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Pseudochaos and Stable-Chaos in Statistical Mechanics and Quantum Physics

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On classical and semiclassical properties of the triangle map

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On classical and semiclassical properties of the triangle map

Workshop ”Pseudochaos and Stable-Chaos in Statistical Mechanics and Quantum Physics”

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Our concerns

- exploring and understanding two-dimensional discrete dynamical systems with polynomial decay of correlations
- understanding long time evolution and spectral statistics for (zero entropy) quantum systems

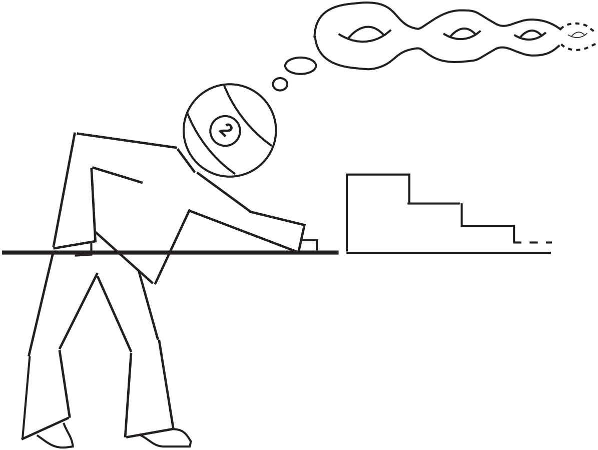
Our two main models

1) the Triangle Map

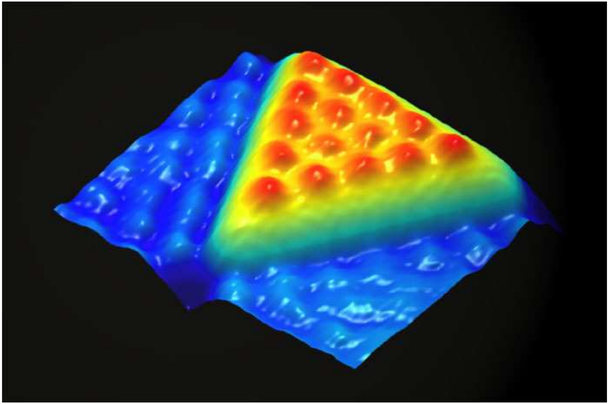
On ergodic and mixing properties of the triangle map,

L. Bunimovich, —-, M. Horvat, S. Isola, T. Prosen (*Physica D* vol. **238**, 395-415 (2009))

What's really in our mind

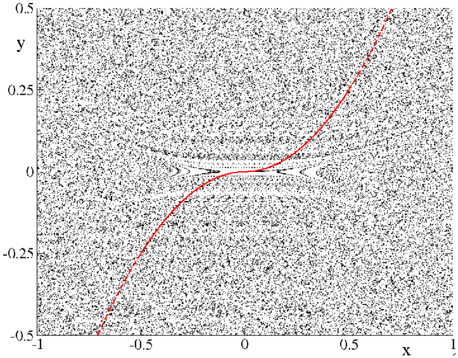


What's really in our mind (www.nano.psu.edu)



the second model

2) Two dimensional extension of the Pomeau-Manneville 1-d family



–*intermittency in two dimension*, R. Artuso, L. Cavallasca and G. Cristadoro (arXiv:0712.3191v1, 2007)

–*work in progress* with R. Artuso and G. Cristadoro

...we start with the *triangle map*: $\phi : \mathbb{T}^2 \to \mathbb{T}^2$

$$\phi(q,p) = (q + p + \alpha\theta(q) + \beta, p + \alpha\theta(q) + \beta)\,,$$

where $(\alpha, \beta) \in \mathbb{R}^2$ and

$$\theta(q) = \begin{cases} 1 & : \quad q \in [0, \frac{1}{2}) \\ -1 & : \quad \text{otherwise} \end{cases}$$

- the map is area-preserving and piecewise parabolic ($\det J = 1$)
- the 'discontinuity set' $\mathcal{D}$ is the codimension one manifold

$$\mathcal{D} = \{\,(0,p)\,|\,p \in [0,1)\,\} \cup \{\,(\tfrac{1}{2},p)\,|\,p \in [0,1)\,\} =: v(0) \cup v(\tfrac{1}{2})\,.$$

the t-th iteration of the map

$$
\begin{aligned}
q_t(q_0, p_0) &= q_0 + p_0\, t + \frac{\beta}{2} t(t+1) + \alpha \sum_{k=1}^{t-1} S_k \,, \\
p_t(q_0, p_0) &= p_0 + \beta\, t + \alpha\, S_t \,,
\end{aligned}
$$

where

$$
S_t := \sum_{k=0}^{t-1} \theta(q_k) \quad \text{satisfies} \quad S_{t+1} = S_t \pm 1 \;\; |S_t| \le t
$$

An *"almost"* random walk

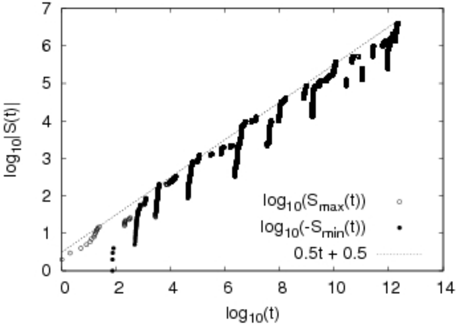


Figure: The extreme values of the sum S_t in the *generic case* $\alpha = 1/e$ and $\beta = (\sqrt{5} - 1)/2$ at $q_0 = 0.1$ and $p_0 = 0.2$.

In the generic case *for all* tested initial points the sum S_t yields the same behaviour as for the standard symmetric random walks.

