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# Advanced Training Course on FPGA Design and VHDL for Hardware Simulation and Synthesis

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## Introduction to two-dimensional digital signal processing

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# *Introduction to two-dimensional digital signal processing*

## Fourier transform definition

**The *Fourier transform* of a complex–valued function $g(x,y)$ of two independent real variables $x$ and $y$ is defined by**

$$F\{g(x,y)\} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x,y)\ \exp\left[-j2\pi(f_X x + f_Y y)\right] dxdy$$

**The transform so defined is itself a complex–valued function of two independent real variables $f_X$ and $f_Y$, generally referred to as *spatial frequencies*. For the above integral to exist, function $g$:**

1. **must be absolutely integrable over the infinite ($x$, $y$) plane**
2. **must have only a finite number of discontinuities and a finite number of maxima and minima in any finite rectangle**
3. **must have no infinite discontinuities.**

**Similarly, the *inverse Fourier transform* of a complex–valued function $G(f_X,f_Y)$ of two independent real variables $f_X$ and $f_Y$ is defined as**

$$F^{-1}\{G(f_X,f_Y)\} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G(f_X,f_Y)\exp\left[j2\pi(f_X x + f_Y y)\right] df_X df_Y$$

**Fourier integral theorem. At each point of continuity of $g$,**

$$\boxed{F^{-1}F\{g(x,y)\} = g(x,y)}$$

**(at each point of discontinuity of $g$, the result is the angular average of the value of $g$, in a small neighborhood of that point).**

## The Dirac delta function

**It is common to represent an *idealized point source of light* as a two–dimensional Dirac Delta function. This is a quantity (actually, a *distribution*) that is zero everywhere except at the origin, where it goes to infinity in a manner so as to encompass a unit volume, that is**

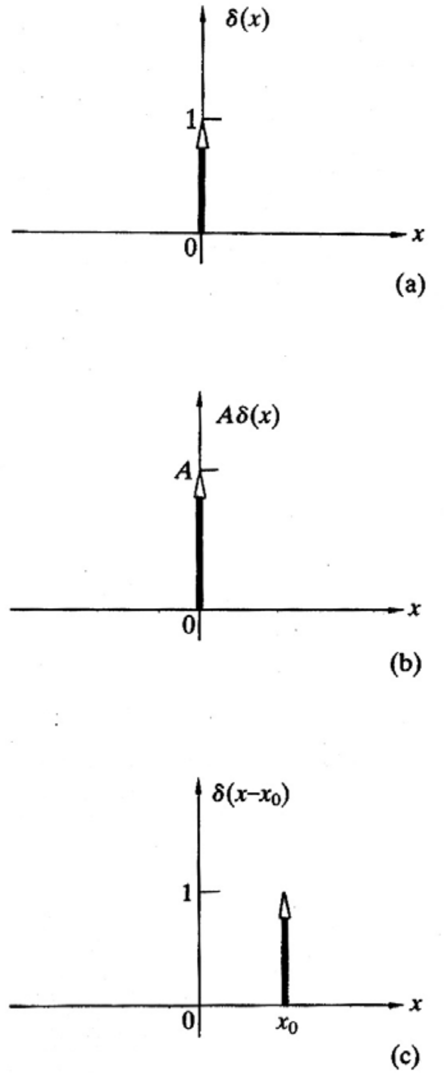
$$\delta(x,y)=\begin{cases}\infty & x=y=0\\ 0 & \text{otherwise}\end{cases}$$

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\delta(x,y)dxdy = 1$$

**The defining characteristic of the delta function is its so called *sifting* property:**

$$\boxed{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x',y')\delta(x-x',y-y')\,dx'dy' = g(x,y)}$$

**at any point $(x, y)$ of continuity of $g$.**




The height of the arrow representing the delta function corresponds to the volume under the function.

**Having an infinite discontinuity, the delta function fails to satisfy condition 3 above for the existence of the Fourier transform. However, it is possible to envisage the delta function as the convergence limit of a sequence of Gaussian pulses such as**

$$\delta(x,y) = \lim_{N\to\infty} N^2 e^{-N^2\pi\left(x^2+y^2\right)}$$

**Each member function of this defining sequence does satisfy the existence requirements, and each has Fourier transform given by**

$$F\left\{N^2 e^{-N^2\pi(x^2+y^2)}\right\} = e^{-\frac{\pi(f_X^2+f_Y^2)}{N^2}}$$

**Accordingly, the *generalized* transform of $\delta$ is found to be**

$$\boxed{F\{\delta(x,y)\} = \lim_{N\to\infty}\left\{e^{-\frac{\pi(f_X^2+f_Y^2)}{N^2}}\right\} = 1}$$

**As the Gaussian pulses become narrower and taller, their transforms grow broader until the pulse is infinitesimal in width, and its transform is infinite in extent, i.e. a constant.**

**Let's write explicitly the Fourier integral theorem** $g(x,y)=F^{-1}F\left\{g(x,y)\right\}$ **as**

$$g(x,y)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} df_X df_Y \exp\left[j2\pi\left(f_X x+f_Y y\right)\right]\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} dx'dy' g(x',y')\exp\left[-j2\pi\left(f_X x'+f_Y y'\right)\right]$$

$$=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} dx'dy' g(x',y')\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} df_X df_Y \exp\left\{j2\pi\left[f_X\left(x-x'\right)+f_Y\left(y-y'\right)\right]\right\}$$

**But we also have**

$$g(x,y)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} dx'dy' g(x',y')\delta(x-x',y-y')$$

**therefore**

$$\boxed{\delta(x-x',y-y')=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\exp\left\{j2\pi\left[f_X\left(x-x'\right)+f_Y\left(y-y'\right)\right]\right\}df_X df_Y}$$

**Hence, we can also write**

$$\delta(x,y)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\exp\left[j2\pi(f_X x+f_Y y)\right]df_X df_Y=F^{-1}\left\{1\right\}$$

**which is known the *integral representation of the delta function*.**

**Similarly, starting from**

$$G(f_X, f_Y) = FF^{-1}\{G(f_X, f_Y)\}$$

**we conclude that**

$$G(f_X, f_Y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} df_X' df_Y' G(f_X', f_Y') \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} dxdy \exp\left\{-j2\pi\left[x\left(f_X - f_X'\right) + y\left(f_Y - f_Y'\right)\right]\right\}$$

**and therefore that**

$$\boxed{\delta(f_X - f_X', f_Y - f_Y') = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \exp\left\{-j2\pi\left[x\left(f_X - f_X'\right) + y\left(f_Y - f_Y'\right)\right]\right\} dxdy}$$

