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2058-S-10

Pseudochaos and Stable-Chaos in Statistical Mechanics and Quantum Physics

21 - 25 September 2009

Anderson Localization for the Nonlinear Schroedinger Equation (NLSE): results and puzzles

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Anderson Localization for the Nonlinear Schrödinger Equation (NLSE): Results and Puzzles

Yevgeny Krivolapov, Hagar Veksler, Avy Soffer, and SF

Experimental Relevance

Nonlinear Optics

Bose Einstein Condensates (BECs)

Competition between randomness and nonlinearity

nature Vol 446 | 1 March 2007 | doi:10.1038/nature05623

LETTERS

Transport and Anderson localization in disordered two-dimensional photonic lattices

Tal Schwartz[1], Guy Bartal[1], Shmuel Fishman[1] & Mordechai Segev[1]

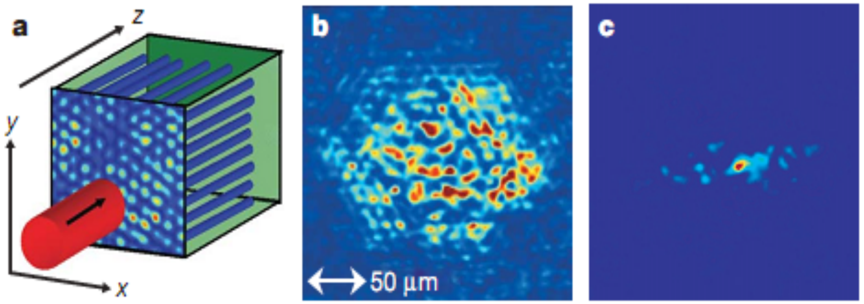
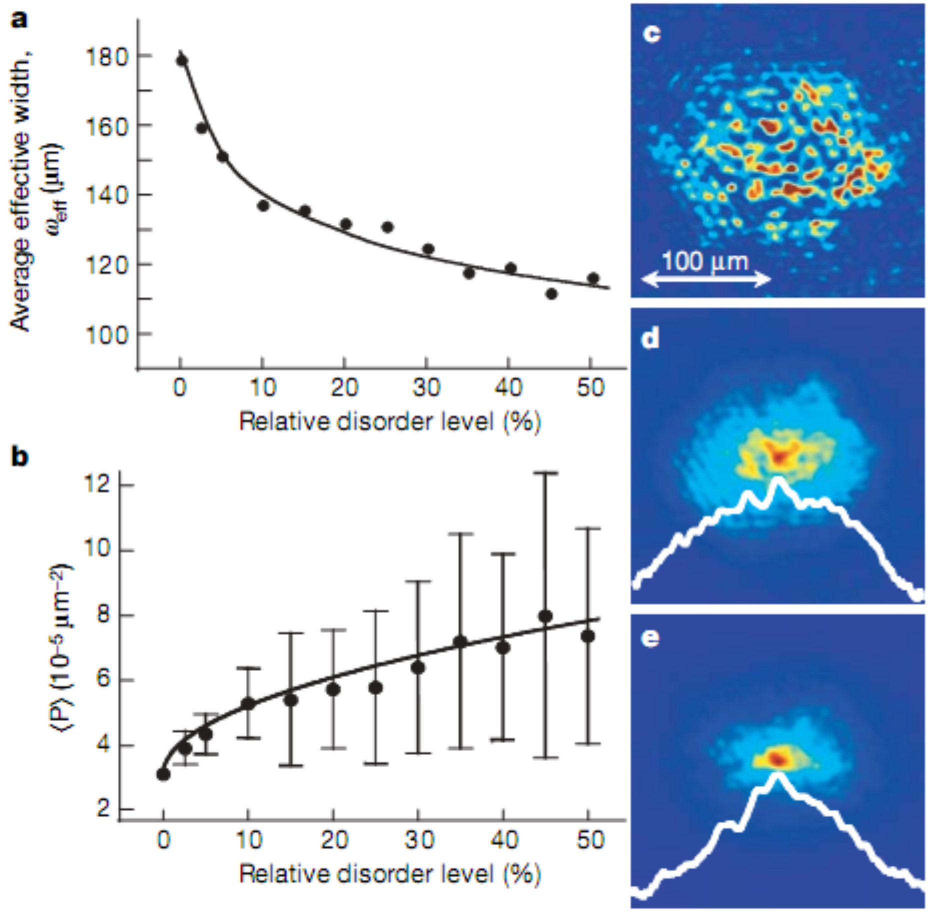


Figure 1 | Transverse localization scheme. a, A probe beam entering a disordered lattice, which is periodic in the two transverse dimensions (x and y) but invariant in the propagation direction (z). In the experiment described here, we use a triangular (hexagonal) photonic lattice with a periodicity of 11.2 μm and a refractive-index contrast of $\sim 5.3 \times 10^{-4}$. The lattice is induced optically, by transforming the interference pattern among three plane waves



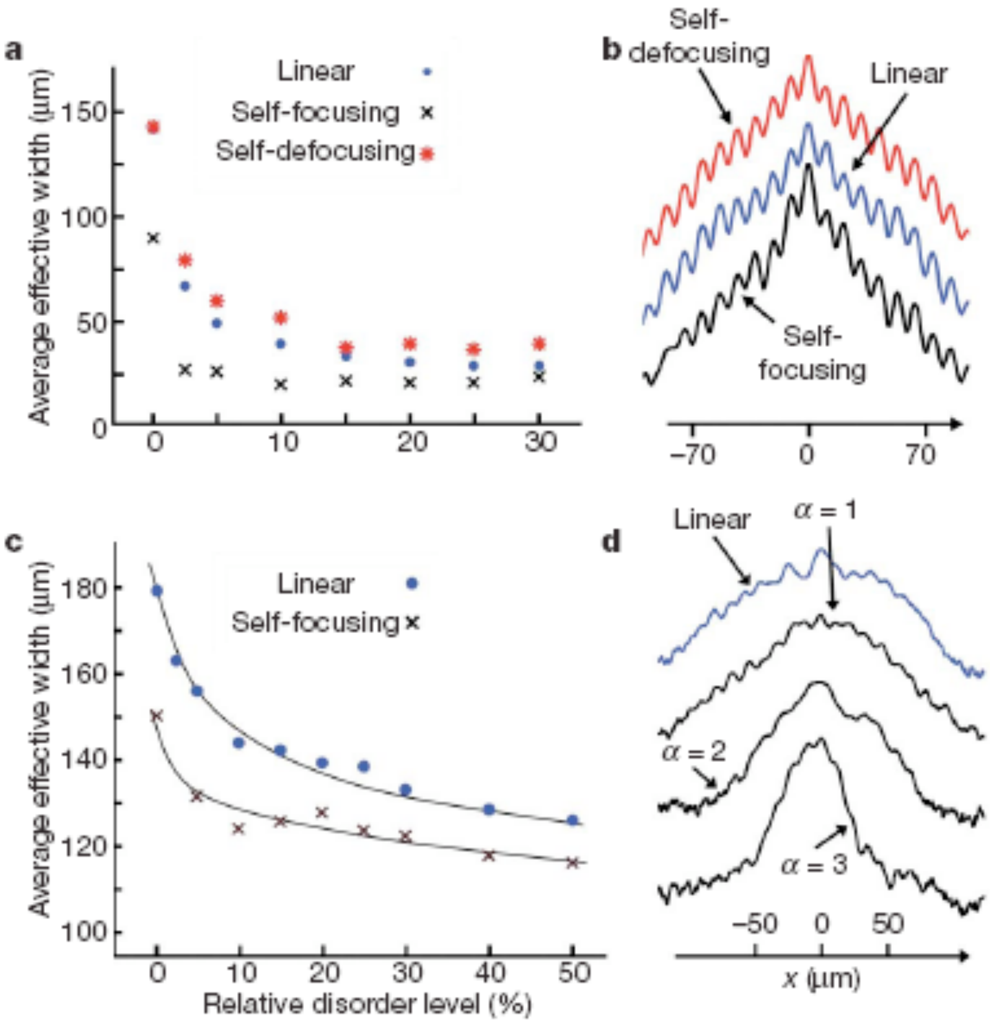
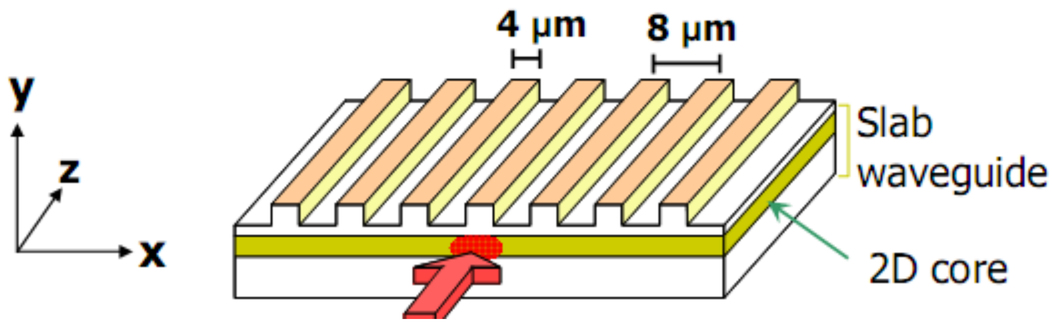


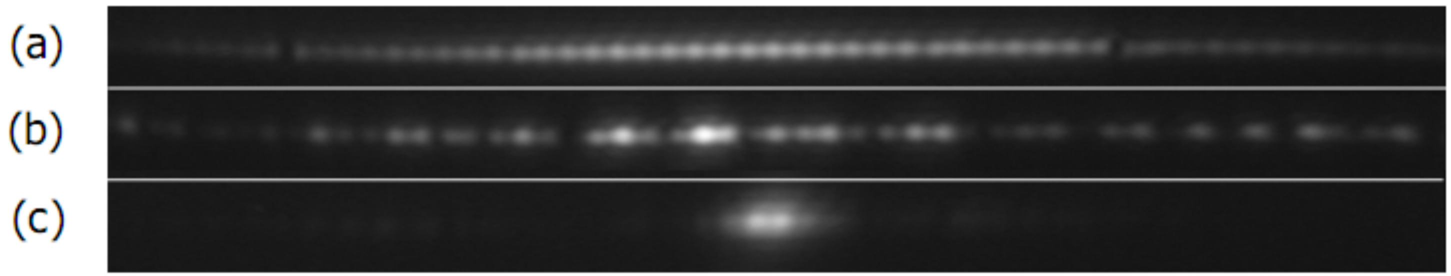
Figure 4 | Numerical (top row) and experimental (bottom row) results,

PRL **100,** 013906 (2008) PHYSICAL REVIEW LETTERS week ending 11 JANUARY 2008

Anderson Localization and Nonlinearity in One-Dimensional Disordered Photonic Lattices