$\beta = 0$: logarithmic diffusion

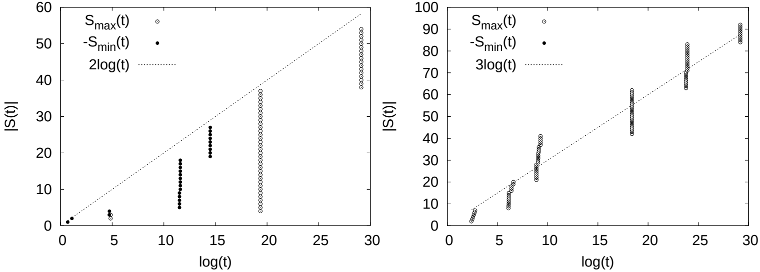
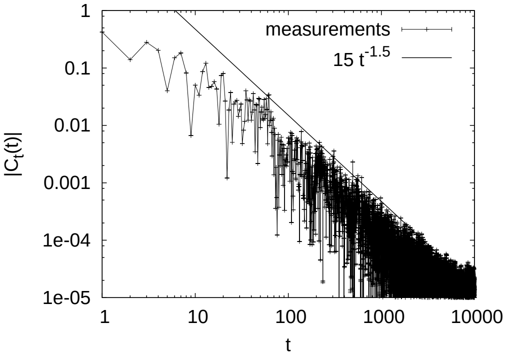


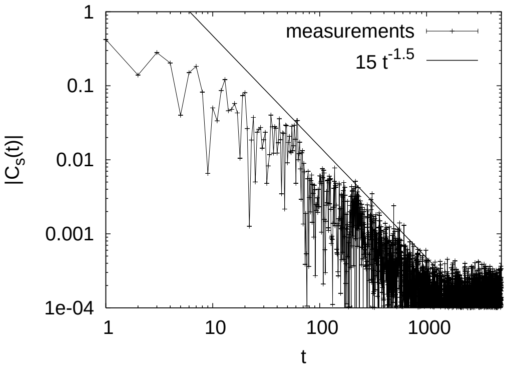
Figure: The points of extreme values $S_{\max}(t)$ and $S_{\min}(t)$ of the sum S_t for $\alpha = \sqrt{2} - 1, (\sqrt{5} - 1)/2$ (a,b) at $q_0 = 0.1$ and $p_0 = 0.2$.

Decay of correlations in the *generic case*: $\alpha, \beta \notin \mathbb{Q}$, $\beta \neq \alpha\mathbb{Z} + \mathbb{Z}$



$$\lim_{N\to\infty} \frac{1}{N-t} \sum_{k=0}^{N-t} f(\phi^k(x_0))f(\phi^{k+t}(x_0)) \approx \mu(f \circ \phi^t \cdot f) = O(t^{-\frac{3}{2}})$$

Decay of correlations in the *generic case*: $\alpha, \beta \notin \mathbb{Q}$, $\beta \neq \alpha\mathbb{Z} + \mathbb{Z}$



$$\lim_{N\to\infty} \frac{1}{N-t} \sum_{k=0}^{N-t} f(\phi^k(x_0)) f(\phi^{k+t}(x_0)) \approx \mu(f \circ \phi^t \cdot f) = O(t^{-\frac{3}{2}})$$

Decay of correlations in the *non generic case*: $\beta = 0$

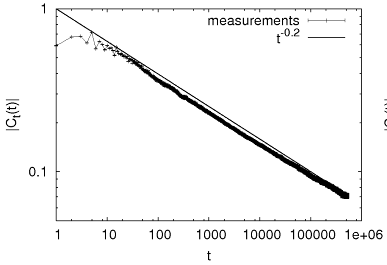


Figure: The auto-correlation $C_t(t)$ using observable $f(q,p) = \sin(2\pi q) + \sin(2\pi p)$ at $\beta = 0$ and $\alpha = \sqrt{2} - 1, e^{-1}$ (a,b), where we take $N = 2^{19}$ and average the result over $m = 10^4$ realisations.

the t-th iteration of the map

$$
\begin{aligned}
q_t(q_0, p_0) &= q_0 + p_0\, t + \frac{\beta}{2}\, t(t+1) + \alpha \sum_{k=1}^{t-1} S_k \ , \\
p_t(q_0, p_0) &= p_0 + \beta\, t + \alpha\, S_t \ ,
\end{aligned}
$$

where

$$
S_t := \sum_{k=0}^{t-1} \theta(q_k) \quad \text{satisfies} \quad S_{t+1} = S_t \pm 1 \ \ |S_t| \le t
$$

$\theta(q)$ is locally constant: the dynamics transforms a horizontal segment to a finite number of horizontal segments,moreover a polygon P is transformed into a finite number of smaller disjoint polygons....

Dynamics of polygons: main questions

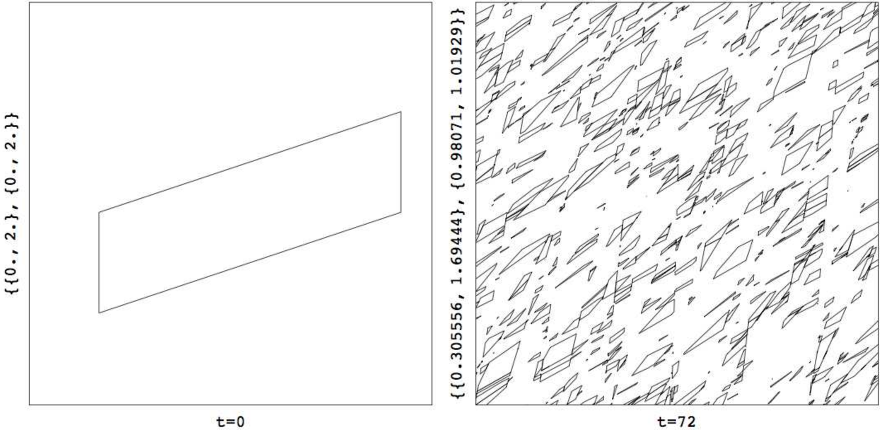
We now consider $\mathcal{A}_t := \phi^t(\mathcal{A})$, where $\mathcal{A} \subset \mathbb{T}^2$ is an arbitrary initial polygon.

