linear approximation to the hyperbolas:

$$P_2(t) = \frac{(3\ln 3 + 2)\left[(a+h_1)^2 + (a-h_2)^2\right]}{(16a+8\pi r)t} + O(1/t^2)$$

Numerical tests



P_d Probability from direct simulation

P_h Probability from integrating exact region (hyperbolas).

P_s Probability from integrating straight line approximation (previous slide)

Left: P_d and P_s

Right: $P_s - P_h$ is order $1/t^2$.

Example 4: Extended billiards

Why does statistical mechanics work? What dynamical properties are required? Consider a single particle colliding with a large collection of non-overlapping convex scatterers and investigate diffusion properties. In decreasing strength of statistical properties, we have:

1. Periodic circles with finite horizon (Lorentz gas)

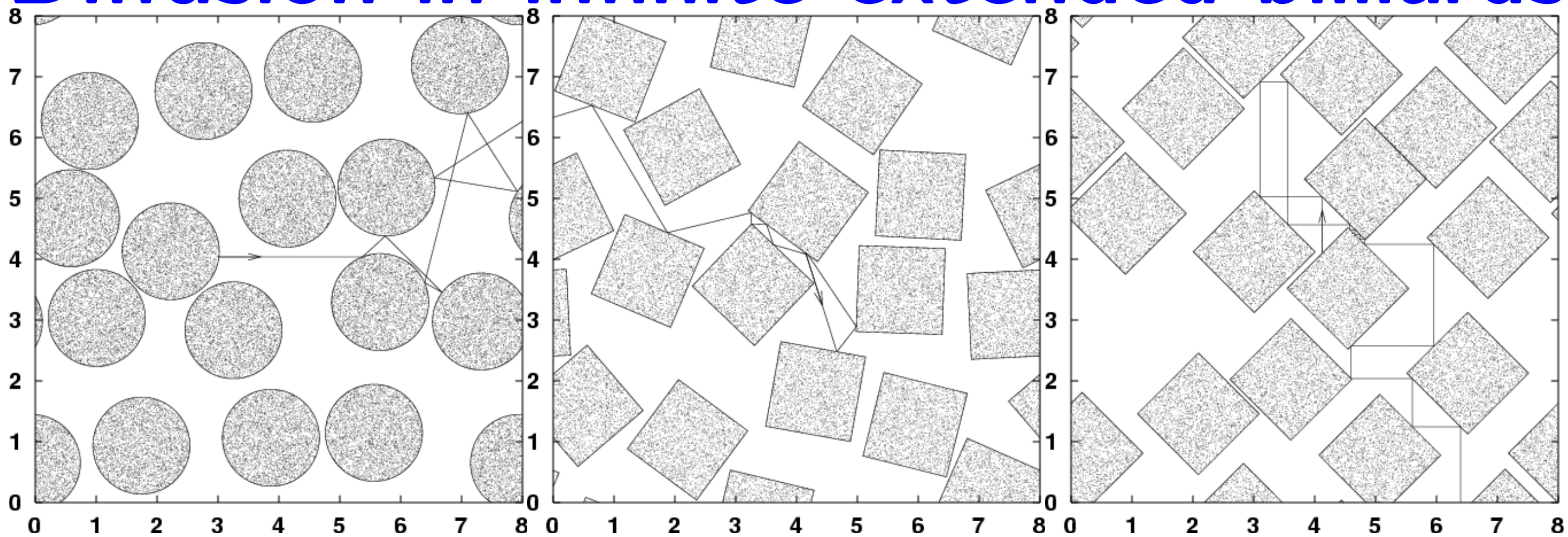
2. Random circles

3. Random squares (modified Ehrenfest model)

4. Periodic collections of squares with finite horizon

5. Fixed squares (original Ehrenfest model)

Diffusion in infinite extended billiards



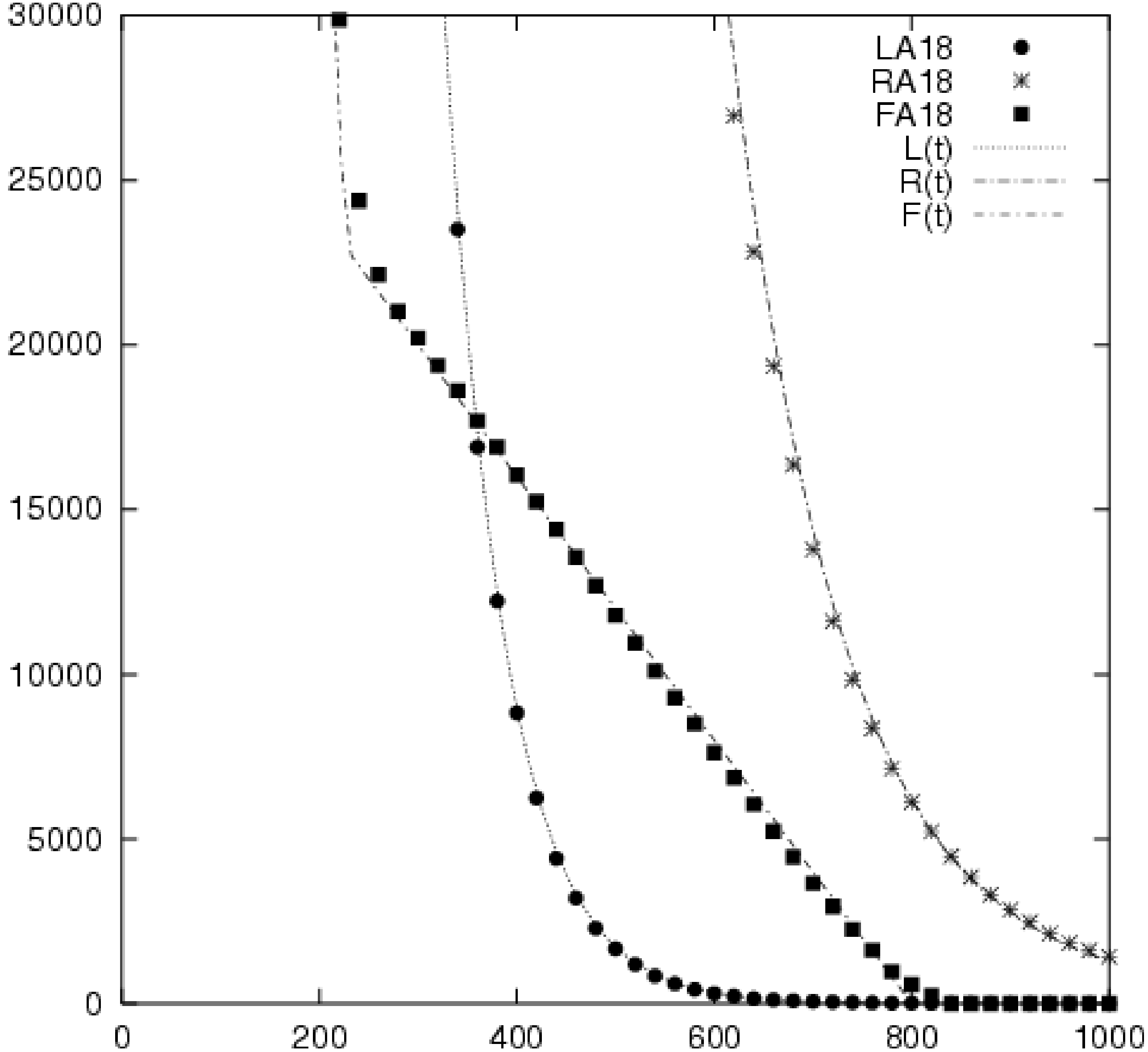
All these but the fixed squares appear to approach the Weiner process in the relevant limit ($\lambda^2 x$, λt as $\lambda \to \infty$), cf the infinite horizon Lorentz gas with logarithmic superdiffusion. [H van Beijeren, E G D Cohen and CD, 1999-2001; more recent work on polygons has looked at instability (van Beijeren 2004), fields (Bianca and Rondoni 2009), many particles (Cecconi, Cencini, Vulpiani 2007; Hoover, Hoover and Bannerman 2009)]

Open Extended billiards

However if we consider an open collection of scatterers, ie fixed x while $t \to \infty$, it is clear the square scatterers will not give the exponential decay of the diffusion equation. The results are:

- Circles (Lorentz): $e^{-\gamma t}$, cf diamond
- Randomly oriented squares: C/t, cf circle
- Fixed orientation squares: Complete decay in finite time (generically no periodic orbits)