Yoav Lahini,[1,*] Assaf Avidan,[1] Francesca Pozzi,[2] Marc Sorel,[2] Roberto Morandotti,[3]
Demetrios N. Christodoulides,[4] and Yaron Silberberg[1]



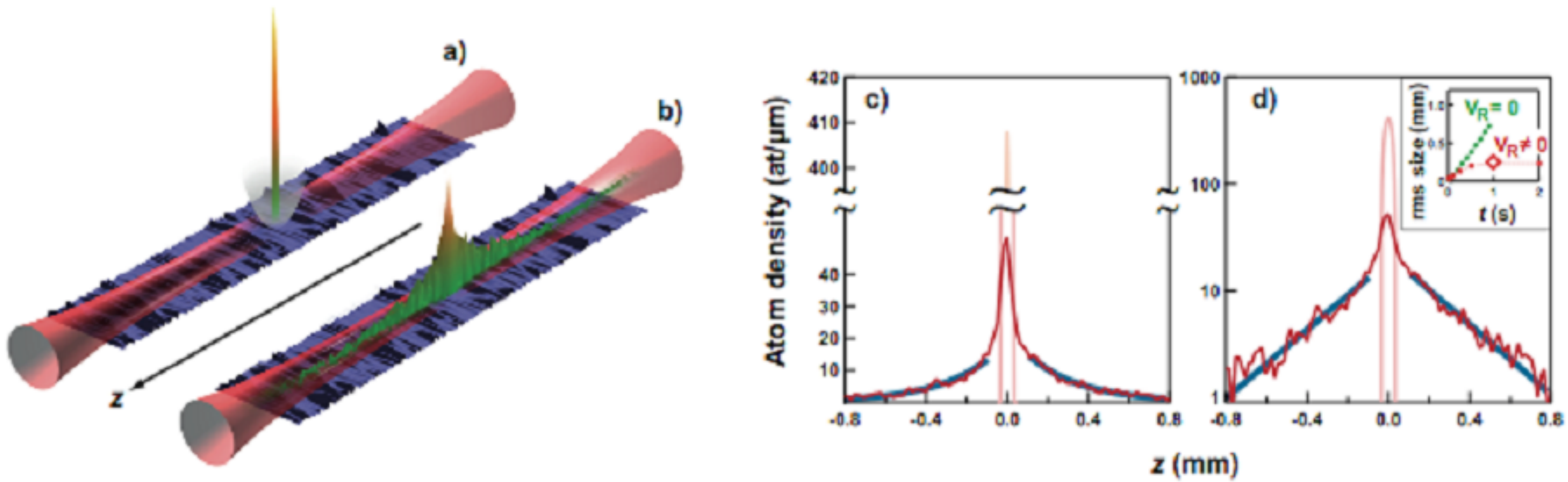


(a) Periodic array – *expansion*
(b) Disordered array - *expansion*
(c) Disordered array - *localization*

Direct observation of Anderson localization of matter-waves in a controlled disorder

Juliette Billy[1], Vincent Josse[1], Zhanchun Zuo[1], Alain Bernard[1], Ben Hambrecht[1], Pierre Lugan[1], David Clément[1], Laurent Sanchez-Palencia[1], Philippe Bouyer[1] & Alain Aspect[1]

[1] *Laboratoire Charles Fabry de l'Institut d'Optique, CNRS and Univ. Paris-Sud, Campus Polytechnique, RD 128, F-91127 Palaiseau cedex, France*



The Nonlinear Schroedinger (NLS) Equation

$$i\frac{\partial}{\partial t}\psi = \mathcal{H}_o\psi + \beta\left|\psi\right|^2\psi$$

1D lattice version

$$\mathcal{H}_0\psi(x) = -\big(\psi(x+1)+\psi(x-1)\big)+\varepsilon(x)\psi(x)$$

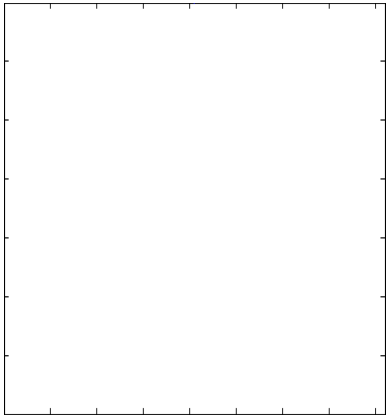
1D continuum version

$$\mathcal{H}_0\psi(x) = -\frac{1}{2}\frac{\partial^2}{\partial x^2}\psi(x)+\varepsilon(x)\psi(x)$$

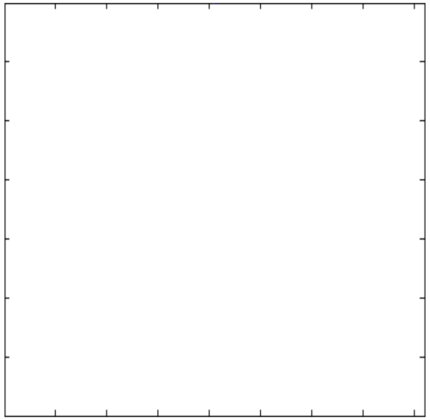
V random $\longrightarrow$ $\mathcal{H}_0$ Anderson Model

$$i\frac{\partial \psi_n}{\partial t} = -\left(\psi_{n+1} + \psi_{n-1}\right) + \varepsilon_n \psi_n + \beta \left|\psi_n^{\,2}\right| \psi_n$$

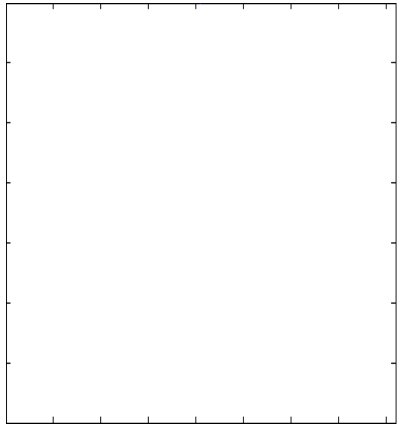
$\beta = 0 \quad \varepsilon_n = 0$



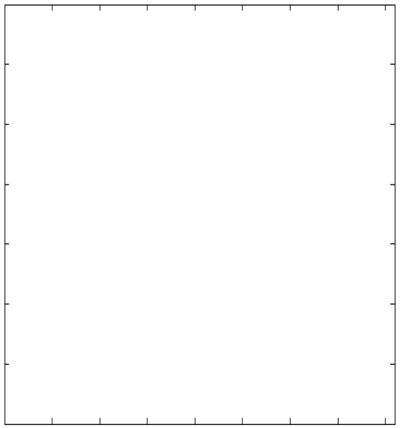
$\beta = 1 \quad \varepsilon_n = 0$



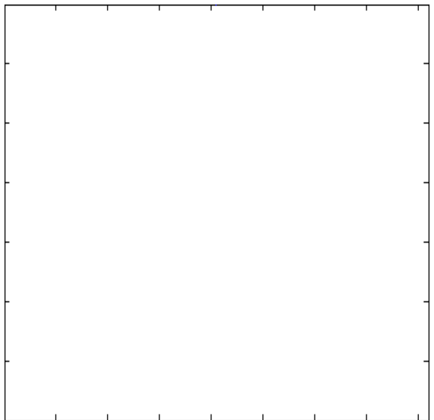
$\beta = -1 \quad \varepsilon_n = 0$



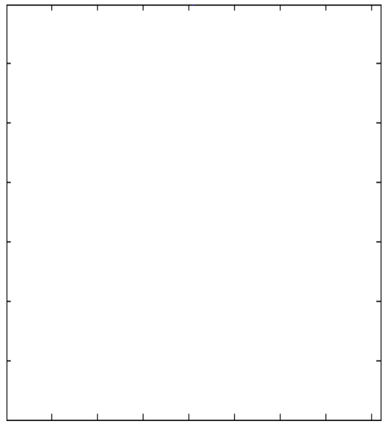
$\beta = 0 \quad \varepsilon_n \neq 0$



$\beta = 1 \quad \varepsilon_n \neq 0$



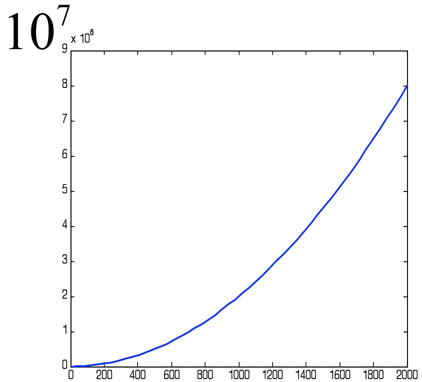
$\beta = -1 \quad \varepsilon_n \neq 0$



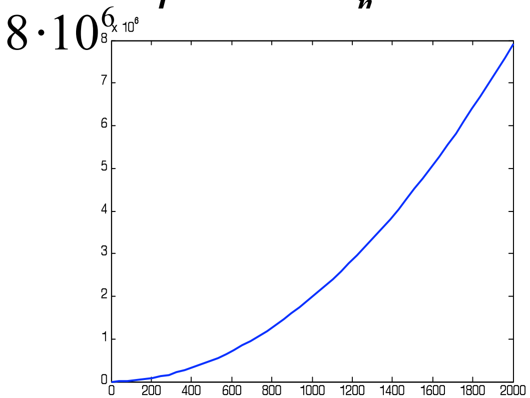
$\xi \approx 6$

$$m_2(t) = \sum_n n^2 \left|\psi_n\right|^2$$

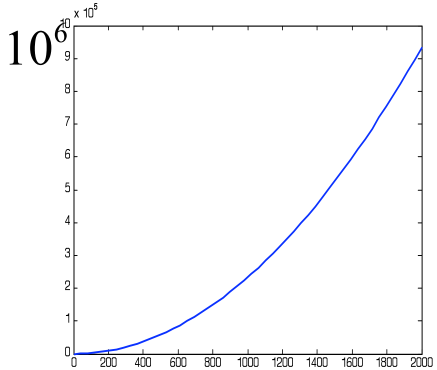
$\beta = 0 \qquad \varepsilon_n = 0$



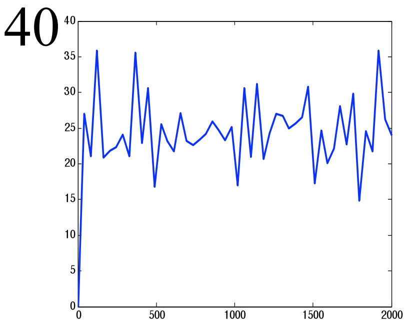
$\beta = 1 \qquad \varepsilon_n = 0$



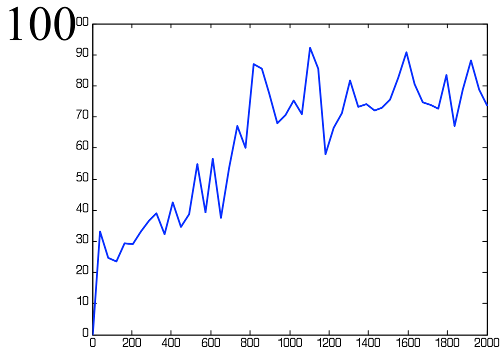
$\beta = -1 \qquad \varepsilon_n = 0$



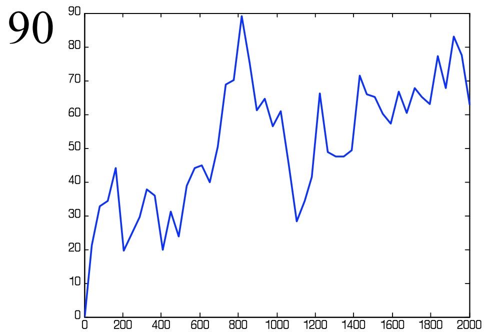
$\beta = 0 \qquad \varepsilon_n \neq 0$



$\beta = 1 \qquad \varepsilon_n \neq 0$



$\beta = -1 \qquad \varepsilon_n \neq 0$



$\beta = 0 \Rightarrow$ localization

Does Localization Survive the Nonlinearity???