**Hence, we can also write**

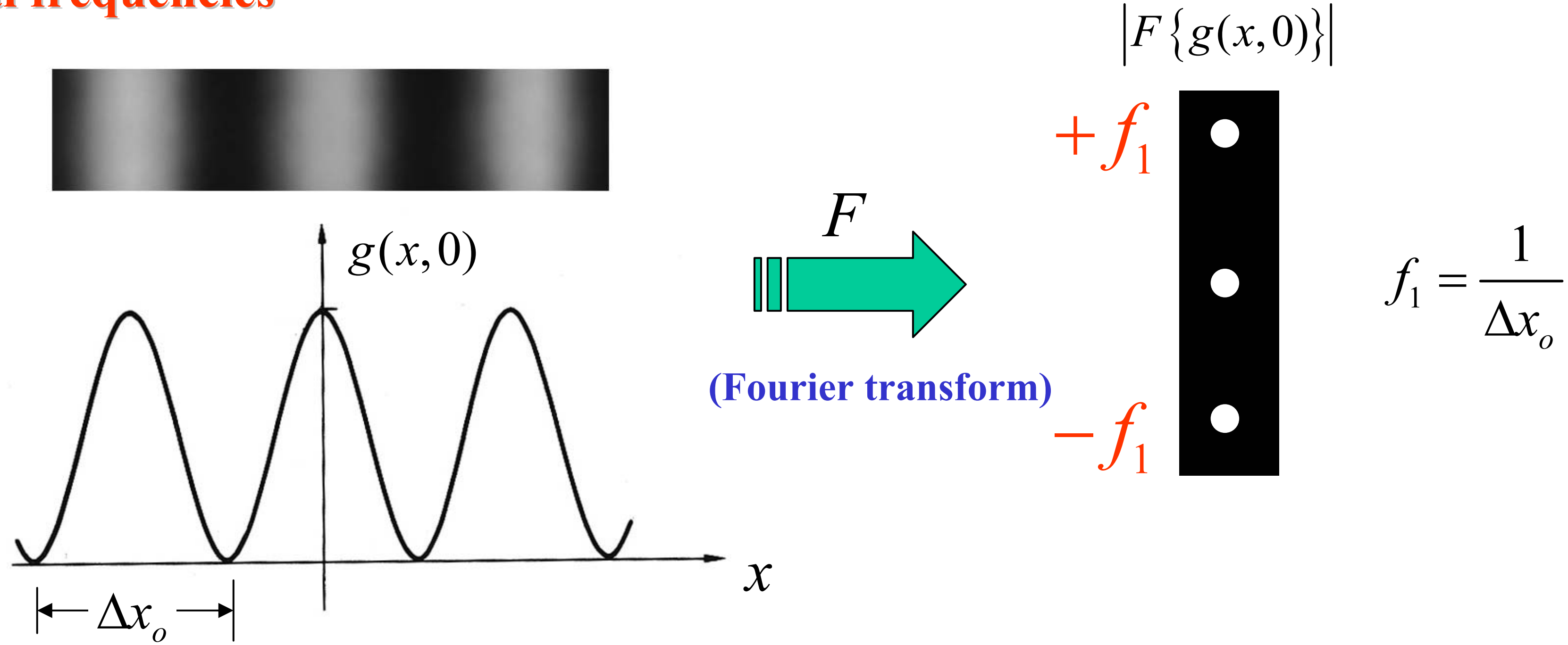
$$\delta(f_X, f_Y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \exp\left[-j2\pi\left(f_X x + f_Y y\right)\right] dxdy = F\{1\}$$

**Finally, we may use the sifting property to evaluate**

$$F\{\delta(x-a, y-b)\} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \exp\left[-j2\pi(f_X x + f_Y y)\right]\delta(x-a, y-b)dxdy = \exp\left[-j2\pi(f_X a + f_Y b)\right]$$

**Therefore, if the delta spike is shifted off the origin, its transform will change phase but not amplitude, which remains equal to one.**

## Spatial frequencies




**For a transparency resembling a sine target with spatial period $\Delta x_o$, there would ideally be only three spots (delta functions) on the transform plane, these being the zero-frequency central peak and the first order or fundamental $f_1 = \pm 1$ on either side of the center.**

**The zero-frequency component with arises because a photographic slide at points of ideal opacity may produce $g = 0$, but cannot provide negative values. Thus the mean value (DC-term) is positive.**

In general, we may regard the two-dimensional Fourier transform as a decomposition of a function $g(x,y)$ into a linear combination, with coefficients, $G(f_X, f_Y)$, of complex exponential functions of the form

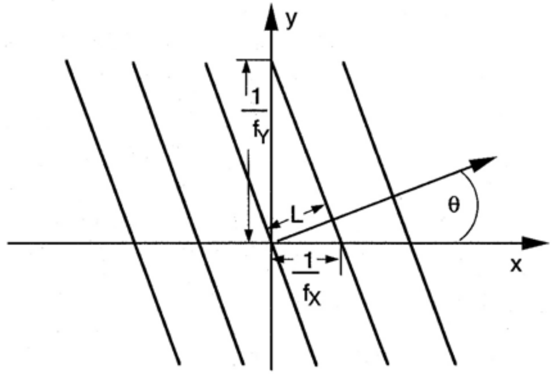
$$\exp\left[j2\pi(f_X x + f_Y y)\right]$$

For any particular pair of spatial frequencies, the phase of the exponential is *zero* or an integer multiple of $2\pi$ radians provide that

$$f_X x + f_Y y = n, \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

i.e., for any integer $n$, the phase is *zero* or an integer multiple of $2\pi$ along the straight lines in the $x$-$y$ plane given by:

$$y = -\frac{f_X}{f_Y}x + \frac{n}{f_Y}$$




Lines of zero phase for the function $e^{j2\pi(f_X x + f_Y y)}$.

and the normals to these lines form an angle $\theta$ with the $x$ axis such that

$$\tan(\theta) = \frac{f_Y}{f_X}$$

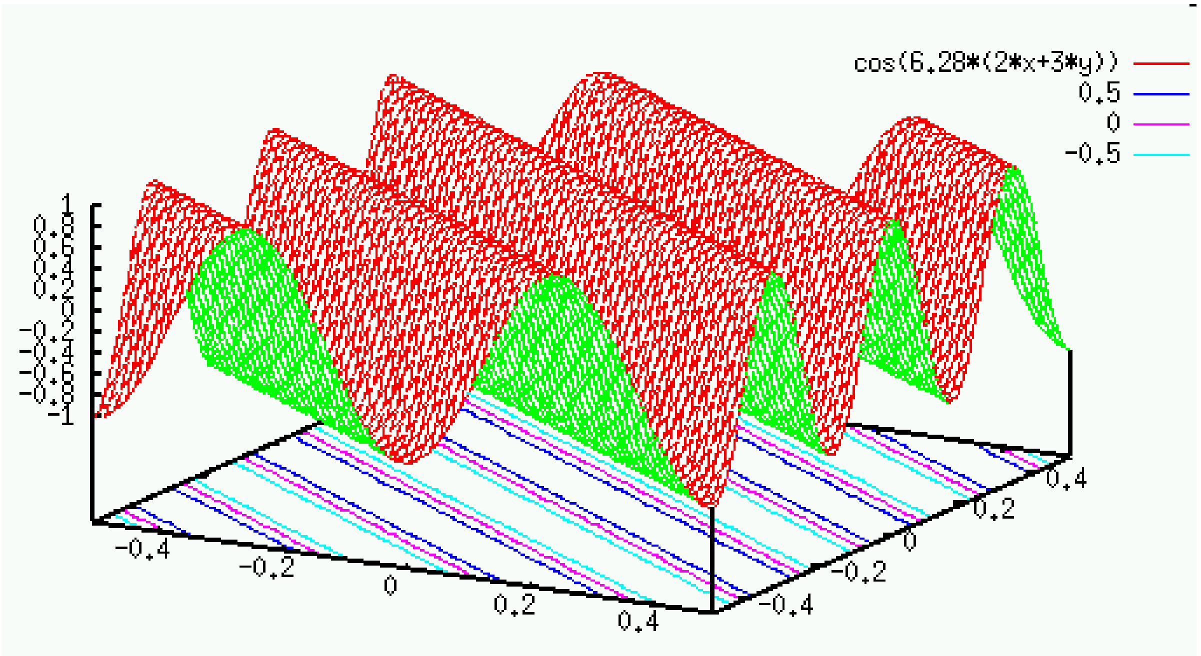
These lines are separated by a distance $L$ (spatial period) given by

