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Towards holographic duality for condensed matter
Lecture IV

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Effective Field Theory and the Fermi Surface, Joseph Polchinski, arXiv:hep-th/9210046 (pages 11 - 36)

A: Nobody ever promised you a rose garden.

Lecture 2: The Landau Theory of Fermi Liquids

Now, as an illustration, we will derive the effective field theory of the low energy excitations in a conductor. As far as I know, this subject is not presented in this way anywhere in the literature. However, it is clear that the essential idea is entirely familiar to those in the field. It is implicit in the writings of Anderson, and recently has been made more explicit by Benfatto and Gallavotti, and Shankar.

Before starting, let me describe one chain of thought which led me to this. Introductions to superconductivity often point out the following remarkable fact. In ordinary metallic superconductors, the size of a Cooper pair may be as large as 10^4 Å. The orbital thus overlaps 10^{10} or more other electrons, with the characteristic electronic interaction energies being as large as an electron volt or more. The BCS theory neglects all but the binding interaction between the paired electrons, which is of order 10^{-3} eV. Yet this leads to results which are not only qualitatively correct but also quantitatively: calculations in BCS theory are supposed to be accurate to order $(m/M)^{1/2}$, where m is the electron mass and M the nuclear mass. Further, while the BCS theory is for weak electron-phonon coupling, it has a strong-coupling generalization, Eliashberg theory, which for arbitrary coupling remains valid to accuracy $(m/M)^{1/2}$. This is again remarkable, because solvable field theories in $3+1$ dimensions are few and far between.

The BCS and Eliashberg theories are derived within the Landau theory of Fermi liquids, which treats a conductor as a gas of nearly free electrons. The justification for this appeals to the notion of 'quasiparticles,' which are dressed electrons, the neglected interactions being absorbed in the dressing. The term quasiparticle is not in common use in particle theory, nor is the notion that a strongly interacting theory can be turned into a nearly free one just by such dressing. What we will see is that in fact this theory is a beautiful example of an effective field theory. The neglected interactions can be regarded as having been *integrated out*, in the usual effective field theory sense. This is possible because of the special kinematics of the Fermi surface. Further, the resulting theory is solvable because there are *almost no* relevant or marginal interactions, in a sense that will be made clear.

We begin by identifying the characteristic scale. The electronic properties of solids are determined by e, $\hbar$ and m, out of which we can construct the energy $e^4 m/\hbar^2 = 27$ eV. Typical electronic energies, such as the width of the conduction electron band, are actually slightly smaller than this, say $E_0 \sim 1$ to 10 eV. The other possible constants, M and c, are so much greater than the electron mass and velocity that we can treat them as infinite. Later we will introduce $1/M$ effects (lattice vibrations). We will see that the fact that solids are near the $M = \infty$ limit is a great simplification. Spin-orbit coupling and other $1/c$ effects also are important in some situations, but not for our discussion.

In a conductor, we can excite a current with an arbitrarily weak electric field, so the spectrum of charged excitations evidently goes down to zero energy. This is the hierarchy of scales that makes effective field theory a useful tool; we want the effective theory describing the excitations with $E << E_0$. Now, at E_0 there are electrons with strong Coulomb interactions. We are not going to try to solve this theory. Rather, we are in a situation similar to the current algebra example, where we will write down the most general effective theory with given fields and symmetries. At this point we need to make a guess—what are the light charged fields? Let us suppose that they are spin-$\frac{1}{2}$ fermions, like the underlying electrons.[4] It must be emphasized that this is only a guess. It can be justified in the artificial limit of a very dilute or weakly interacting system, but with strong interactions it is possible

[4] I will therefore call them electrons.

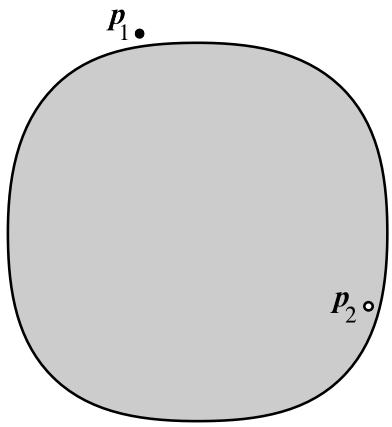

Figure 1: Fermi sea (shaded) with two low-lying excitations, an electron at $\mathbf{p}_1$ and a hole at $\mathbf{p}_2$.

that something very different might emerge. All we can do here is to check the guess for consistency (naturalness), and compare it with experiment.