Work in progress, several open questions

Aim: find a clear estimate, at least for a finite (long) time

Does Localization Survive the Nonlinearity???

- Yes, if there is spreading the magnitude of the nonlinear term decreases and localization takes over.
- No, assume wave-packet width is Δx then the relevant energy spacing is $1/\Delta x$, the perturbation because of the nonlinear term is $\beta|\psi|^2 \approx \beta/\Delta x$ and all depends on β (Shepelyansky)
- No, but does not depend on β
- No, but it depends on realizations
- Yes, because some time dependent quasi-periodic localized perturbation does not destroy localization

Does Localization Survive the Nonlinearity?

- No, the NLSE is a chaotic dynamical system.
- No, but localization asymptotically preserved beyond some front that is logarithmic in time

Numerical Simulations

- In regimes relevant for experiments looks that localization takes place
- Spreading for long time (Shepelyansky, Pikovsky, Molina, Kopidakis, Komineas, Flach, Aubry)
- We do not know the relevant space and time scales
- All results in Split-Step
- No control (but may be correct in some range)
- Supported by various heuristic arguments

Pikovsky, Sheplyansky

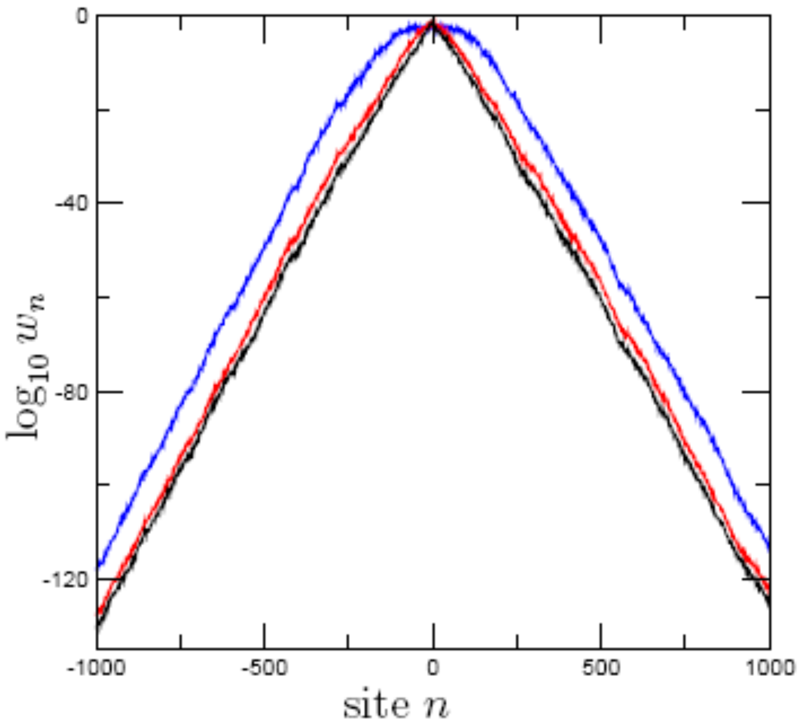


FIG. 2: (color online) Probability distribution w_n over lattice sites n at $W = 4$ for $\beta = 1$, $t = 10^8$ (top blue/solid curve) and $t = 10^5$ (middle red/gray curve); $\beta = 0, t = 10^5$ (bottom black curve; the order of the curves is given at $n = 500$). At $\beta = 0$ a fit $\ln w_n = -(\gamma|n| + \chi)$ gives $\gamma \approx 0.3$, $\chi \approx 4$. The values of $\log_{10} w_n$ are averaged over the same disorder

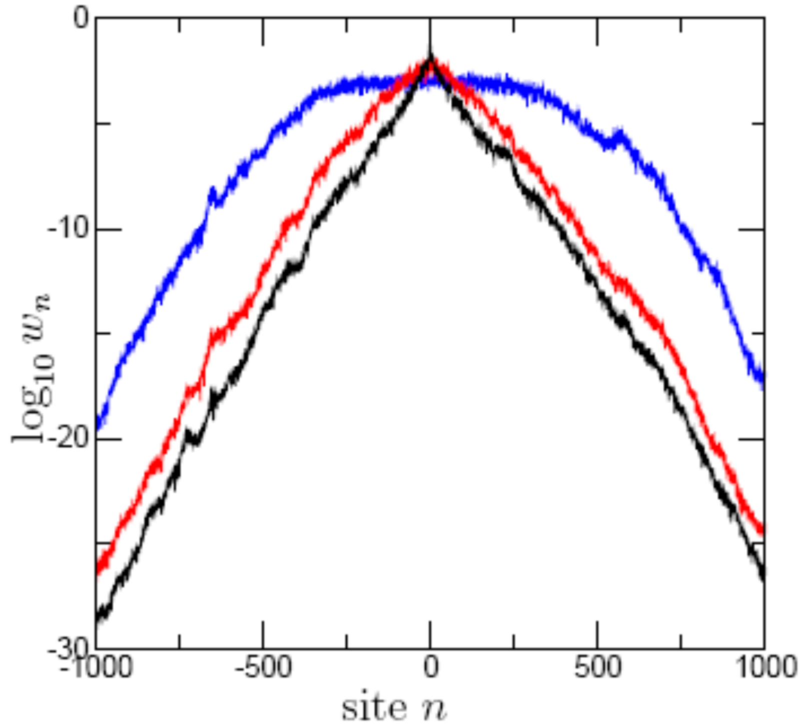
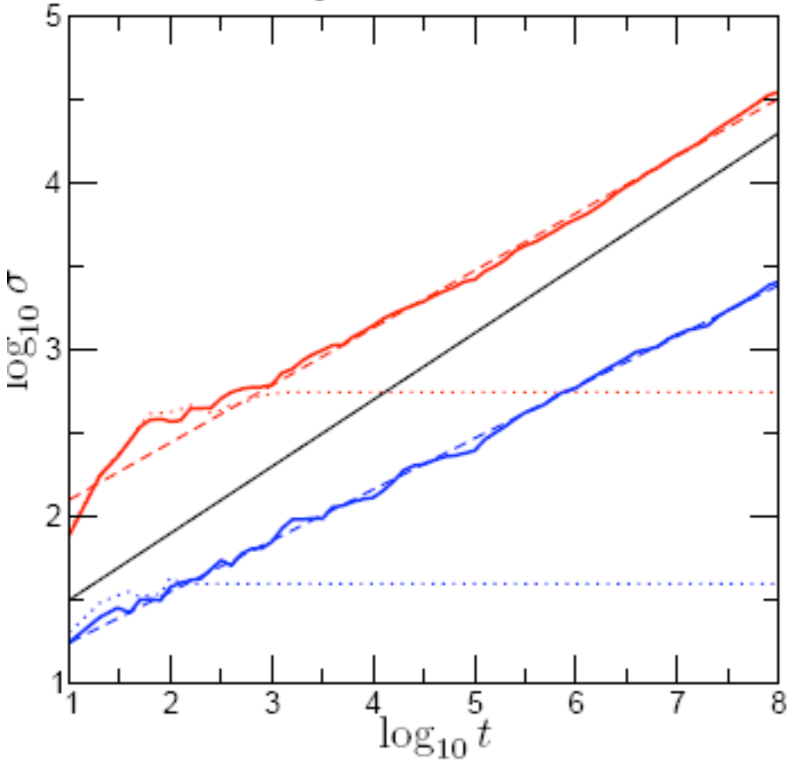


FIG. 3: (color online) Same as in Fig. 2 but with $W = 2$. At $\beta = 0$ a fit $\ln w_n = -(\gamma|n| + \chi)$ gives $\gamma \approx 0.06$, $\chi \approx -3$. The values of $\ln w_n$ are averaged over the same disorder realizations as in Fig. 1.

Slope does not change (contrary to Fermi-Ulam-Pasta)

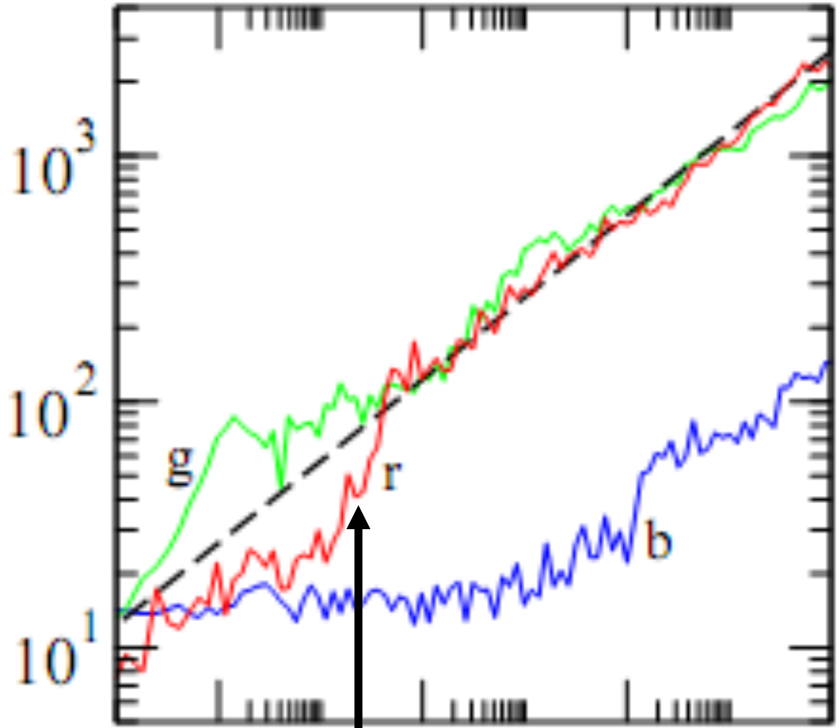
Pikovsky, Shepelyansky

$$\log < x^2 >$$



S.Flach, D.Krimer and S.Skokos

$$\log < x^2 >$$



t

t_s

β b, g, r
0.1, 1, 5

Test of some arguments

Modified NLSE

H. Veksler, Y. Krivolapov, SF

$$i\frac{\partial}{\partial t}\psi = \mathcal{H}_o\psi + \beta\left|\psi\right|^p\psi$$