$$L = \frac{1}{\sqrt{f_X^2 + f_Y^2}}$$

The corresponding (radial) **spatial frequency** is

$$w = L^{-1} = \sqrt{f_X^2 + f_Y^2}$$

## Examples




**The function**

$$g(x,y) = \cos[2\pi(2x+3y)]$$

**on top, presents with $f_X = 2$ cycles per unit distance in $x$ and $f_Y = 3$ cycles per unit distance in $y$. By contrast,**

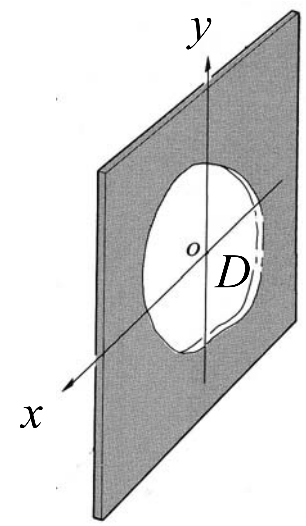
$$g(x,y) = \cos[2\pi(3x+2y)]$$

**(not shown) has 3 cycles per unit distance in $x$ and 2 cycles per unit distance in $y$. But along their individual directions,**

$$\theta = \text{atan}\left(f_Y / f_X\right)$$

**their (radial) spatial frequencies are the same**

**The aperture (pupil) function $g(x,y)$ is defined as:**




$$g(x,y) = \begin{cases} 1 & \text{at points in the domain } D \\ 0 & \text{otherwise} \end{cases}$$

**Thus we may write**

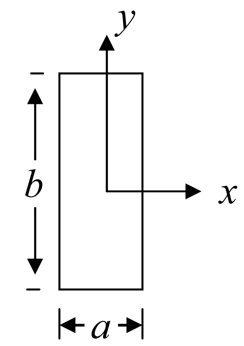
$$G(f_X, f_Y) = F\{g(x,y)\}$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x,y)\ \exp[-j2\pi(f_X x + f_Y y)]\,dxdy$$

$$= \iint_D \exp[-j2\pi(f_X x + f_Y y)]\ df_X df_Y$$

**For a rectangular aperture, we have**

$$G(f_X, f_Y) = \int_{-b/2}^{b/2} \exp\left(-j2\pi f_Y y\right) dy \int_{-a/2}^{a/2} \exp\left(-j2\pi f_X x\right) dx$$




**where $a$ and $b$ are positive constants.**

**The integral in $x$ is**

$$\int_{-a/2}^{a/2} \exp\left(-j2\pi f_X x\right)\, dx = \frac{1}{2\pi f_X} \int_{-\theta_a}^{\theta_a} \exp\left(-j\theta\right)\, d\theta$$

**where**

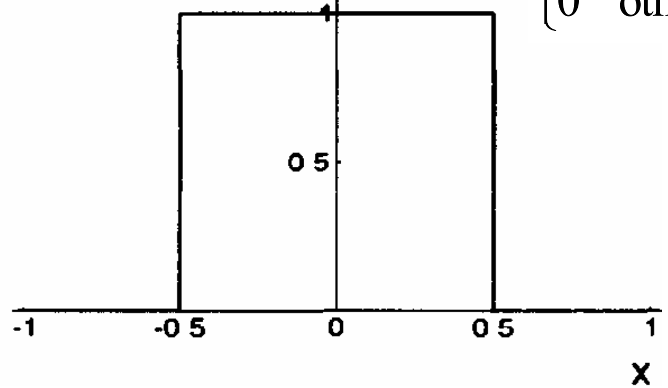
$$\theta_a = 2\pi f_X\, a/2 = \pi f_X a$$

**thus**

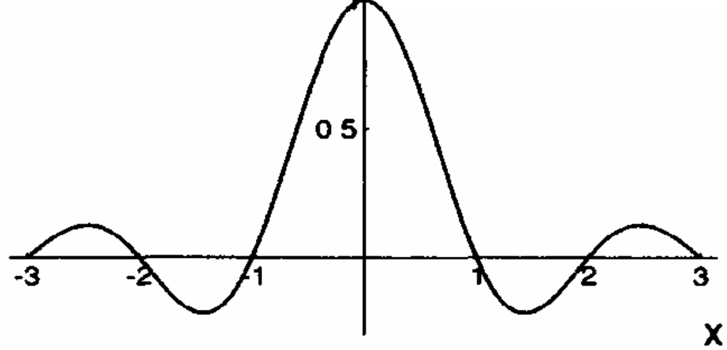
$$\int_{-a/2}^{a/2} \exp\left(-j2\pi f_X x\right)\, dx = \frac{1}{-j2\pi f_X}\left[e^{-j\theta_a} - e^{j\theta_a}\right] = \frac{2}{2\pi f_X}\left[\frac{e^{j\theta_a} - e^{-j\theta_a}}{2j}\right] = a\frac{\sin\left(\pi f_X a\right)}{\pi f_X a}$$

**Defining**

$$\mathrm{rect}(x) = \begin{cases} 1 & |x| \le \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$




$$\mathrm{sinc}(x) = \frac{\sin\left(\pi x\right)}{\pi x}$$



**we may rewrite the previous result as**

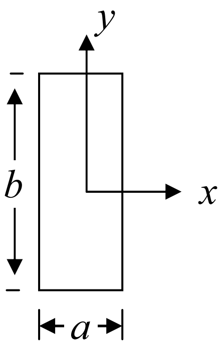
$$\int_{-a/2}^{a/2} \exp\left(-j2\pi f_X x\right)\, dx = \int_{-\infty}^{\infty} \mathrm{rect}\left(x/a\right)\exp\left(-j2\pi f_X x\right)\, dx = a\ \mathrm{sinc}\left(f_X a\right)$$