After some time t, the set $\mathcal{A}_t$ is composed of many, say $N(t)$, *pieces*.

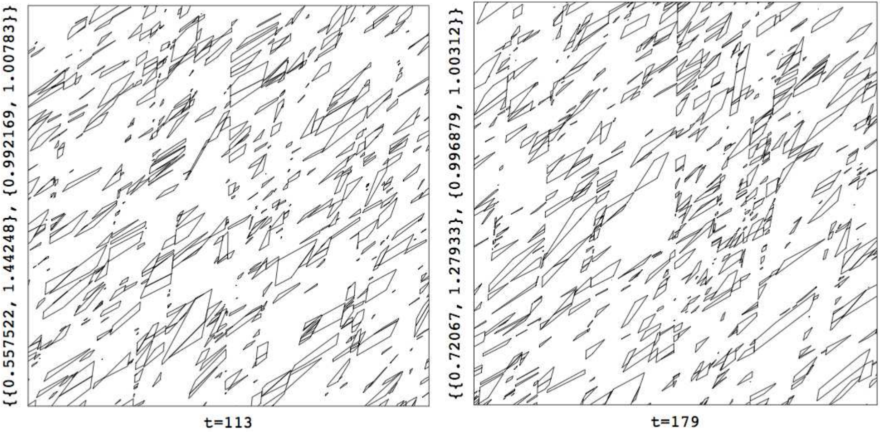
The following main questions are addressed in our numerical experiment:

- do there exist well defined statistical distributions of geometric properties of these little polygonal pieces ?
- what are the scalings of geometric properties of little pieces as t grows? Are asymptotic, $t \to \infty$, statistical distributions universal, i.e. independent of the geometry and volume of the initial set $\mathcal{A}$?
- Are this pieces distributed uniformly in phase space $\mathbb{T}^2$ i.e. that the number of pieces inside an arbitrary ('window') set $\mathcal{B} \subset \mathbb{T}^2$ with area $\mu(\mathcal{B})$ is, after long time t, equal to $\mu(\mathcal{B})N(t)$?

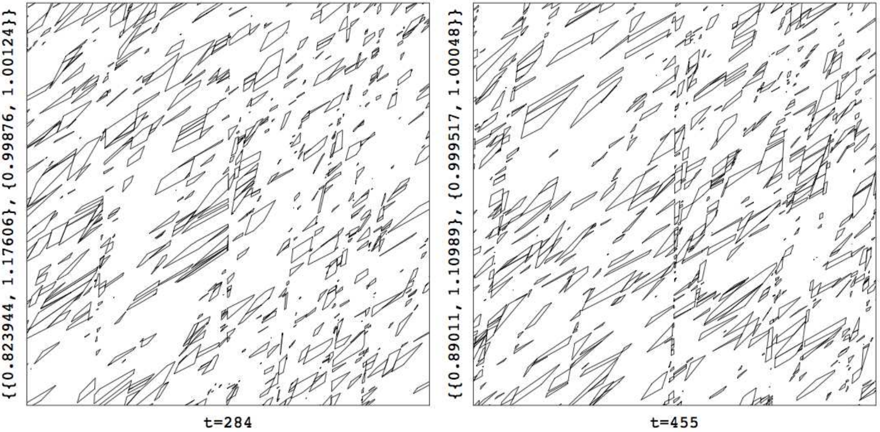
Dynamics of polygons: some numerical results