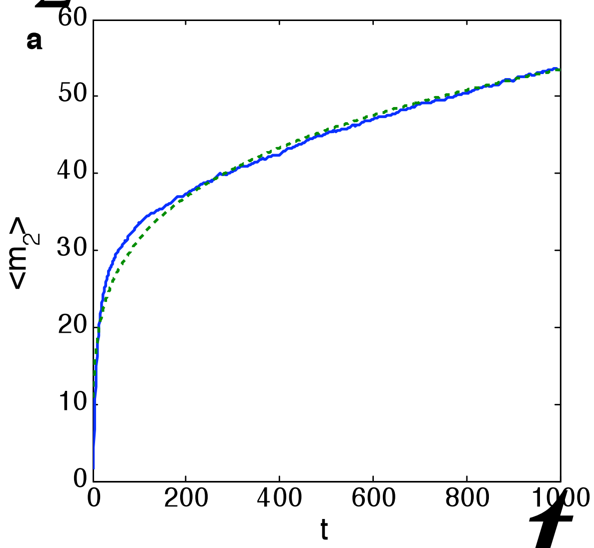
$$m_2 = \left\langle x^2 \right\rangle \sim t^\alpha$$

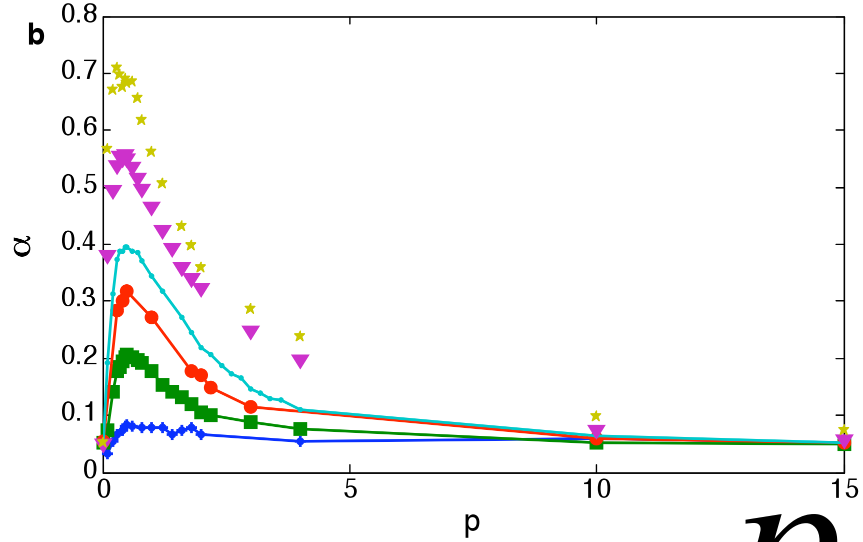
Shepelyansky-Pikovsky original arguments –No spreading for $p > 2$

Flach, Krimer and Sokos $\alpha = \frac{1}{p+1}$

$\alpha = 1$ for $p = 0$

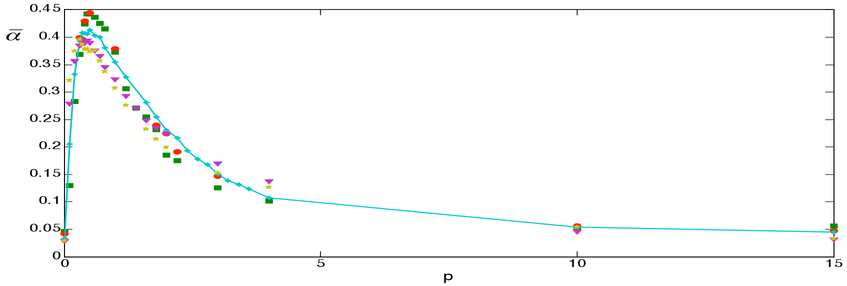
m_2 α





t p

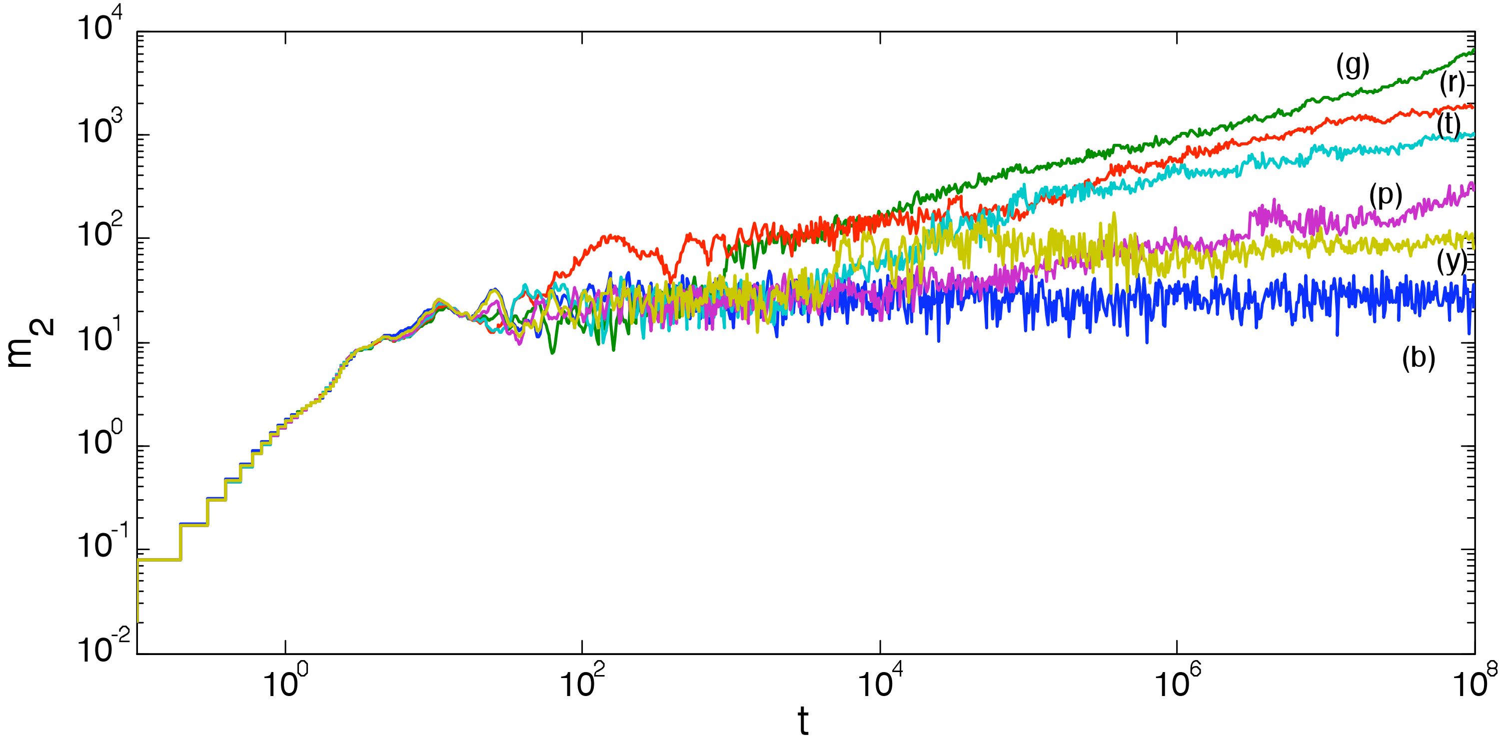
$\bar{\alpha}$



$$\bar{\alpha} = c_1(\beta)\alpha + c_2(\beta)$$

p

m_2



t

p= 1.5, 2, 2.5, 4, 8, 0 (top to bottom)

Also M. Mulansky

Summary of test:

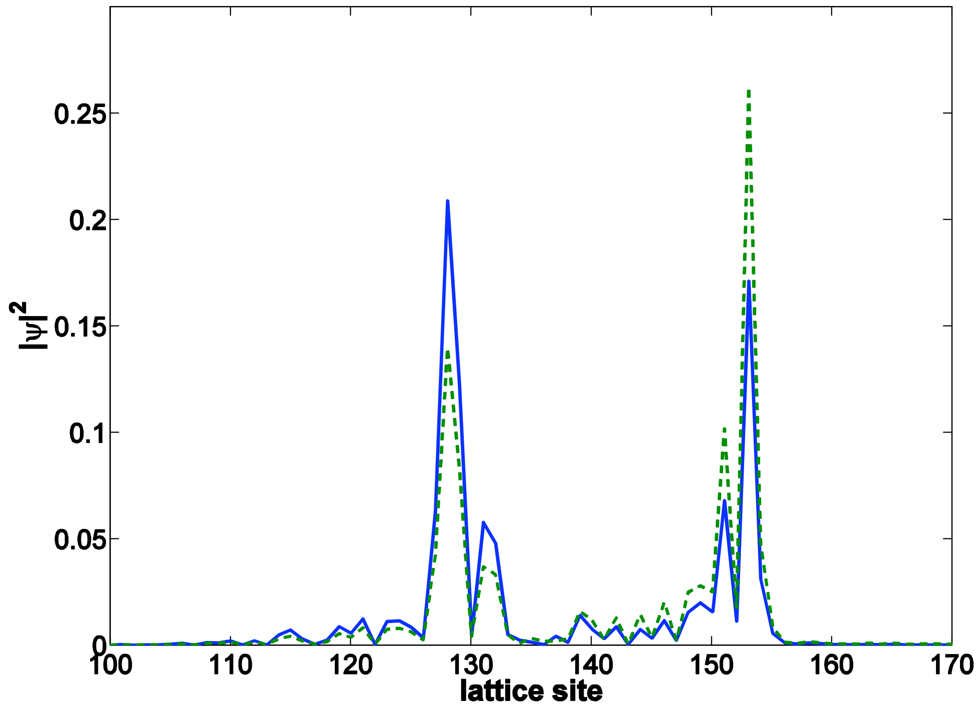
1. Nothing happens at $p = 2$
2. approach to localization at $p = 0$