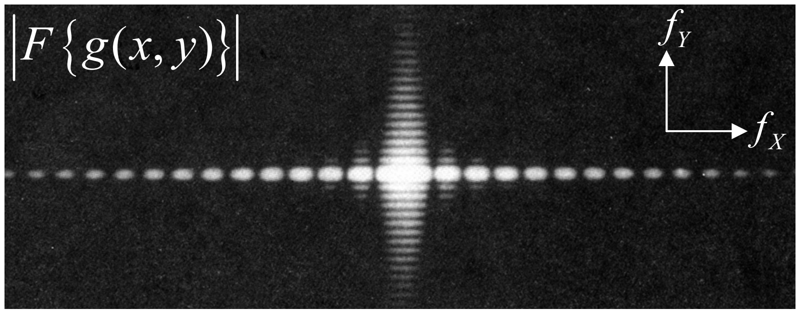
**Consequently**

$$\int_{-b/2}^{b/2} \exp\left(-j2\pi f_Y\, y\right)\, dy = \int_{-\infty}^{\infty} \mathrm{rect}\left(y/b\right)\exp\left(-j2\pi f_Y\, y\right)\, dy = b\ \mathrm{sinc}\left(f_Y b\right)$$

**so that, finally**

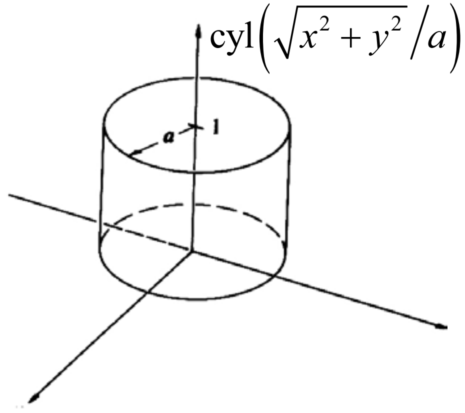
$$F\left\{\mathrm{rect}(x/a)\mathrm{rect}(y/b)\right\} = ab\ \mathrm{sinc}\left(f_X a\right)\mathrm{sinc}\left(f_Y b\right)$$



To represent the pupil function for a circular aperture we can exploit the radially symmetrical *cylinder* (or *circle*) function, which is defined as

$$\mathrm{cyl}(r) = \begin{cases} 1 & r \le 1 \\ 0 & \text{otherwise} \end{cases}$$



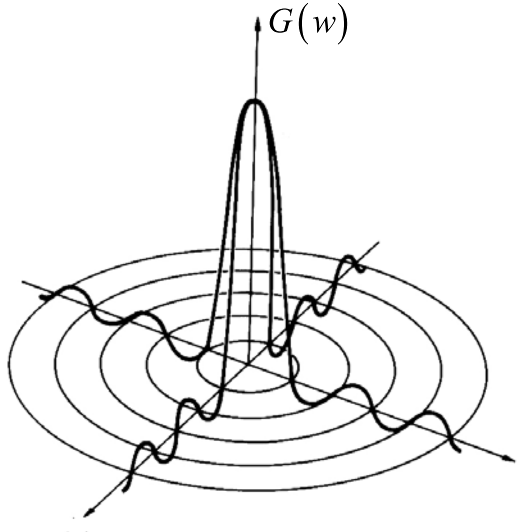
(a)

Hence, for a circular pupil with radius $a$, we write

$$g(x,y) = \mathrm{cyl}\left(\frac{\sqrt{x^2+y^2}}{a}\right)$$

and its Fourier transform is the radially symmetric function

$$G(w) = \pi a^2 \left[\frac{2J_1(aw)}{aw}\right]$$

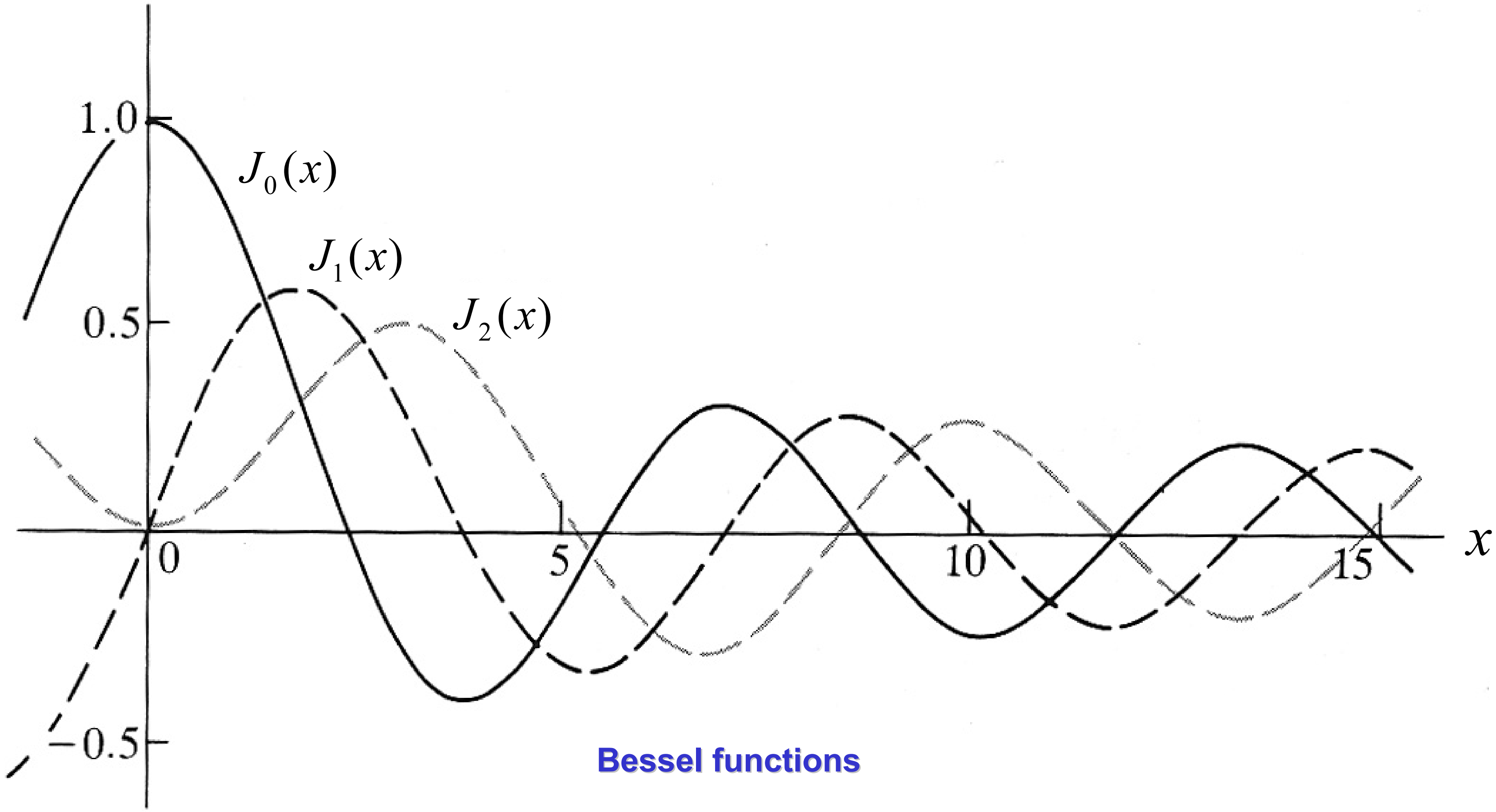


(b)

where the radial frequency $w$ is given by

$$w = \sqrt{f_X^2 + f_Y^2}$$

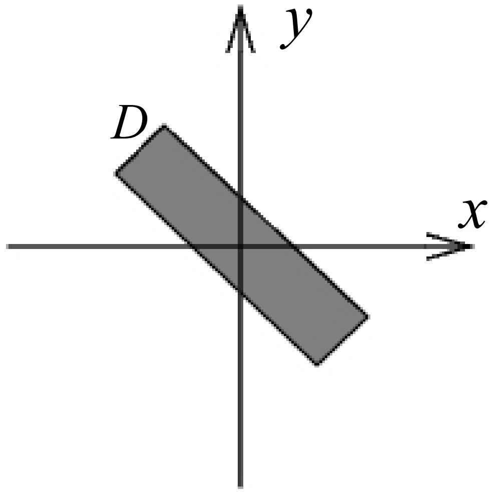
and $J_1$ is the Bessel function of order 1 (of the first kind).