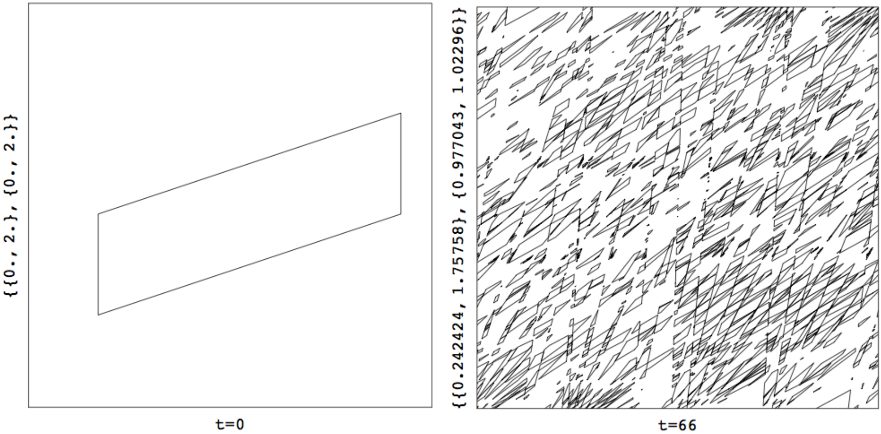


Dynamics of polygons: some numerical results

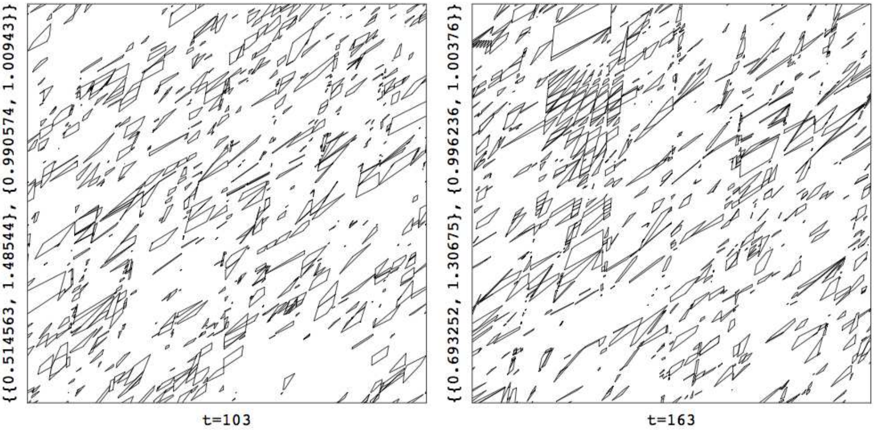


Dynamics of polygons: some numerical results

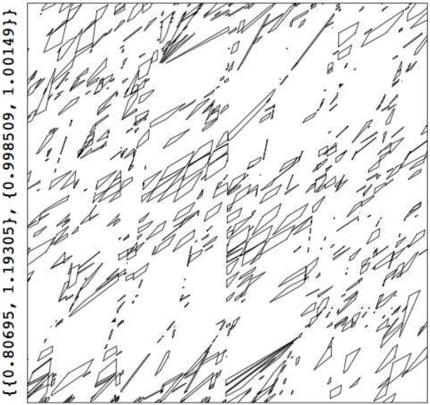



{{0., 2.}, {0., 2.}}
t=0
{{0.242424, 1.75758}, {0.977043, 1.02296}}
t=66

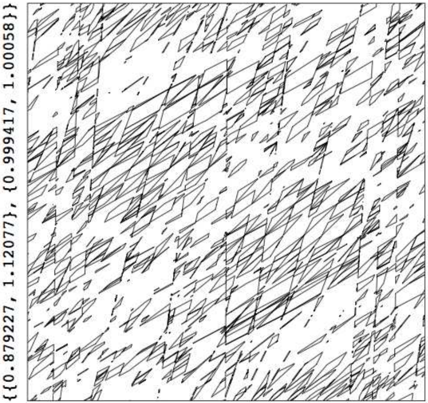
Dynamics of polygons: some numerical results



Dynamics of polygons: some numerical results

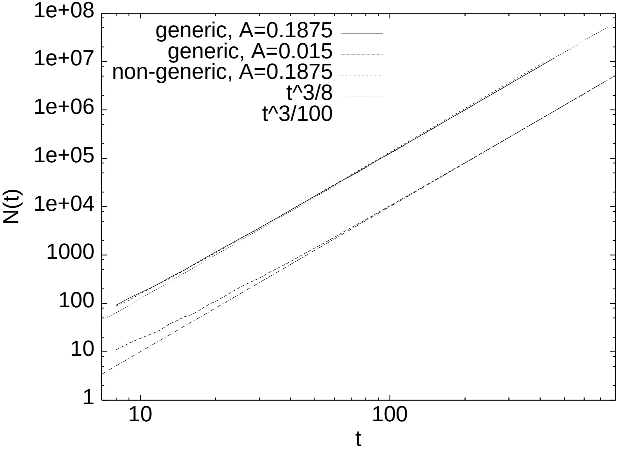


t=259



t=414

The growth of $N(t)$



$$N(t) \sim \widetilde{C} t^3 , \qquad \widetilde{C} \propto \mu(\mathcal{A}) .$$

Statistical distribution of geometric measures

- the area $a := \mu(\mathcal{P})$

$$\langle a \rangle = 30.0/t^3 \,, \qquad \sigma_a = 1.12 \, \langle a \rangle$$

- the q-diameter $\delta_q := q_2 - q_1$

$$\langle \delta_q \rangle = 7.2/t \,, \qquad \sigma_{\delta_q} = 0.59 \, \langle \delta_q \rangle$$

- the p-diameter $\delta_p := p_2 - p_1$

$$\langle \delta_p \rangle = 14.4/t^2 \,, \qquad \sigma_{\delta_p} = 0.52 \, \langle \delta_p \rangle$$

- the average slope of the piece $s := \delta_p / \delta_q$

$$\langle s \rangle = 2.2/t \,, \qquad \sigma_s = 0.37 \, \langle s \rangle$$

Self-similarity of pieces

- the q-diameter $\delta_q := q_2 - q_1$

$$\langle \delta_q \rangle = 7.2/t \,, \qquad \sigma_{\delta_q} = 0.59 \, \langle \delta_q \rangle$$

- the p-diameter $\delta_p := p_2 - p_1$

$$\langle \delta_p \rangle = 14.4/t^2 \,, \qquad \sigma_{\delta_p} = 0.52 \, \langle \delta_p \rangle$$

This suggest the transformation $q' = qt$ and $p' = pt^2$.