Begin by examining the free action

$$\int dt\, d^3\mathbf{p} \left\{ i\psi^\dagger_\sigma(\mathbf{p})\partial_t\psi_\sigma(\mathbf{p}) - (\varepsilon(\mathbf{p}) - \varepsilon_{\rm F})\psi^\dagger_\sigma(\mathbf{p})\psi_\sigma(\mathbf{p}) \right\}. \tag{12}$$

Here σ is a spin index and $\varepsilon_{\rm F}$ is the Fermi energy. The single-electron energy $\varepsilon(\mathbf{p})$ would be $p^2/2m$ for a free electron, but in the spirit of writing down the most general possible action we make no assumption about its form.[5] The ground state of this theory is the Fermi sea, with all states $\varepsilon(\mathbf{p}) < \varepsilon_{\rm F}$ filled and all states $\varepsilon(\mathbf{p}) > \varepsilon_{\rm F}$ empty. The Fermi surface is defined by $\varepsilon(\mathbf{p}) = \varepsilon_{\rm F}$. Low lying excitations are obtained by adding an electron just above the Fermi surface, or removing one (producing a hole) just below, as shown in figure 1.

Now we need to ask how the fields behave as we scale all energies by a factor $s < 1$. In the relativistic case, the momentum scaled with the energy,

[5] A possible $\mathbf{p}$-dependent coefficient in the time-derivative term has been absorbed into the normalization of $\psi_\sigma(\mathbf{p})$.

but here things are very different. As figure 1 makes clear, as the energy scales to zero we must scale the momenta *toward the Fermi surface*. To do this, write the electron momentum as

$$\mathbf{p} = \mathbf{k} + \mathbf{l}, \tag{13}$$

where $\mathbf{k}$ is vector on the Fermi surface and $\mathbf{l}$ is a vector orthogonal to the Fermi surface. Then when $E \to sE$, the momenta scale $\mathbf{k} \to \mathbf{k}$ and $\mathbf{l} \to s\mathbf{l}$. Expand the single particle energy

$$\varepsilon(\mathbf{p}) - \varepsilon_{\mathrm{F}} = l v_{\mathrm{F}}(\mathbf{k}) + O(l^2), \tag{14}$$

where the Fermi velocity $\mathbf{v}_{\mathrm{F}} = \partial_{\mathbf{p}}\varepsilon$. Scaling

$$dt \to s^{-1}dt, \quad d\mathbf{k} \to d\mathbf{k}, \quad d\mathbf{l} \to s d\mathbf{l}, \quad \partial_t \to s\partial_t, \quad l \to sl, \tag{15}$$

each term in the action

$$\int dt\, d^2\mathbf{k}\, d\mathbf{l} \left\{ i\psi_\sigma^\dagger(\mathbf{p})\partial_t\psi_\sigma(\mathbf{p}) - l v_{\mathrm{F}}(\mathbf{k})\psi_\sigma^\dagger(\mathbf{p})\psi_\sigma(\mathbf{p}) \right\} \tag{16}$$

scales as s^1 times the scaling of $\psi^\dagger\psi$. The fluctuations of ψ thus scale as $s^{-1/2}$.

Now we play the effective field theory game, writing down *all* terms allowed by symmetry and seeing how they scale. If we find a relevant term we lose: the theory is unnatural. The symmetries are

1. Electron number.

2. The discrete lattice symmetries. Actually, in the action (12), we have treated translation invariance as a continuous symmetry, so that momentum is exactly conserved. Because the electrons are moving in a periodic potential, they can exchange discrete amounts of momentum with the lattice. Including these terms, the free action can be rediagonalized, with the result that the integral over momentum becomes a

sum over bands and an integral over a fundamental region (Brillouin zone) for each band. This does not affect the analysis in any essential way, so for simplicity we will treat momentum as exactly conserved. In addition, the action is constrained by any discrete point symmetries of the crystal.