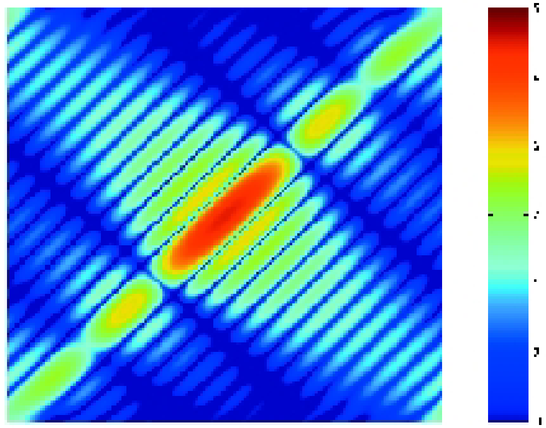


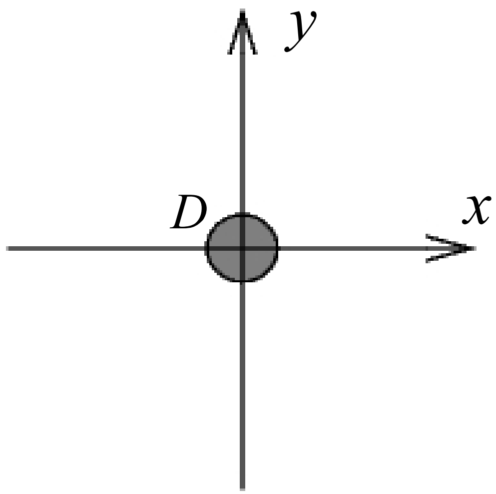

**Bessel functions**

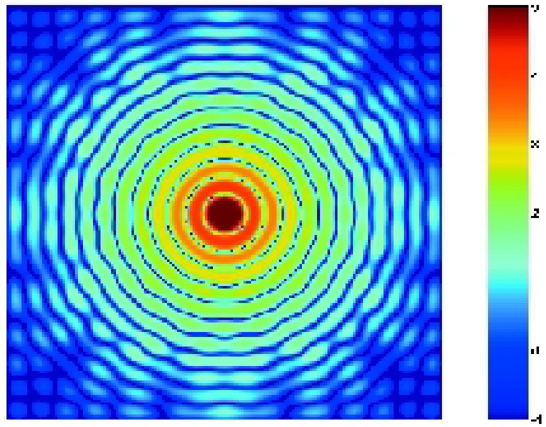
$$g(x,y)=\begin{cases}1 & \text{if } (x,y) \text{ belongs to } D\\ 0 & \text{otherwise}\end{cases}$$