Therefore, we introduce t-invariant quantities

$$\chi_a = t^3 a \,, \quad \chi_q = t\delta_q \,, \quad \chi_p = t^2 \delta_p \,, \quad \chi_s = ts \,.$$

and study (for example) the *Cumulative number distributions*

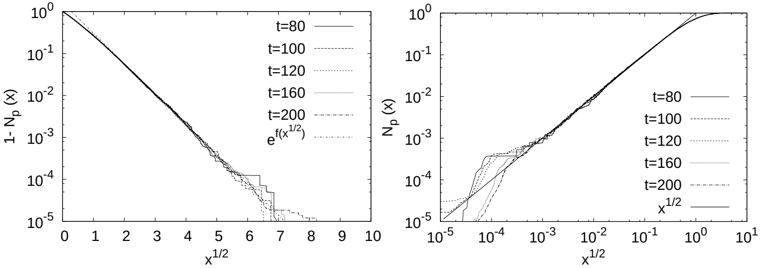
$$N_Q(\chi) = \frac{\#\{Q(\mathcal{P}) \leq \chi\}}{N(t)} \,, \qquad Q \text{ which is one of } \chi_a, \chi_x, \chi_y, \chi_s$$

Qualitative feature of the number distributions $N_Q(\chi)$

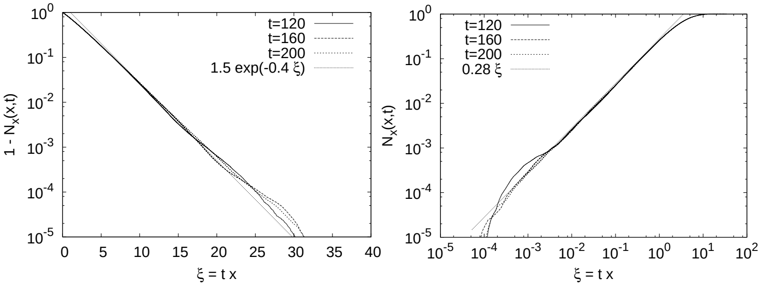
It has been checked with great numerical accuracy that distributions $N_Q(\chi)$ are asymptotically (for large t) independent:

- of time t
- of the size and shape of the initial set $\mathcal{A}$
- of the particular values of generic irrationals α, β.

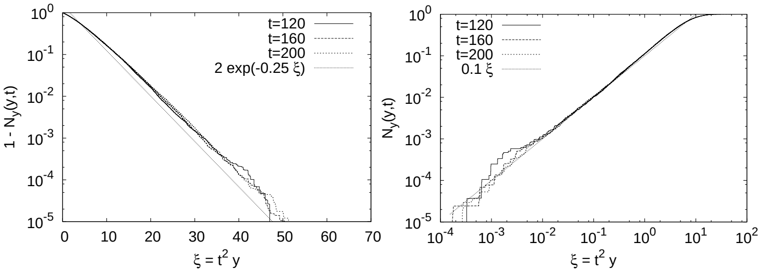
Qualitative feature of the number distributions $N_Q(\chi)$



Qualitative feature of the number distributions $N_Q(\chi)$

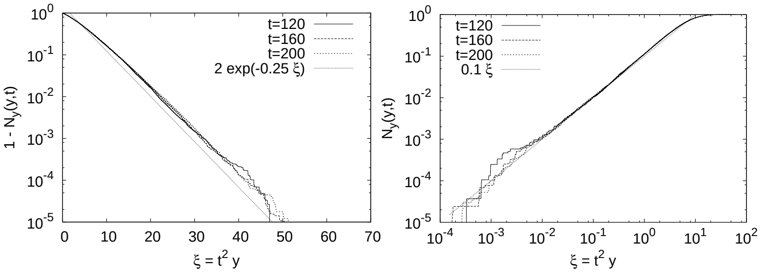


Qualitative feature of the number distributions $N_Q(\chi)$



Numerical data can be described even globally quite well by an exponential fit $N_{x,y}(\chi) = 1 - \exp(-\gamma_{x,y}\chi)$ with some exponents γ_x, γ_y.

Qualitative feature of the number distributions $N_Q(\chi)$



As shown for a simplified special case: these distributions (and the scaling) can be derived from the fixed point condition for certain dynamic renormalisation group equations.

The later is found for the random triangle model.

A finite Markov approximation

Consider:

$$\mathcal{D}^{(t)} = \bigcup_{n=0}^{t-1} \phi^{(n)}(\mathcal{D}).$$

The set of lines $\mathcal{D}^{(t)}$ can be considered as a boundary set for a finite set of $N(t)$ polygons, denoted by $\mathcal{M}^{(t)}$.

Note that

$$\phi : \mathcal{M}^{(t)} \to \mathcal{M}^{(t+1)} \qquad \text{and} \qquad \mathcal{D}^{(t')} \subseteq \mathcal{D}^{(t)} \text{ if } t' \leq t.$$

We now define a probabilistic process in terms of a Markov transition matrix $M^{(t)}$ of dimension $N(t)$.

A finite Markov approximation

- Most elements $\mathcal{A} \in \mathcal{M}^{(t)}$ are mapped again onto a single element $\mathcal{B} \in \mathcal{M}^{(t)}$: transition matrix reads $M^{(t)}_{\mathcal{A}',\mathcal{A}} = \delta_{\mathcal{A}',\mathcal{B}}$.
- for $N'(t) \propto N(t+1) - N(t) \propto t^2$ elements $\mathcal{A} \in \mathcal{M}^{(t)}$ they are cut and split into two distinct polygons of $\mathcal{M}^{(t)}$, say $\mathcal{B}_1$ and $\mathcal{B}_2$.