Singular limit - crossover verified

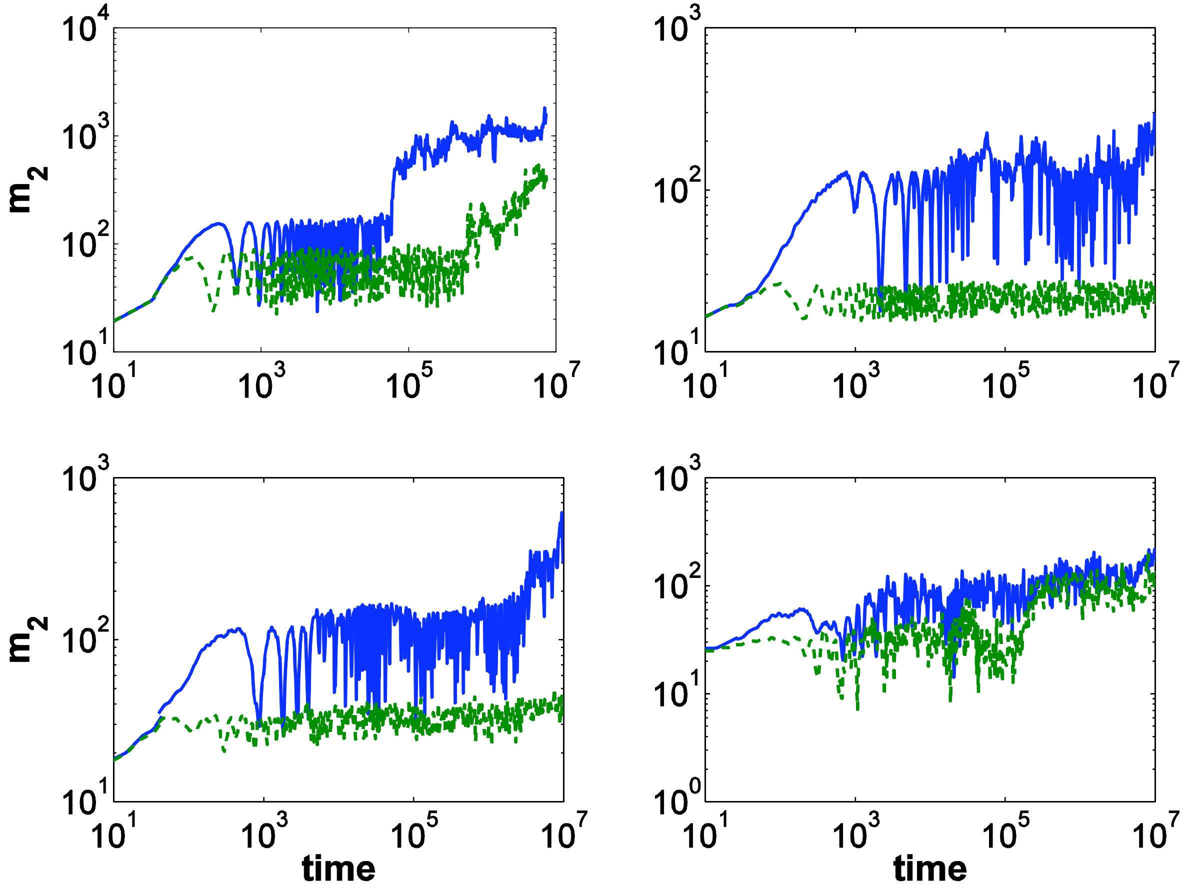
These theories may have a range of validity

The role of double-humped states

A pair of double humped states for the linear system $(\beta = 0)$. The states are marked with blue solid line and green dashed line.



The second moment as function of time for a representative double humped (solid blue) and broken (dashed green) realizations for wave packets started in the vicinity of O.



Effective Noise Theories

- D. Shepeyansky and A. Pikovsky
- Ch. Skokos, D.O. Krimer, S. Komineas and S. Flach

$$\psi\left(x,t\right)=\sum_{m} c_m\left(t\right)e^{-iE_m t}u_m\left(x\right)$$

$$i\frac{\partial}{\partial t}c_n=\beta\sum_{m_1,m_2,m_3} V_n^{m_1,m_2,m_3}c_{m_1}{}^{*}c_{m_2}c_{m_3}e^{i(E_n+E_{m_1}-E_{m_2}-E_{m_3})t}$$

Overlap $\qquad V_n^{m_1,m_2,m_3}=\sum_x u_n(x)u_{m_1}(x)u_{m_2}(x)u_{m_3}(x)$

$$\left|V_n^{m_1m_2m_3}\right|\leq[const]e^{-\frac{1}{3}\gamma\left(\left|x_n-x_{m_1}\right|+\left|x_n-x_{m_2}\right|+\left|x_n-x_{m_3}\right|\right)}$$

of the range of the localization length $\quad \xi$

$$i\frac{\partial}{\partial t}c_n = \beta \sum_{m_1,m_2,m_3} V_n^{m_1,m_2,m_3} c_{m_1}{}^* c_{m_2} c_{m_3} e^{i(E_n + E_{m_1} - E_{m_2} - E_{m_3})t}$$

Assume $\left|c_{m_1}^2\right| \approx \left|c_{m_2}^2\right| \approx \left|c_{m_3}^2\right| \approx \rho$ **initially** $\left|c_n^2\right| \ll \rho$

$$i\frac{\partial}{\partial t}c_n \approx P\beta\rho^{3/2} f(t)$$

$f(t)$ **Random uncorrelated**

Assume $P = A\beta\rho$ **!!!**

$$\left\langle \left|c_n^2\right| \right\rangle = P^2\beta^2\rho^3 t$$

Equilibration time

Equilibrium $\left\langle \left|c_n^2\right| \right\rangle = \rho$

$$T_{eq} = \frac{1}{A^2\beta^4\rho^4}$$

$$D = \frac{1}{T_{eq}} = A^2 \beta^4 \rho^4$$

$$\frac{1}{\rho^2} \sim m_2 = Dt = A^2 \beta^4 \rho^4 t$$

$$\frac{1}{\rho^2} \sim m_2 = \left[A^2 \beta^4 t \right]^{1/3}$$

Consistent $T_{eq} \sim t^{2/3} \ll t$

Can it go on forever?

Perturbation Theory

The nonlinear Schroedinger Equation on a Lattice in 1D

$$i\frac{\partial}{\partial t}\psi = \mathcal{H}_o\psi + \beta\left|\psi\right|^2\psi$$

$$\mathcal{H}_0\psi(x) = -\Big(\psi(x+1)+\psi(x-1)\Big)+\varepsilon(x)\psi(x)$$

ε_n random $\longrightarrow$ $\mathcal{H}_0$ Anderson Model

Eigenstates $\mathcal{H}_0 u_m(x) = E_m u_m(x)$

$$\psi(x,t) = \sum_m c_m(t)e^{-iE_m t}u_m(x)$$

The states are indexed by their centers of localization

u_m Is a state localized near x_m

$$\left|u_m(x)\right| \le D_{\omega,\varepsilon} e^{\varepsilon|x_m|} e^{-\gamma|x-x_m|}$$

$\omega -$ realization

for realizations of measure $1-\delta$

$$\left|D_{\omega,\varepsilon}\right| \le D_{\delta,\varepsilon} < \infty$$

This is possible since nearly each state has a Localization center and in a box size M there are approximately M states

Indexing by "center of mass" more stable

$$i\frac{\partial}{\partial t}c_n = \beta \sum_{m_1,m_2,m_3} V_n^{m_1,m_2,m_3} c_{m_1}{}^* c_{m_2} c_{m_3} e^{i(E_n+E_{m_1}-E_{m_2}-E_{m_3})t}$$

Overlap $$V_n^{m_1,m_2,m_3} = \sum_x u_n(x)u_{m_1}(x)u_{m_2}(x)u_{m_3}(x)$$

$$\left|V_n^{m_1m_2m_3}\right| \le V_\delta^{\varepsilon,\varepsilon'} e^{\varepsilon\left(|x_n|+|x_{m_1}|+|x_{m_2}|+|x_{m_3}|\right)} e^{-\frac{1}{3}\left(\gamma-\varepsilon'\right)\left(|x_n-x_{m_1}|+|x_n-x_{m_2}|+|x_n-x_{m_3}|\right)}$$

of the range of the localization length ξ

perturbation expansion

$$c_n(t) = c_n{}^{(o)} + \beta c_n{}^{(1)} + \beta^2 c_n{}^{(2)} + \ldots\ldots + \beta^{N-1} c_n{}^{(N-1)} + \beta^N Q_n$$

Iterative calculation of $c_n{}^{(l)}$

start at $c_n^{(0)} = c_n(t=0) = \delta_{n0}$

start at $c_n^{(0)} = \delta_{n0}$

Step 1

$$\frac{\partial}{\partial t} c_n^{(1)} = -iV_n^{000} e^{i(E_n - E_0)t}$$

$n \neq 0 \longrightarrow c_n^{(1)} = V_n^{000}\left(\frac{1 - e^{i(E_n - E_0)t}}{E_n - E_0}\right)$

$n = 0 \Rightarrow \frac{\partial}{\partial t} c_0^{(1)} = -iV_0^{000}$

Secular term to be removed by $E_n \Rightarrow E_n' = E_n + \beta E_n^{(1)} + \ldots.$

here $E_n^{(1)} = V_0^{000}$

Secular terms can be removed in all orders by

$$E_n \Rightarrow E_n^{'} = E_n + \beta E_n^{(1)} + \beta^2 E_n^{(2)} + \ldots\ldots$$

New ansatz

$$\psi_n(t) = \sum_m c_m(t) e^{-iE_m^{'} t} u_n^m$$

The problem of small denominators

$$n \neq 0 \longrightarrow c_n^{(1)} = V_n^{000} \left(\frac{1 - e^{i(E_n - E_0)t}}{E_n - E_0} \right)$$

The Aizenman-Molchanov approach (fractional moments)

Step 2

$$i\partial_t c_n^{(2)} = \sum_{m_1,m_2,m_3} V_n^{m_1 m_2 m_3} c_{m_1}^{*(1)} c_{m_2}^{(0)} c_{m_3}^{(0)} e^{i\left(E_n + E_{m_1} - E_{m_2} - E_{m_3}\right)t} +$$