3. Spin $SU(2)$. In the $c \to \infty$ limit, physics is invariant under independent rotations of space and spin, so spin $SU(2)$ acts as an internal symmetry.

Starting with terms quadratic in the fields, we have first

$$\int dt\, d^2\mathbf{k}\, d\mathbf{l}\, \mu(\mathbf{k}) \psi^\dagger_\sigma(\mathbf{p}) \psi_\sigma(\mathbf{p}). \tag{17}$$

Combining the scaling of the various factors, this goes as $s^{-1+1-2/2} = s^{-1}$. This resembles a mass term, and it is relevant. Notice, though, that it can be absorbed into the definition of $\varepsilon(\mathbf{p})$. We should expand around the Fermi surface appropriate to the full $\varepsilon(\mathbf{p})$. Thus, the *existence* of a Fermi surface is natural, but it is unnatural to assume it to have any very precise shape beyond the constraints of symmetry. Adding one time derivative or one factor of l makes the operator marginal, scaling as s^0; these are the terms already included in the action (16). Adding additional time derivatives or factors of l makes an irrelevant operator.

Turning to quartic interactions, the first is

$$\begin{aligned} \int dt\, d^2\mathbf{k}_1\, d\mathbf{l}_1\, d^2\mathbf{k}_2\, d\mathbf{l}_2\, d^2\mathbf{k}_3\, d\mathbf{l}_3\, d^2\mathbf{k}_4\, d\mathbf{l}_4\, V(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) \\ \psi^\dagger_\sigma(\mathbf{p}_1)\psi_\sigma(\mathbf{p}_3)\psi^\dagger_{\sigma'}(\mathbf{p}_2)\psi_{\sigma'}(\mathbf{p}_4)\delta^3(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4). \end{aligned} \tag{18}$$

This scales as $s^{-1+4-4/2} = s$, times the scaling of the delta-function. Let us first be glib, and argue that

$$\begin{aligned} \delta^3(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4) &= \delta^3(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4 + \mathbf{l}_1 + \mathbf{l}_2 - \mathbf{l}_3 - \mathbf{l}_4) \\ &\sim \delta^3(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4). \end{aligned} \tag{19}$$

That is, we ignore the $\mathbf{l}$ compared to the $\mathbf{k}$, since the former are scaling to zero. The argument of the delta-function then does not depend on s, so the delta-function goes as s^0 and the overall scaling is s^1. Pending a more careful treatment later, the operator (18) is irrelevant. It is then easy to see that any further interactions are even more irrelevant.

To summarize, with our assumption about the nature of the charge carriers the most general effective theory has only irrelevant interactions, becoming more and more free as $E \to 0$. The assumption of a nearly free electron gas is thus internally consistent, and in fact is a good description of most conductors. It should be emphasized that this is just a reformulation of a simple and standard solid state argument, to the effect that the kinematics of the Fermi surface plus the Pauli exclusion principle exclude all but a fraction E/E_0 of possible final states in any scattering process.

There are two complications to discuss before the analysis is complete. The first is phonons. Because a crystal spontaneously breaks spacetime symmetries, the low energy theory must include the correponding Goldstone excitations. We therefore introduce a phonon field $\mathbf{D}(\mathbf{r})$, which is equal to $M^{1/2}$ times the local displacement of the ions from their equilibrium positions.[6] The free action for this field is

$$\frac{1}{2}\int dt\, d^3\mathbf{q}\, \Big\{\partial_t D_i(\mathbf{q})\partial_t D_i(-\mathbf{q}) - M^{-1}\Delta_{ij}(\mathbf{q})D_i(\mathbf{q})D_j(-\mathbf{q})\Big\}. \qquad (20)$$

In the time derivative term, the M in the kinetic energy of the ions cancels against the factors of $M^{-1/2}$ from the definition of $\mathbf{D}$; again we have made a finite rescaling to eliminate a $\mathbf{q}$-dependent coefficient. The restoring force, on the other hand, is to first approximation independent of M and so the M^{-1} comes from the definition of $\mathbf{D}$.