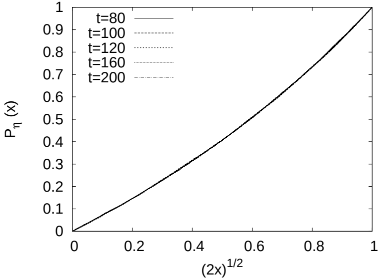
  The transition matrix read

$$M^{(t)}_{\mathcal{A},\mathcal{A}'} = \frac{1}{\mu(\mathcal{A})} \left(\delta_{\mathcal{A}',\mathcal{B}_1}\mu(\mathcal{B}_1) + \delta_{\mathcal{A}',\mathcal{B}_2}\mu(\mathcal{B}_2)\right) \ .$$

- Moreover, to each split of a polygon we associate the relative splitting strength as

$$\eta(\mathcal{A}) := \min\{\mu(\mathcal{B}_1), \mu(\mathcal{B}_2)\}/\mu(\mathcal{A}) \leq \frac{1}{2} \, .$$

Cumulative distribution of splitting strengths $P_\eta(x)$

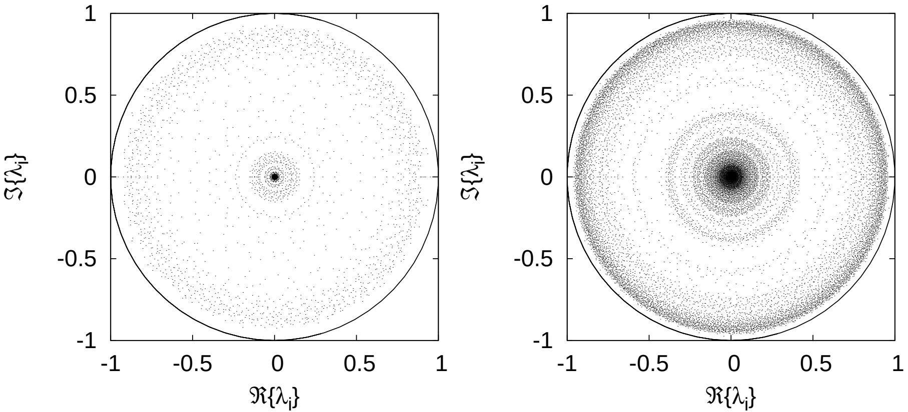


$$N_\eta(x) = y + ay(1-y)(y+b) + \epsilon\,, \quad y = \sqrt{2x}\,, \quad |\epsilon| < 5\cdot 10^{-3}, a \approx 0.1,\, b \approx 2$$

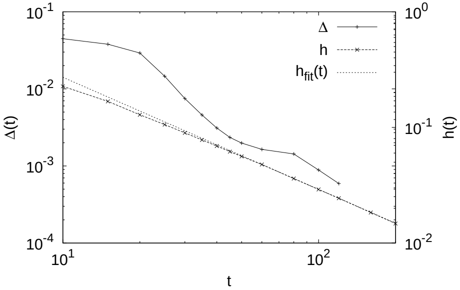
The first two central moments of the limiting distribution are

$$\overline{\eta} = 0.2 \pm 0.002\,, \qquad \sigma_\eta = 0.155 \pm 0.004\,.$$

The corresponding Markov process



Spectral Gap and Kolmogorov-Sinai entropy



- Spectral gap

$$\Delta(t) = 1 - |\max_i\{\lambda_i(t) \neq 1\}| \sim C_{\mathrm{gap}}\, t^{-1} .$$

- Entropy: $\tau = 0.97 \pm 0.03$

$$h(t) = -\sum_{\mathcal{A}\in\mathcal{M}^{(t)}} \mu(\mathcal{A}) \sum_{\mathcal{A}'\in\mathcal{M}^{(t)}} M^{(t)}_{\mathcal{A},\mathcal{A}'} \log\left(M^{(t)}_{\mathcal{A},\mathcal{A}'}\right) \sim C_{\mathrm{entropy}} t^{-\tau}$$

The binary coding

We associate to an individual binary code $\omega = (\omega_0, \dots, \omega_{t-1}) \in \{0,1\}^t$ a set of points Λ_ω defined as

$$\Lambda_\omega = \left\{ x \,:\, \phi^k(x) \in \Lambda_{\omega_k},\, 0 \le k < t \right\} = \bigcap_{k=0}^{t-1} \phi^{-k}\left(\Lambda_{\omega_k}\right) .$$

We denote by $L_t \subseteq \{0,1\}^t$ the set of ϕ-admissible words of length t, i.e.

$$L_t = \left\{ \omega \in \{0,1\}^t \,:\, \Lambda_\omega \neq \emptyset \right\} .$$

The partitions $\Lambda^{(t)}$ based on the binary partition Λ is then defined as a collection of all non-empty Λ_ω

$$\Lambda^{(t)} = \left\{ \Lambda_\omega \right\}_{\omega \in L_t} .$$

The binary coding

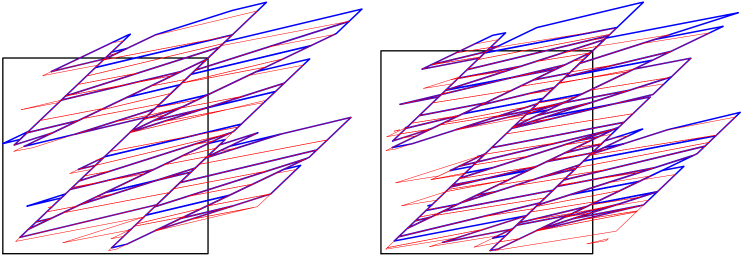


Figure: In the upper two plots we show polygonal boundaries of the partitions $\Lambda^{(t)}$ for two subsequent time steps, namely $\mathcal{D}^{(t)}$ (blue thick lines) and on top of them $\mathcal{D}^{(t+1)}$ (red thin lines), for $t = 3$ (left plot) and $t = 4$ (right plot).