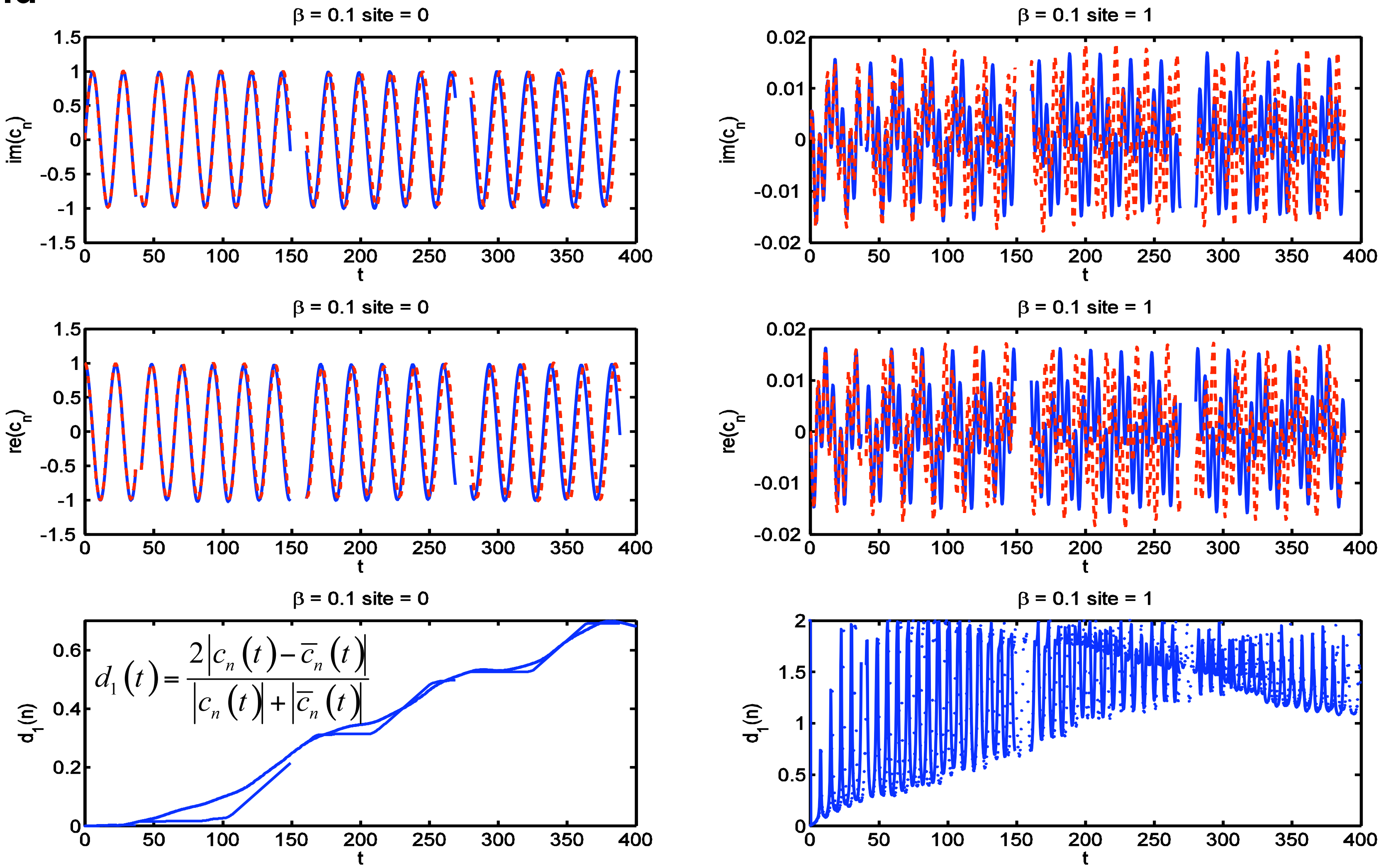
$$2\sum_{m_1,m_2,m_3} V_n^{m_1 m_2 m_3} c_{m_1}^{*(0)} c_{m_2}^{(1)} c_{m_3}^{(0)} e^{i\left(E_n + E_{m_1} - E_{m_2} - E_{m_3}\right)t}$$

$$E_n^{(1)} = 2V_n^{n00} \qquad n \neq 0$$

$$
i\partial_t c_n^{(2)} = \begin{cases} \displaystyle\sum_{m\neq 0} \frac{V_0^{m00}V_m^{000}}{E'_m - E'_0}\left(e^{i\left(E'_m - E'_0\right)t} + 2e^{i\left(E'_0 - E'_m\right)t}\right) & n = 0 \\ \displaystyle\frac{2V_n^{n00}V_n^{000}}{E'_n - E'_0} e^{i\left(E'_n - E'_0\right)t} + & \\ \displaystyle\sum_{m\neq 0,n} \frac{2V_n^{m00}V_m^{000}}{E'_m - E'_0} e^{i\left(E'_n - E'_m\right)t} + & \\ \displaystyle\sum_{m\neq 0} \frac{V_n^{m00}V_m^{000}}{E'_m - E'_0}\left(e^{i\left(E'_n + E'_m - 2E'_0\right)t} - 3e^{i\left(E'_n - E'_0\right)t}\right) & n \neq 0 \end{cases}
$$

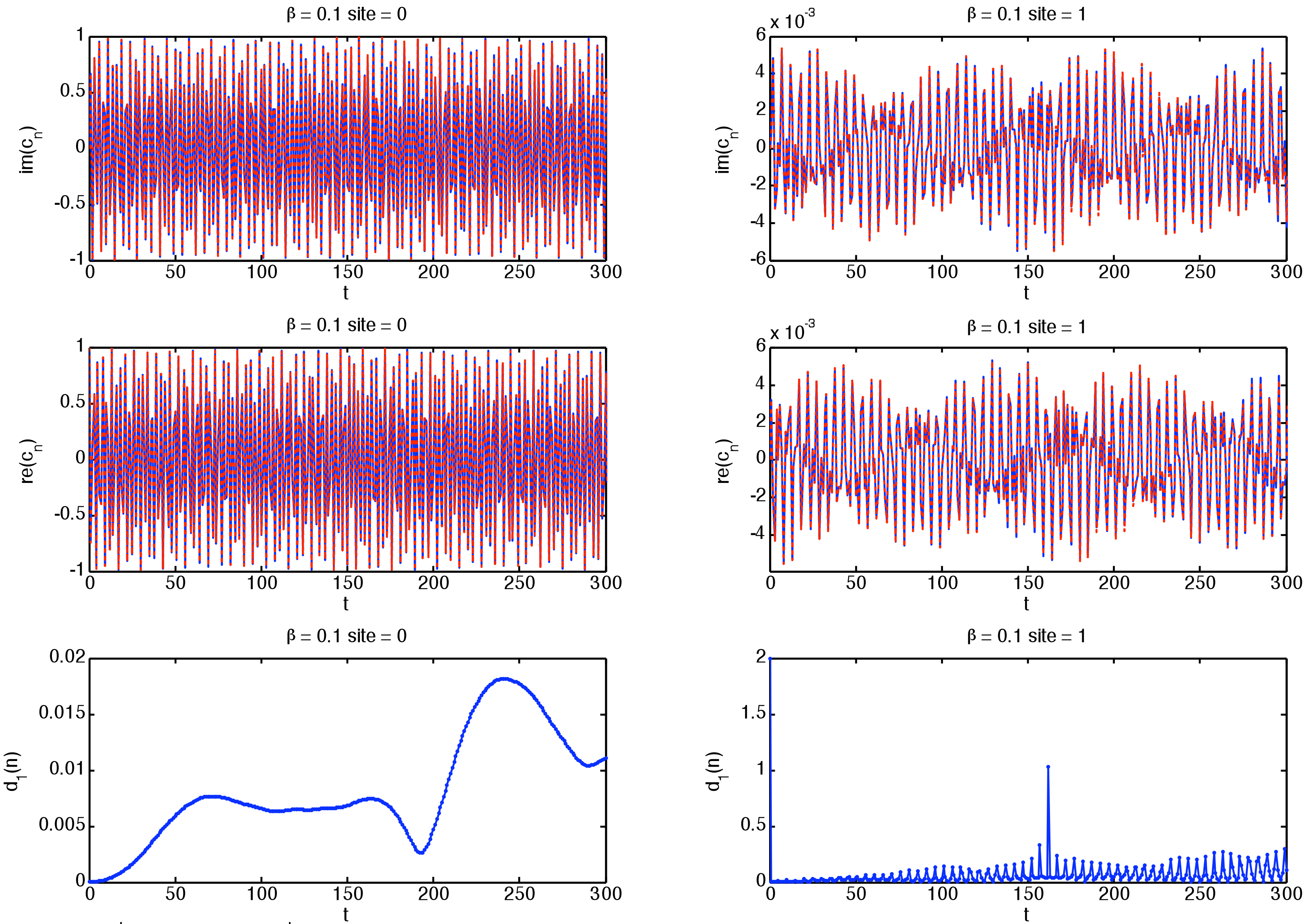
Secular term removed

Second order



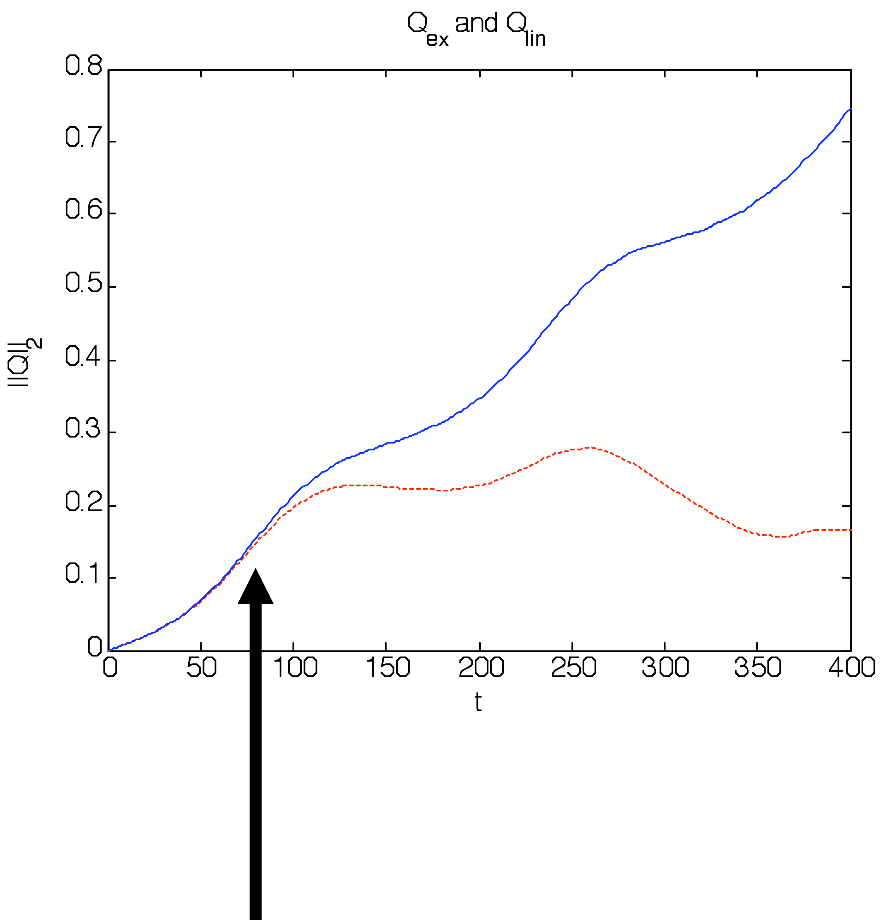
In the first and the second rows of the figure are presented the imaginary and real parts, respectively, of the numerical solution of (1.1) (blue, solid) and the perturbative approximation (red, dashed) as function of time for c_0 (left column) and c_1 (right column). In the third row the relative difference, $d_1(t)$, is plotted. The parameters of the plot are, $J = 0.25$, $\Delta = 1$ and $\beta = 0.1$ and lattice size of 128. See (1.1) for the definition of the constants.