[6] Notice that a crystal actually breaks *nine* spacetime symmetries: three translational, three rotational and three Galilean. For internal symmetries, Goldstone's theorem gives a one-to-one correspondence between broken symmetries and Goldstone bosons. This need not be true for spacetime symmetries, and three Goldstone fields are sufficient to saturate all the broken symmetry Ward identities.

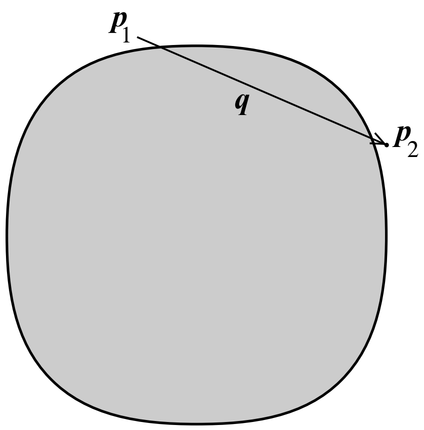

Figure 2: An electron of momentum $\mathbf{p}_1$ absorbs a phonon of large momentum $\mathbf{q}$ but remains near the Fermi surface.

Now examine the scaling of the phonon field. We determine the scaling from the kinetic term; except at very low frequenies (to be discussed) this term dominates because of the M^{-1} in the restoring force. How does $\mathbf{q}$ scale? Figure 2 shows a typical phonon-electron interaction, such as is responsible for BCS superconductivity. As the electron momenta scale towards the Fermi surface, $\mathbf{q}$ approaches a nonzero limit, so $\mathbf{q} \propto s^0$. The integration and time derivatives in the kinetic term (20) then scale as s^1, so the phonon field scales as $s^{-1/2}$.

The restoring force is relevant, scaling as s^{-2}, so in spite of its small coefficient it will dominate at energies below

$$E_1 = (m/M)^{1/2} E_0. \tag{21}$$

This is the Debye energy, the characteristic energy scale of the phonons. The restoring force is like a mass term, making the phonons decouple below E_1. Of course, Goldstone's theorem guarantees that the eigenvalues of $\Delta_{ij}(\mathbf{q})$ vanish as $\mathbf{q} \to 0$, so there are still some phonons present at arbitrarily low energy. But their effects are doubly suppressed, by the phonon phase space and because, as Goldstone bosons, their interactions are proportional to $\mathbf{q}$. The long-wavelength phonons can therefore be neglected for most purposes.

Now consider interactions, starting with

$$\int dt\, d^3\mathbf{q}\, d^2\mathbf{k}_1\, d\mathbf{l}_1\, d^2\mathbf{k}_2\, d\mathbf{l}_2\, M^{-1/2} g_i(\mathbf{q}, \mathbf{k}_1, \mathbf{k}_2) \qquad (22)$$
$$D_i(\mathbf{q}) \psi^\dagger_\sigma(\mathbf{p}_1) \psi_\sigma(\mathbf{p}_2) \delta^3(\mathbf{p}_1 - \mathbf{p}_2 - \mathbf{q}),$$

where an electron emits or absorbs a phonon. The electron-ion force is to first approximation independent of M, so the explicit $M^{-1/2}$ is from the definition of $\mathbf{D}$. This scales as $s^{-1+2-3/2} = s^{-1/2}$ if we treat the delta-function as before, so it is relevant. When the phonons decouple at E_1, the coupling has grown by $(E_1/E_0)^{-1/2} = (m/M)^{-1/4}$. However, combined with the small dimensionless coefficient $(m/M)^{1/2}$ of the interaction (22), this leaves an overall suppression of $(m/M)^{1/4}$.

There are two ways to proceed to lower energies. The first is simply to note that the restoring force dominates the kinetic term below E_1, and so should be used to determine the scaling. Then $\mathbf{D}$ scales as $s^{+1/2}$ and so does the interaction (22); it is irrelevant below E_1. Alternately, we can integrate the phonon out. The leading interaction induced in this way is again the four-Fermi term (18), which by the earlier analysis is irrelevant. Further interactions are even more suppressed, so the inclusion of phonons has not changed the free electron picture: we find an electron-phonon interaction which reaches a maximum of order $(m/M)^{1/4}$ at the Debye energy and then falls.