$$\left|F\{g(x,y)\}\right|$$

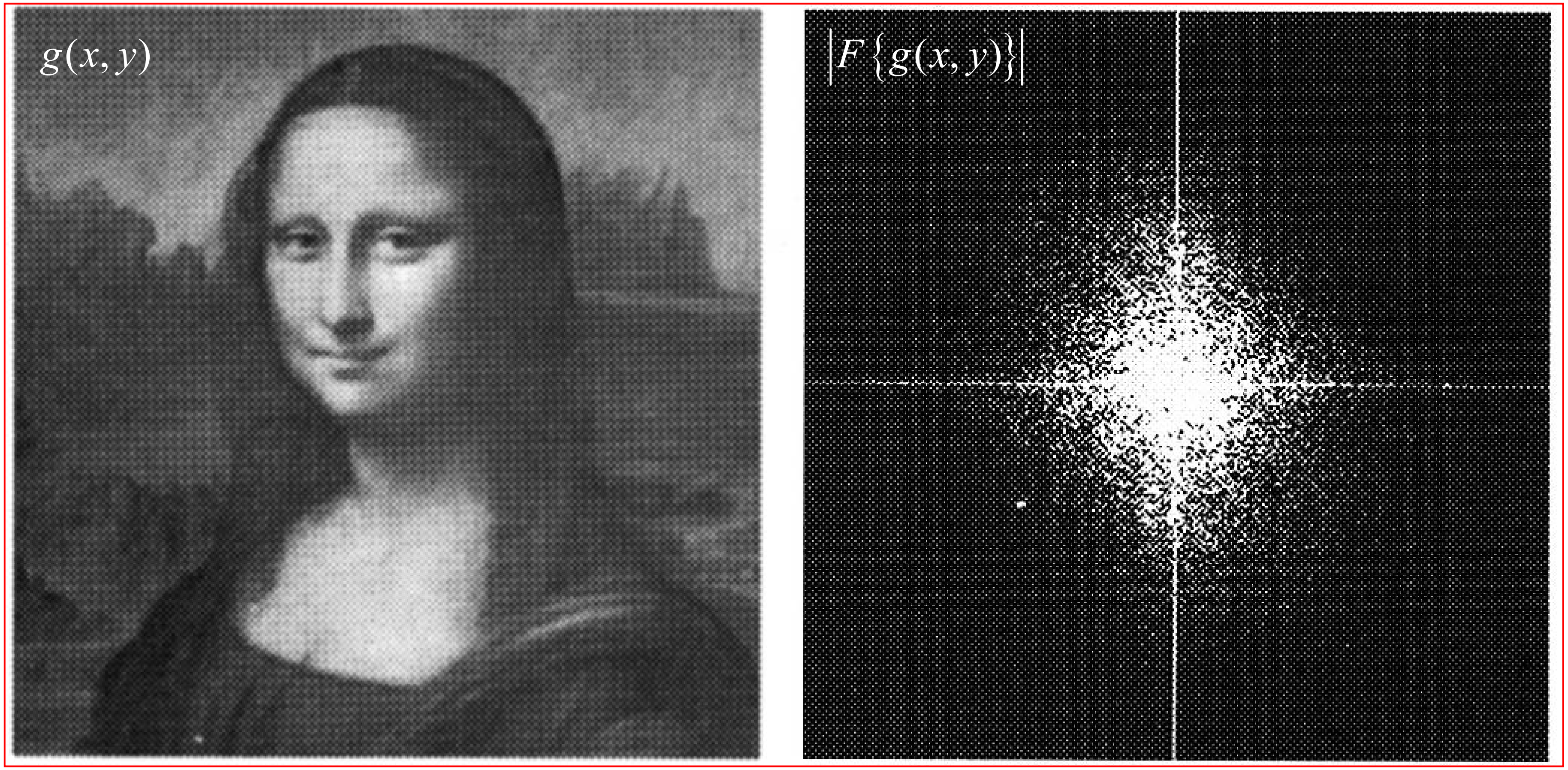


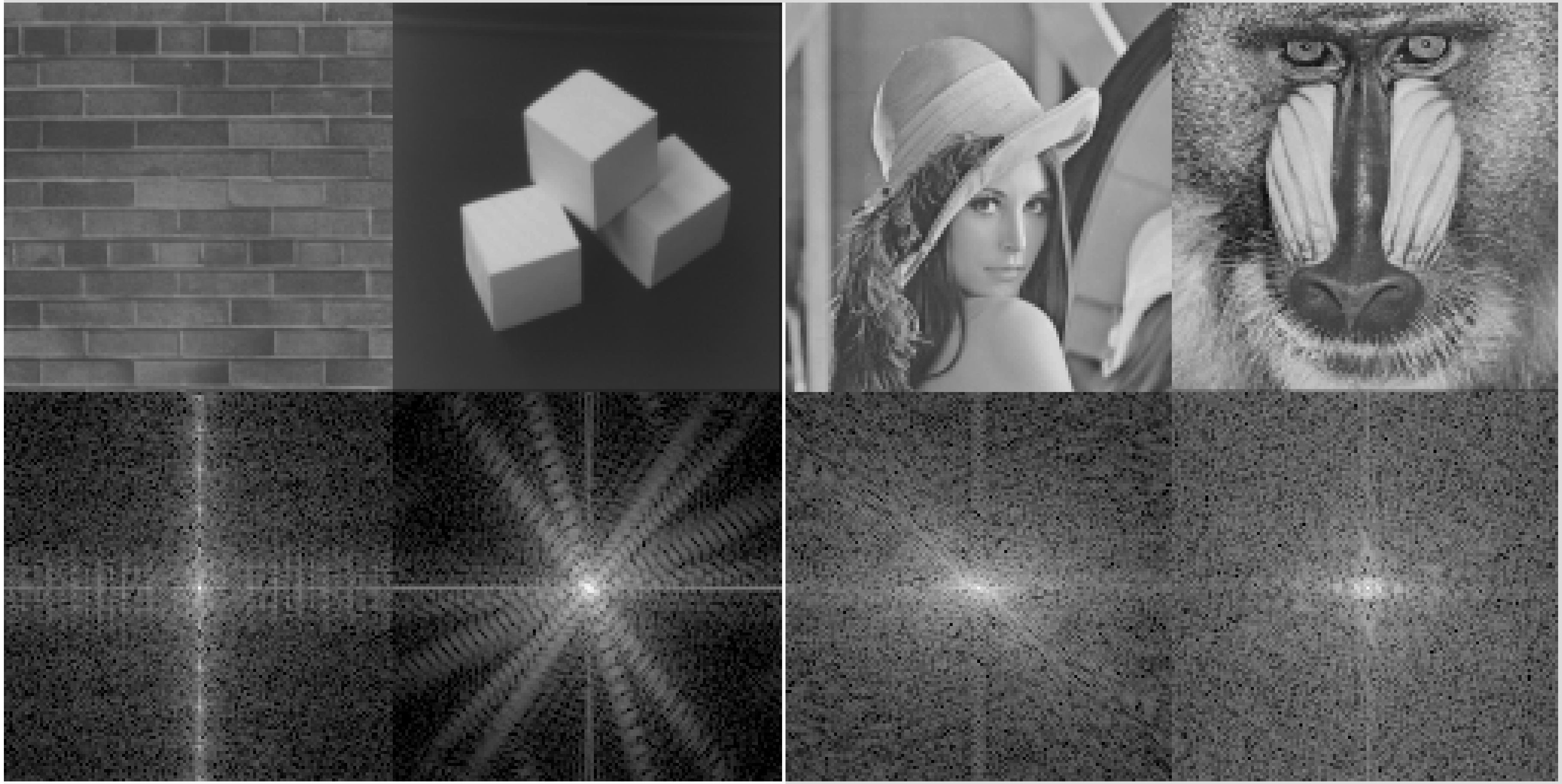


“Apertures” (left) and corresponding two-dimensional Fourier transforms amplitudes, color coded according to the color scale bars.




Left: a famous picture. Right: corresponding two-dimensional Fourier transform. The bright narrow central cross arises from the sharp boundary edges of the picture.



**Images (top) and corresponding two-dimensional Fourier transforms (bottom).**

## Fourier transform theorems

**Linearity theorem.** **The transform of a weighted sum of two (or more) functions** $g$ **and** $h$ **is the identically weighted sum of their individual transform, that is**

$$F\{\alpha g + \beta h\} = \alpha F\{g\} + \beta F\{h\}$$

**Proof. This theorem follows directly from the linearity of the integrals that define the Fourier transform.**

**Shift theorem.** **Translation in the space domain introduces a linear phase shift in the frequency domain, that is**

$$F\{g(x-a, y-b)\} = F\{g(x,y)\}\exp\left[-j2\pi(f_X a + f_Y b)\right]$$

**Proof:**

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x-a, y-b)\ \exp\left[-j2\pi(f_X x + f_Y y)\right]\ dxdy$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x', y')\ \exp\left\{-j2\pi\left[f_X(x'+a) + f_Y(y'+b)\right]\right\}\ dx'dy'$$

$$= F\{g(x,y)\}\exp\left[-j2\pi(f_X a + f_Y b)\right]$$

## The Parseval theorem.

$$\boxed{G(f_X,f_Y)=F\{g(x,y)\}\;\Rightarrow\;\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}|g(x,y)|^2dxdy=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}|G(f_X,f_Y)|^2df_Xdf_Y}$$

**Proof:**

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}|g(x,y)|^2dxdy=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}g(x,y)g^*(x,y)\;dxdy$$

$$=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}dxdy\left[\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}d\xi d\eta G(\xi,\eta)\exp\left[j2\pi(x\xi+y\eta)\right]dxdy\right]$$

$$\times\left[\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}d\alpha d\beta G^*(\alpha,\beta)\exp\left[-j2\pi(x\alpha+y\beta)\right]dxdy\right]$$

$$=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}d\xi d\eta G(\xi,\eta)\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}d\alpha d\beta\;G^*(\alpha,\beta)$$

$$\times\left[\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\exp\left\{j2\pi\left[x(\xi-\alpha)+y(\eta-\beta)\right]\right\}dxdy\right]$$

$$=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}d\xi d\eta G(\xi,\eta)\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}d\alpha d\beta\;G^*(\alpha,\beta)\;\delta(\xi-\alpha,\eta-\beta)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}d\xi d\eta|G(\xi,\eta)|^2$$

**The <u>convolution</u> $c$ of two functions, $g$ and $h$, is the result of a commutative operation, denoted by the symbol ⊗, such that**

$$c(x,y)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x',y')\ h(x-x',y-y')\ dx'dy'$$