Order 4 red=approximate, blue exact

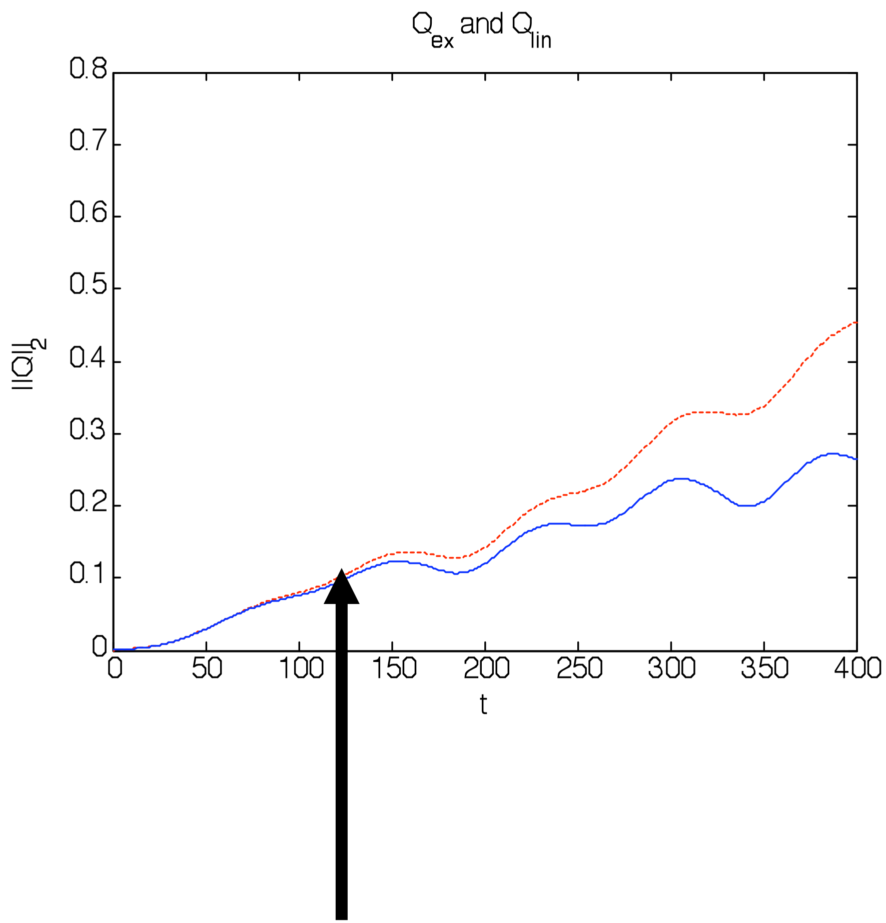


$$d_1(t) = \frac{2\left|c_n(t) - \bar{c}_n(t)\right|}{\left|c_n(t)\right| + \left|\bar{c}_n(t)\right|}$$

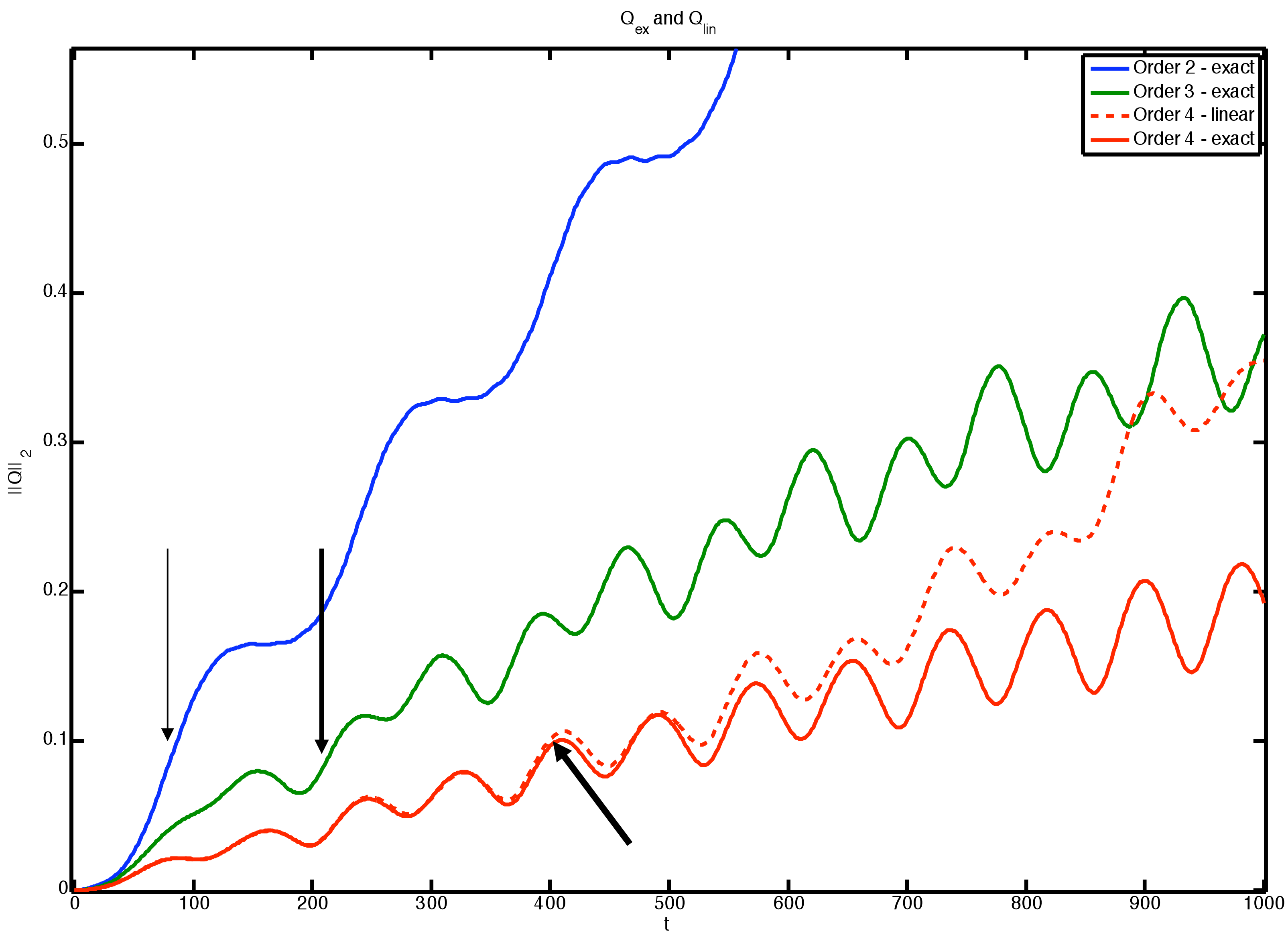
Second order



Third order



Blue=exact, red=linear

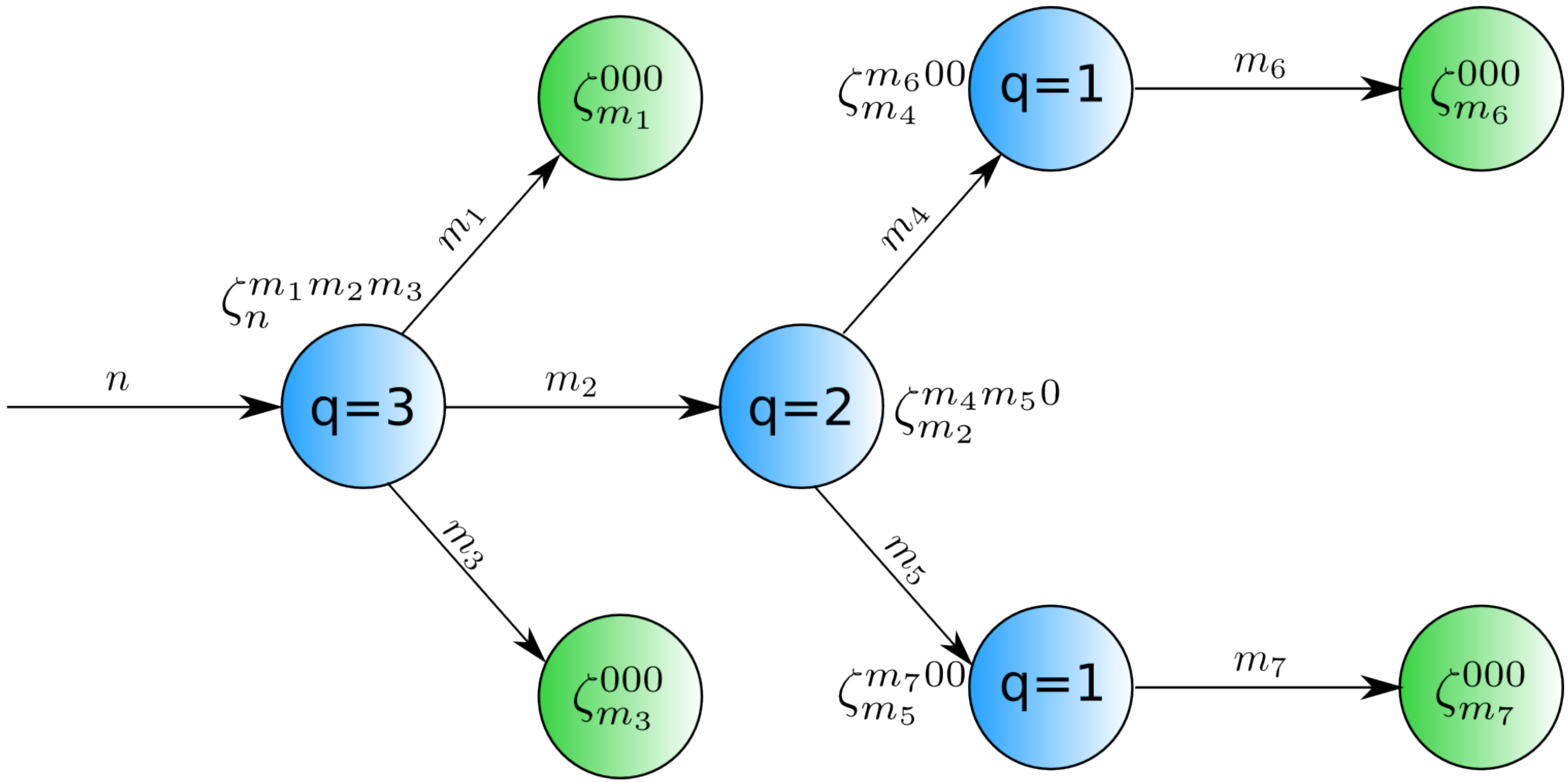


Q_{ex} and Q_{lin}
Order 2 - exact
Order 3 - exact
Order 4 - linear
Order 4 - exact
$||Q||_2$
t
0
0.1
0.2
0.3
0.4
0.5
0
100
200
300
400
500
600
700
800
900
1000

Aim: to use perturbation theory to obtain a numerical solution that is controlled a posteriori

Perturbation theory steps

- Expansion in nonlinearity
- Removal of secular terms
- Control of denominators
- Probabilistic bound on general term
- Control of remainder



An example of a graph that is used to construct the general term. The graph describes an 8-th order term, $\zeta_n^{m_1 m_2 m_3} \zeta_{m_1}^{000} \zeta_{m_3}^{000} \zeta_{m_2}^{m_4 m_5 0} \zeta_{m_4}^{m_6 00} \zeta_{m_5}^{m_7 00} \zeta_{m_6}^{000} \zeta_{m_7}^{000}$.