The binary coding

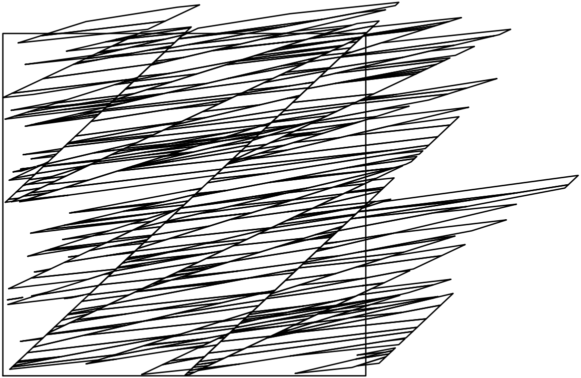


Figure: A finer partition, i.e. polygonal boundaries $\mathcal{D}^{(8)}$.

The binary coding

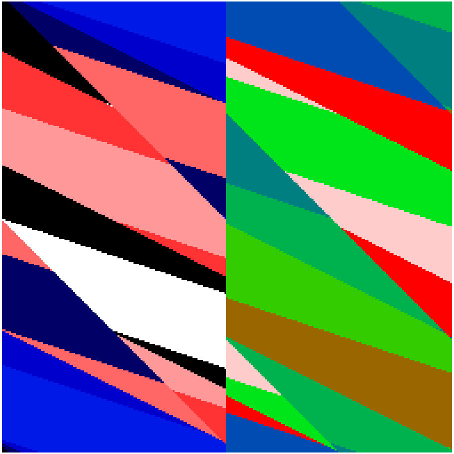


Figure: The partition of the triangle map at time $t = 4$ individually coloured obtained at $\alpha = 1/e$, $\beta = (\sqrt{5} - 1)/2$.

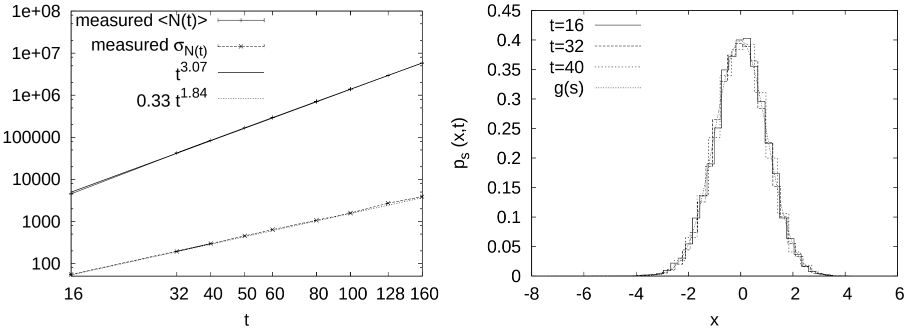
Stochastic model: the random triangle map

$$
\begin{aligned}
p_{t+1} &= p_t + \alpha\,\mathrm{sgn}(\xi_t - q_t) + \beta \quad \bmod 1\,, \\
q_{t+1} &= q_t + p_{t+1} \quad \bmod 1\,,
\end{aligned}
$$

where $\xi_t \in [0,1]$ is a u.d. random variable (triangle map: $\xi_t = 1/2$)

We study the behaviour of the natural binary partitions $\Lambda^{(t)}(\Gamma)$ with time t across realisations of the random sequence $\Gamma = \{\xi_t\}_{t\in\mathbb{Z}^*}$.

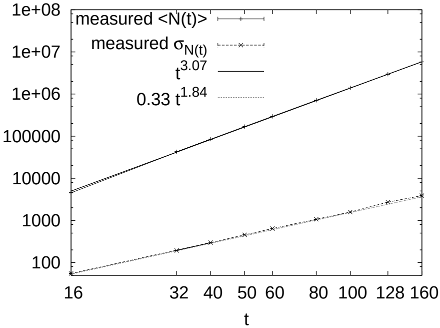
Number of partitions in the random triangle map

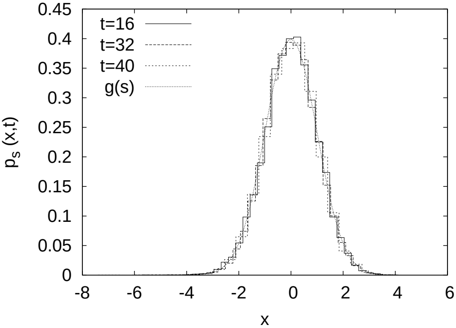


- $$\overline{N}(t) := \langle N(t,\Gamma)\rangle_\Gamma \sim C_1\, t^\rho, \quad \sigma_N^2(t) := \left\langle N(t,\Gamma)^2 \right\rangle_\Gamma - \overline{N}(t)^2 \sim C_2\, t^{2r},$$

  $\rho \doteq 3.07 \pm 0.02$ and $r \doteq 1.84 \pm 0.01$.

Number of partitions in the random triangle map





- $$p_s(x,t) = \langle \delta\left(x - s(t,\Gamma)\right)\rangle_\Gamma = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right) ,$$

  for the rescaled number of elements of the partitions $s(t,\Gamma) = (N(t,\Gamma) - \overline{N}(t))/\sigma_N(t)$.