**Convolution theorem.** **The Fourier transform of the convolution is the product of the individual Fourier transforms, that is**

$$\boxed{F\{g\otimes h\}=F\{g\}F\{h\}}$$

**Proof:**

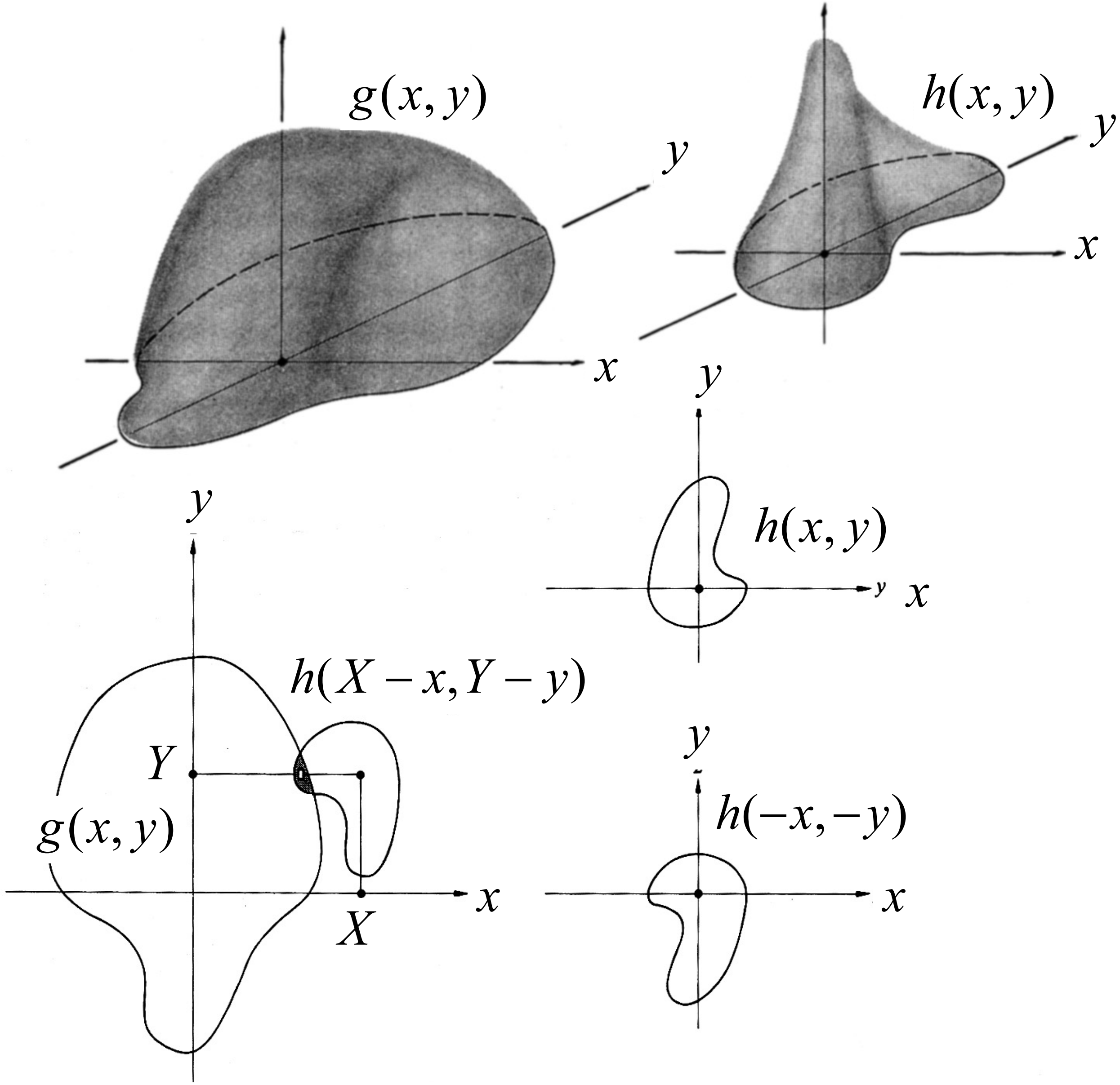
$$F\left\{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x',y')\ h(x-x',y-y')\ dx'dy'\right\}$$

$$=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x',y')\ F\{h(x-x',y-y')\}\ dx'dy'$$

$$=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x',y')\ \exp\left[-j2\pi(f_X x'+f_Y y')\right]dx'dy'F\{h(x,y)\}$$

$$=F\{g\}F\{h\}$$

## Graphical representation of the 2D convolution process



# Table 1

**Properties of the Fourier transform ($a$, $b$, $f_{x0}$ and $f_{y0}$ are real nonzero constants; $k$ and $l$ are nonnegative integers).**

| Property | $g(x,y)$ | $G(f_x,f_y)$ |
|---|---|---|
| Linearity | $au_1(x,y) + bu_2(x,y)$ | $aU_1(f_x,f_y) + bU_2(f_x,f_y)$ |
| Convolution | $u_1(x,y) * u_2(x,y)$ | $U_1(f_x,f_y)U_2(f_x,f_y)$ |
| Correlation | $u_1(x,y) \circ u_2(x,y)$ | $U_1(f_x,f_y)U_2^*(f_x,f_y)$ |
| Modulation | $u_1(x,y)u_2(x,y)$ | $U_1(f_x,f_y) * U_2(f_x,f_y)$ |
| Separable function | $u_1(x)u_2(y)$ | $U_1(f_x)U_2(f_y)$ |
| Space shift | $u(x-x_0, y-y_0)$ | $\mathrm{e}^{-j2\pi(f_x x_0 + f_y y_0)} \cdot U(f_x,f_y)$ |
| Frequency shift | $g(x,y) = \mathrm{e}^{j2\pi(f_{x0}x + f_{y0}y)} \cdot u(x,y)$ | $G(f_x,f_y) = U(f_x - f_{x0}, f_2 - f_{y0})$ |
| Differentiation in space domain | $\dfrac{\partial^k}{\partial x^k}\dfrac{\partial^\ell}{\partial y^\ell}u(x,y)$ | $(2\pi j f_x)^k (2\pi j f_y)^\ell U(f_x,f_y)$ |
| Differentiation in frequency domain | $(-j2\pi x)^k(-j2\pi y)^\ell u(x,y)$ | $\dfrac{\partial^k}{\partial f_x^k}\dfrac{\partial^\ell}{\partial f_y^\ell}U(f_x,f_y)$ |
| Laplacian in the space domain | $\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right)u(x,y)$ | $-4\pi^2(f_x^2 + f_y^2)U(f_x,f_y)$ |
| Laplacian in the frequency domain | $-4\pi^2(x^2 + y^2)u(x,y)$ | $\left(\dfrac{\partial^2}{\partial f_x^2} + \dfrac{\partial^2}{\partial f_y^2}\right)U(f_x,f_y)$ |
| Square of signal | $\lvert u(x,y)\rvert^2$ | $U(f_x,f_y) * U^*(f_x,f_y)$ |
| Square of spectrum | $u(x,y) * u^*(x,y)$ | $\lvert U(f_x,f_y)\rvert^2$ |
| Rotation of axes | $u(\pm x, \pm y)$ | $U(\pm f_x, \pm f_y)$ |
| Parseval's theorem | $\int\limits_{-\infty}^{\infty}\int u(x,y)g^*(x,y)\mathrm{d}x\mathrm{d}y =$ | $\int\limits_{-\infty}^{\infty}\int U(f_x,f_y)G^*(f_x,f_y)\mathrm{d}f_x\mathrm{d}f_y$ |
| Real $u(x,y)$ | $U(f_x,f_y) = U^*(-f_x,-f_y)$ | |
| Real and even $u(x,y)$ | $U(f_x,f_y)$is real and even | |
| Real and odd $u(x,y)$ | $U(f_x,f_y)$is imaginary and odd | |