Bounding the General Term

$$i\partial_t c_n^{(k)} = -\sum_{s=0}^{k-1} E_n^{(k-s)} c_n^{(s)} +$$

$$+\sum_{m_1 m_2 m_3} V_n^{m_1 m_2 m_3} \left[\sum_{r=0}^{k-1} \sum_{s=0}^{k-1-r} \sum_{l=0}^{k-1-r-s} c_{m_1}^{(r)*} c_{m_2}^{(s)} c_{m_3}^{(l)} \right] e^{i\left(E_n^{'} + E_{m_1}^{'} - E_{m_2}^{'} - E_{m_3}^{'}\right)t}$$

results in products of the form

$$\overbrace{\zeta_n^{m_1 m_2 m_3} \zeta_{m_1}^{m_4 m_5 m_6} \dots \zeta_{m_5}^{000} \zeta_{m_6}^{000}}^{k \text{ terms}}$$

$$\zeta_n^{m_1 m_2 m_3} \equiv \frac{V_n^{m_1 m_2 m_3}}{E_n^{'} - \left\{E^{'}\right\}_{m_i}}$$

Using Cauchy-Schwartz inequality for member of product

$$\left\langle \left| \zeta_n^{m_1 m_2 m_3} \right|^s \right\rangle \le \left\langle \frac{1}{\left| E_n^{'} - \left\{ E^{'} \right\}_{m_i} \right|^{2s}} \right\rangle^{1/2} \left\langle \left| V_n^{m_1 m_2 m_3} \right|^{2s} \right\rangle^{1/2}$$

where $0 < s < \dfrac{1}{2}$

Small denominators

If the shape of the squares of the eigenfunctions sufficiently different, using a recent result of Aizenman and Warzel we propose

Conjecture 3. For the Anderson model, the joint distribution of R eigenenergies is bounded,

$$p\left(E_1, E_2, \ldots, E_R\right) \leq \bar{D}_R$$

where $\bar{D}_R \propto R! < \infty$.

Corollary 4. Given $0 < s < 1$, for $f = \sum_{k=1}^{R} c_k E_{i_k}$, where c_k are integers the following mean is bounded from above

$$\left\langle \frac{1}{|f|^s} \right\rangle \leq D_R < \infty.$$

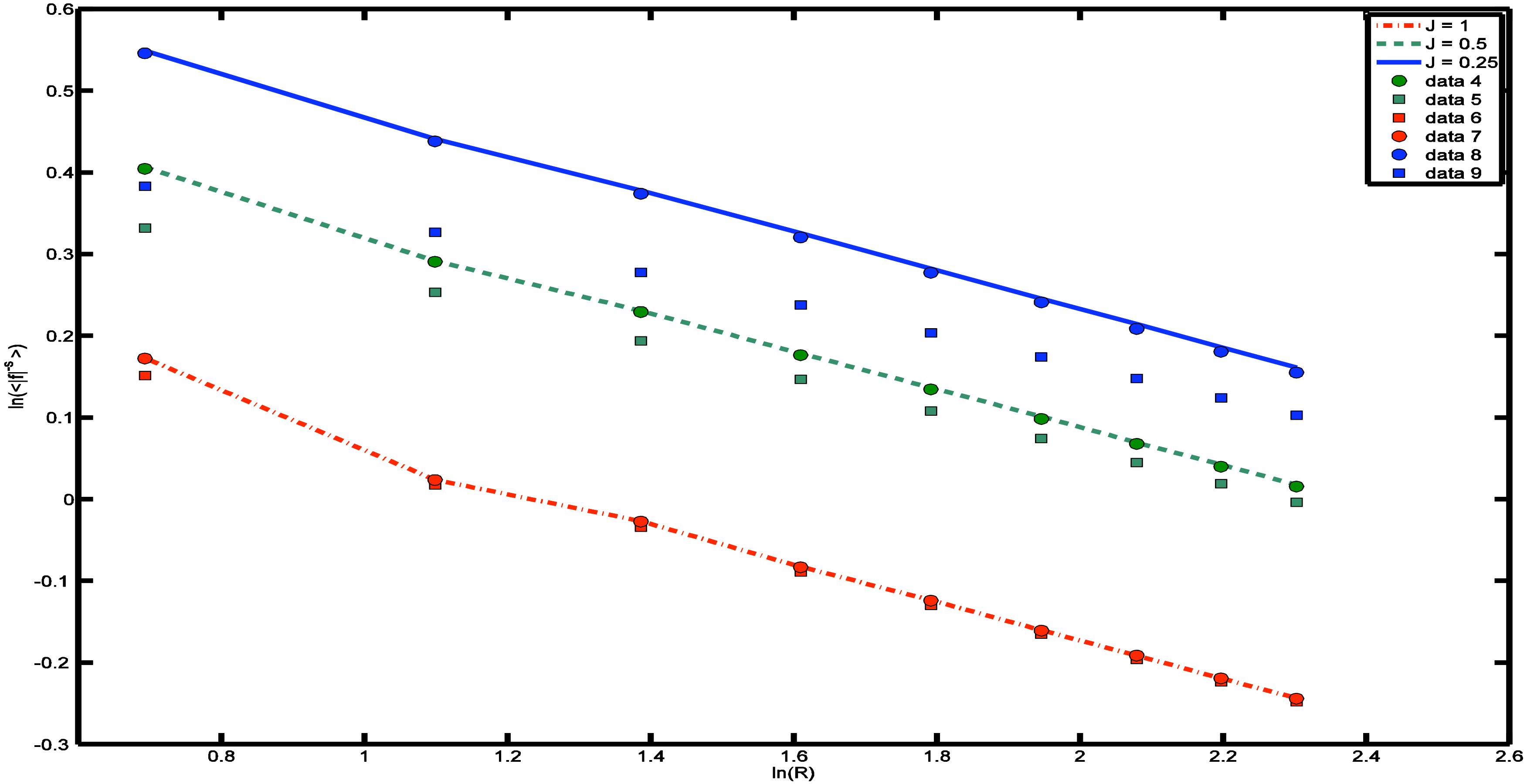
where $D_R \propto \bar{D}_R$.

Assuming sufficient independence between states localized far away

Conjecture 5. In the limit of $R \to \infty$, for $0 < s < 1$ and for $f = \sum_{k=1}^{R} c_k E_{i_k}$, where c_k are integers

$$\left\langle \frac{1}{|f|^s} \right\rangle \asymp \frac{1}{R^{s/2}}$$

Conjecture 6. Corollary 4 and Conjecture 5 hold also if the E_i are replaced by the renormalized energies E'_i.



The logarithm of $\left\langle |f|^{-1/2} \right\rangle$ as a function of the logarithm of R. The lines designate denominators with $\beta = 0$, with the solid line (blue) is for $J = 0.25$ the dashed line (green) is for $J = 0.5$ and the dot-dashed line (red) is for $J = 1$. The solid circles and the squares are data with $\beta = 1$, and E'_n calculated up to the second in β, such that different colors represent different J, in the similar manner as for the lines. The solid squares are for parameters similar to the ones with the solid circles, but with the restriction that at least one of the states that corresponds to E'_n which is localized near the origin.

Probabilistic Bound on general term

Theorem 10. For a given k and $\delta, \varepsilon, \varepsilon', \eta' > 0$

$$\Pr\left(\left|c_n^{(k)}\right| \geq \left(F_\delta^{(k)}\right)^k e^{ck^2+c'k} e^{-(\gamma-\varepsilon-\varepsilon'-\eta')|x_n|}\right) \leq e^{-c'} e^{-\eta'|x_n|/k}.$$

where $F_\delta^{(k)}$ which is proportional to D_δ and c and c' are constants. It can be seen that $c \backsimeq 2$ and later we set $c' = cN$.

The Bound deteriorates with order

The remainder

$$c_n(t) = c_n^{(o)} + \beta c_n^{(1)} + \beta^2 c_n^{(2)} + \ldots\ldots + \beta^{N-1} c_n^{(N-1)} + \beta^N Q_n$$

Bound by a bootstrap argument

$$\left|Q_n(t)\right| \leq M(t) \cdot e^{-(\gamma - \varepsilon - \varepsilon')|x_n|} = 2t \cdot C_\delta e^{6cN^2} e^{-(\gamma - \varepsilon - \varepsilon')|x_n|}$$

The Bound on the remainder

$$\left|\beta^N Q_n\right| \leq const \cdot e^{6cN^2+N\ln\beta+\ln t} e^{-(\gamma-\varepsilon-\varepsilon')|x_n|}.$$

Note that for a given t and β there is an optimum N for which the remainder is minimal. Additionally, for any fixed time and order N, $\lim_{\beta\to 0}\left|\beta^N Q_n\right|/\beta^{N-1}=0$, which shows that the series is in fact an asymptotic one

One can show that for strong disorder C_δ and the constant are multiplied by

$$f(\gamma)\xrightarrow[\gamma\to\infty]{}0$$

It is probably proportional to $\exp(-\gamma)$

Front logarithmic in time $\bar{x}\propto\frac{1}{\gamma}\ln t$

Bound on error

- Solve linear equation for the remainder of order N
- If bounded to time t_0 perturbation theory accurate to that time.
- Order of magnitude estimate $\beta^N t_0 \sim 1$ if asymptotic $\beta^N N! \sim 1$ hence $t_0 \sim N!$ for optimal order (up to constants).

Perturbation theory steps

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Summary

1. A perturbation expansion in β was developed
2. Secular terms were removed
3. A bound on the general term was derived
4. Results were tested numerically
5. A bound on the remainder was obtained, indicating that the series is asymptotic.
6. For limited time tending to infinity for small nonlinearity, front logarithmic in time $\overline{x} \propto \ln t$ or $(\ln t)^2$
7. Improved for strong disorder

Open problems

1. Can the logarithmic front be found for arbitrary long time?
2. Is the series asymptotic or convergent? Under what conditions?
3. Can the series be re-summed?
4. Can the bound on the general term be improved?
5. Rigorous proof of the various conjectures on the linear, Anderson model
6. How to use to produce an a posteriori bound?

References

- S. Fishman, Y. Krivolapov and A. Soffer, Perturbation Theory for the Nonlinear Schroedinger Equation with a Random Potential, To be published in Nonlinearity (arXiv 4901.4951)
- References 33-40 there