Observation: A mixture of circles and squares should behave similar to the stadium (not necessarily ergodic): Instability determined by chaotic orbits (from circles), while escape determined by slowest decay (from squares).



LA18
RA18
FA18
L(t)
R(t)
F(t)
30000
25000
20000
15000
10000
5000
0
0
200
400
600
800
1000

Summary of results

- Diamond (cf extended circles):

$$\gamma = -\lim_{t\to\infty}\frac{1}{t}\ln P(t) = \frac{\epsilon}{\langle T\rangle} + O(\epsilon^2)$$

$$\gamma_{AB} = \gamma_A + \gamma_B - \frac{1}{\langle T\rangle}\left\{\sum_{j=-\infty}^{\infty}\langle u_A^0 u_B^j\rangle + \sum_{n=3}^{\infty}[Q_{nAB}(0) - Q_{nA}(0) - Q_{nB}(0)]\right\}$$

- Circle (cf extended random squares):

$$\begin{aligned} P_\infty &= \lim_{t\to\infty} tP(t) \\ &= \frac{1}{8\pi}\sum_{n=1}^{\infty} n[\phi(n) - \mu(n)][g\left(\frac{2\pi}{n} - \theta' - \epsilon\right) + g(\theta' - \epsilon)] \\ &= \frac{2}{\epsilon} + o(\epsilon^{1/2-\delta}) \qquad \text{1 hole; assumes RH} \end{aligned}$$

- Stadium (cf extended circles and squares):

$$P_\infty = \frac{(3\ln 3 + 4)\left[(a+h_1)^2 + (a-h_2)^2\right]}{(16a + 8\pi r)}$$

Other remarks

The combination ϵt appears in $P(t)$ for

- All billiards, sufficiently small t.
- The diamond, small ϵ and large t (at least).
- The circle, small ϵ and all t (numerically).
- The stadium, not for large t.

Periodic orbits play an important role for

- The diamond, correlations increase when the hole covers short periodic orbits
- The circle, is completely determined by periodic orbits at long times
- The stadium, is dominated by its marginal family of periodic orbits plus the neighbourhood.
- Fixed squares: No periodic orbits, no long time survival probability!

The future

- Finite time scaling and dynamical effects.
- Other dynamical behaviour, eg mixed systems.
- Higher dimensions.
- Exotic billiards.
- Quantum connections.
- Applications.

Thank you for your attention!