## Table 2

**Transform pairs for some functions separable in rectangular coordinates**

| Function | Transform |
| --- | --- |
| $\exp[-\pi(a^2x^2+b^2y^2)]$ | $\frac{1}{\lvert ab\rvert}\exp\left[-\pi\left(\frac{f_X^2}{a^2}+\frac{f_Y^2}{b^2}\right)\right]$ |
| $\mathrm{rect}(ax)\,\mathrm{rect}(by)$ | $\frac{1}{\lvert ab\rvert}\mathrm{sinc}(f_X/a)\,\mathrm{sinc}(f_Y/b)$ |
| $\Lambda(ax)\,\Lambda(by)$ | $\frac{1}{\lvert ab\rvert}\mathrm{sinc}^2(f_X/a)\,\mathrm{sinc}^2(f_Y/b)$ |
| $\delta(ax,by)$ | $\frac{1}{\lvert ab\rvert}$ |
| $\exp[j\pi(ax+by)]$ | $\delta(f_X-a/2,\ f_Y-b/2)$ |
| $\mathrm{sgn}(ax)\,\mathrm{sgn}(by)$ | $\frac{ab}{\lvert ab\rvert}\frac{1}{j\pi f_X}\frac{1}{j\pi f_Y}$ |
| $\mathrm{comb}(ax)\,\mathrm{comb}(by)$ | $\frac{1}{\lvert ab\rvert}\mathrm{comb}(f_X/a)\,\mathrm{comb}(f_Y/b)$ |
| $\exp[j\pi(a^2x^2+b^2y^2)]$ | $\frac{j}{\lvert ab\rvert}\exp\left[-j\pi\left(\frac{f_X^2}{a^2}+\frac{f_Y^2}{b^2}\right)\right]$ |
| $\exp[-(a\lvert x\rvert+b\lvert y\rvert)]$ | $\frac{1}{\lvert ab\rvert}\frac{2}{1+(2\pi f_X/a)^2}\frac{2}{1+(2\pi f_Y/b)^2}$ |

## Proof of the Fourier integral theorem

**Let**

$$F\left\{g(x,y)\right\} = G(f_X, f_Y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x',y')\ \exp\left[-j2\pi(f_X x' + f_Y y')\right]dx'dy'$$

**and define**

$$g_R(x,y) = \iint_{A_R} G(f_X, f_Y)\exp\left[j2\pi(f_X x + f_Y y)\right]df_X df_Y$$

**where $A_R$ is a circle of radius $R$, centered at the origin of the frequency plane. Then**

$$g_R(x,y) = \iint_{A_R} df_X df_Y \exp\left[j2\pi(f_X x + f_Y y)\right]$$

$$\times \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} dx'dy'\ \ g(x',y')\ \exp\left[-j2\pi(f_X x' + f_Y y')\right]$$

**Exchanging the order of integration, we may write**

$$g_R(x,y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} dx'dy'\ \ g(x',y')\ \iint_{A_R} df_X df_Y \exp\left\{j2\pi\left[f_X\left(x-x'\right) + f_Y\left(y-y'\right)\right]\right\}$$

**Then, defining**

$$\mathbf{r} = re^{j\theta}, \qquad r = \sqrt{\left(x-x'\right)^2 + \left(y-y'\right)^2}$$

$$\mathbf{w} = we^{j\phi}, \qquad w = \sqrt{f_X^2 + f_Y^2}$$

**the integral over the spatial frequencies becomes**

$$\iint_{A_R} df_X df_Y \exp\left\{ j2\pi\left[ f_X\left(x-x'\right) + f_Y\left(y-y'\right)\right]\right\} = \int_0^R \int_0^{2\pi} d\phi dw\ w\ \exp\left(j\mathbf{r}\cdot\mathbf{w}\right)$$

$$= \int_0^R dw\ w \int_0^{2\pi} d\phi \exp\left[ j\ rw\cos\left(\phi-\theta\right)\right] = R\left[\frac{J_1\left(2\pi rR\right)}{r}\right]$$

**Assume that $(x,y)$ is a point of continuity of $g$. Then**

$$\lim_{R\to\infty} g_R(x,y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} dx'dy'\ g(x',y')\ \lim_{R\to\infty} R\left[\frac{J_1\left(2\pi rR\right)}{r}\right]$$

**Note that for sufficiently large $R$, the circularly symmetric function**

$$h(r) = \left[ \frac{J_1(2\pi r R)}{r} \right]$$

**vanishes practically everywhere, except in the proximity of $r = 0$. Therefore it is possible to envisage the *delta* function also as the convergence limit of the sequence**

$$\delta(x - x', y - y') = \lim_{R\to\infty} R \left[ \frac{J_1(2\pi r R)}{r} \right]$$

**hence**

$$\lim_{R\to\infty} g_R(x,y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} dx' dy' \;\; g(x',y') \; \delta(x - x', y - y') = g(x,y)$$

**But, from the definition of $g_R$, we also have**

$$\lim_{R\to\infty} g_R(x,y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G(f_X, f_Y)\exp\left[ j2\pi(f_X x + f_Y y)\right] df_X df_Y = F^{-1}\left\{ G(f_X, f_Y) \right\}$$

**therefore**

$$\boxed{F^{-1} F\left\{ g(x,y) \right\} = g(x,y)}$$

**Assume now a point of discontinuity of $g$. This point can be located anywhere, for instance at the origin, so that**

$$g_R(0,0) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} dxdy\ g(x,y)\ R\left[\frac{J_1\left(2\pi rR\right)}{r}\right]$$

**where**

$$r = \sqrt{x^2 + y^2}$$

**Again, for sufficiently large $R$, the quantity in square brackets is significantly different from zero only in a small area around the origin. In this small region, $g$ depends approximately on the angle $\theta$ alone, thus**

$$g_R(0,0) \approx \int_0^{2\pi} g_o\left(\theta\right) d\theta \int_0^{\infty} R\left[\frac{J_1\left(2\pi rR\right)}{r}\right] rdr$$

**Where $g_o$ represents the dependence of $g$ from $\theta$ near the origin. Since**

$$\int_0^{\infty} J_1\left(2\pi rR\right) Rdr = \frac{1}{2\pi}\int_0^{\infty} J_1\left(\alpha\right)\ d\alpha = \frac{1}{2\pi}$$

**we have shown that**

$$\lim_{R\to\infty} g_R(0,0) = \frac{1}{2\pi}\int_0^{2\pi} g_o\left(\theta\right)d\theta$$

**i.e. that at each point of discontinuity of g, the two successive transformations**

$$F^{-1}F\left\{g(x,y)\right\}$$

**produce, as a results, the angular average of *g* in a small neighborhood of that point.**