Mixing properties in the stochastic model

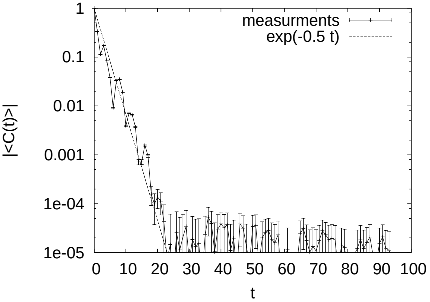
We study

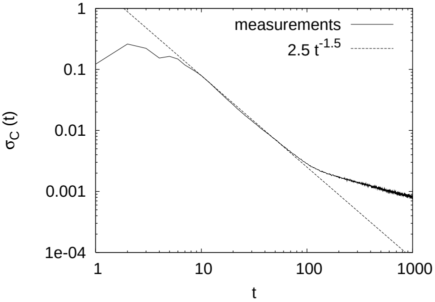
$$C(t,\Gamma)[f] = \int_{\mathbb{T}^2} \mathrm{d}q\, \mathrm{d}p\, f(q,p) f(q_t(q,p,\Gamma), p_t(q,p,\Gamma)) \,,$$

and

$$\overline{C}(t) := \langle C(t,\Gamma)\rangle_\Gamma \,, \qquad \sigma_C^2(t) := \left\langle C(t,\Gamma)^2 \right\rangle_\Gamma - \overline{C}(t)^2$$

Mixing properties in the stochastic model

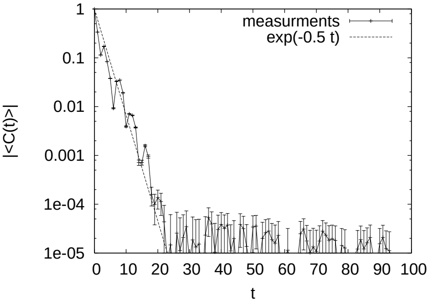


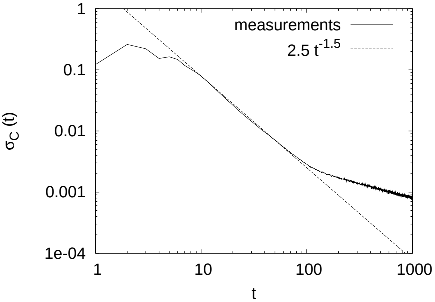


Not surprisingly:

$$\overline{C}(t) = \exp(-|O(t)|) ,$$

Mixing properties in the stochastic model





But

$$\sigma_C(t) = O(t^{-\frac{3}{2}}) \, .$$

This is the same power-law decay as the one found for the correlation in generic case of the deterministic triangle....

Correlation decay: some heuristic

$$C_{\mathcal{A},\mathcal{B}}(t) = \mu(\mathcal{B} \cap \phi^{(t)}(\mathcal{A})) - \mu(\mathcal{B})\mu(\mathcal{A}) \,.$$

$$N_{\mathcal{A}}(t) = \frac{\mu(\mathcal{A})}{\langle a \rangle (t)} \sim \mathrm{const}\, t^3 \qquad \text{for} \quad t \gg 1 \,.$$

$$N_{\mathcal{A},\mathcal{B}}(t) \approx \frac{\mu(\mathcal{A})\mu(\mathcal{B})}{\langle a \rangle (t)} \,.$$

And hence

$$\sigma_{C_{\mathcal{A},\mathcal{B}}}(t) = \sqrt{N_{\mathcal{A},\mathcal{B}}(t)}\, \langle a \rangle (t) = \sqrt{\mu(\mathcal{A})\mu(\mathcal{B}) \langle a \rangle (t)} = O(t^{-\frac{3}{2}}) \,,$$

Semiclassical properties of the triangle map

- *A semi-classical study of the Casati-Prosen triangle map*, —-, S O'Keefe and B Winn, *Nonlinearity* **18**, 1073-1094 (2005)

Egorov estimate for the Casati-Prosen map:

Theorem

Let $f \in C^\infty(\mathbb{T}^2)$ *be compactly supported away from the set* $\phi^{-1}(\{0, 1/2\} \times \mathbb{T})$*. Then for any* $R > 0$*,*

$$\left\| U_{\alpha,\beta}^{-1} \hat{f} U_{\alpha,\beta} - \widehat{f \circ F_{\alpha,\beta}} \right\| = \mathrm{O}_{R,f}(N^{-R})$$

as $N \to \infty$

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Schnirelman theorem:

Theorem

$$\lim_{N\to\infty} \langle\psi_{j_N}|\hat{f}|\psi_{j_N}\rangle = \int_{\mathbb{T}^2} f(q,p)\,d\mu,$$

where $j_N \in J_N$ (density 1 set), and $|\psi_j\rangle$ is a normalised basis of eigenvectors of $U_{\alpha,\beta}$.

Spectral statistics

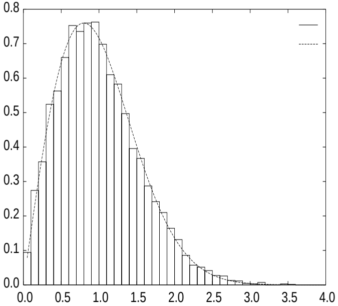

Figure: Spectral statistics for $\alpha = (\sqrt{5} - 1)/2$ and $\beta = \sqrt{2}$, $N = 7001$. On the left is the nearest neighbour density, and on the right the form factor.

Value distribution of eigenvectors

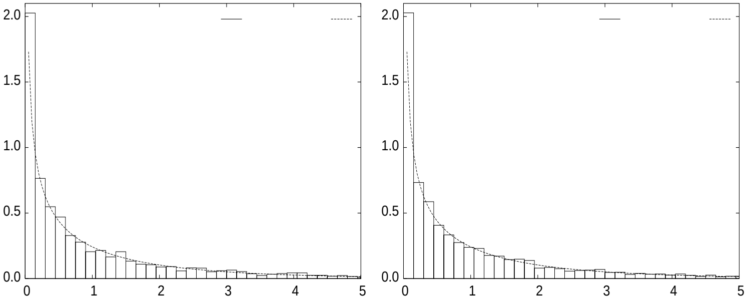

Figure: Value distribution of eigenvectors for $\alpha = (\sqrt{5} - 1)/2$ and $\beta = \sqrt{2}$ (left) and $\alpha = (\sqrt{5} - 1)/2$ and $\beta = 0$ (right), at $N = 4001$.