If this were the whole story, it would be rather boring. It is difficult to see how we can ever get an interesting collective effect like superconductivity in the low energy theory if all interactions are getting weaker and weaker. However, there is an important subtlety in the kinematics, so that our treatment (19) of the delta-function is not always valid. This simplest way to see this is pictorial (figure 3). Consider a process where electrons of momenta

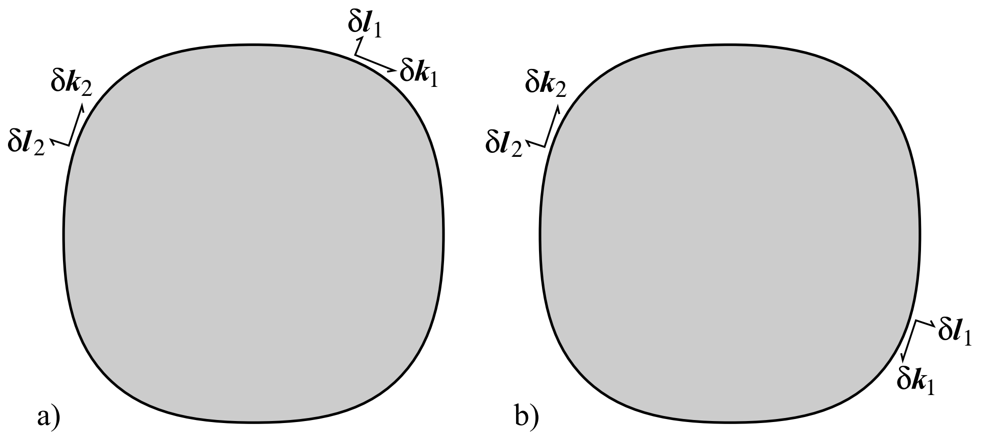

Figure 3: a) For two generic points near a two-dimensional Fermi surface, the tangents $\delta\mathbf{k}_i$ are linearly independent. b) For diametrically opposite points on a parity-symmetric Fermi surface, the tangents are parallel.

$\mathbf{p}_{1,2}$ scatter into momenta $\mathbf{p}_{3,4}$. Expand

$$\mathbf{p}_3 = \mathbf{p}_1 + \delta\mathbf{k}_3 + \delta\mathbf{l}_3, \qquad \mathbf{p}_4 = \mathbf{p}_2 + \delta\mathbf{k}_4 + \delta\mathbf{l}_4. \tag{23}$$

The momentum delta-function in d_s space dimensions is then

$$\delta^{d_\mathrm{s}}(\delta\mathbf{k}_3 + \delta\mathbf{k}_4 + \delta\mathbf{l}_3 + \delta\mathbf{l}_4). \tag{24}$$

Now, for generic momenta, shown in figure 3a, $\delta\mathbf{k}_3$ and $\delta\mathbf{k}_4$ are linearly independent and our neglect of $\delta\mathbf{l}_3$ and $\delta\mathbf{l}_4$ is justified. An electron of momentum $\mathbf{p}_1$ absorbs a phonon of large momentum $\mathbf{q}$ but remains near the Fermi surface. Incidentally, while the picture is two-dimensional, it is easy to see that this argument applies equally for all $d_\mathrm{s} \geq 2$: the possible variations $\delta\mathbf{k}_3$, $\delta\mathbf{k}_4$ span the full d_s-space. However, if $\mathbf{p}_1 = -\mathbf{p}_2$, so that the total momentum is zero, then $\delta^{d_\mathrm{s}}(\delta\mathbf{k}_3 + \delta\mathbf{k}_4)$ is *degenerate*, since one component of the argument vanishes automatically. In this case, one component of the delta-function *does* constrain the $\mathbf{l}$, and so scales inversely to $\mathbf{l}$, as s^{-1}. The four-Fermi interaction then scales as s^0; it is *marginal*.[7]

The rule which emerges is that in any process, if the external momenta are such that the total momentum $\mathbf{P}$ of two the lines in a four-Fermi vertex is constrained to be zero, that vertex is marginal.[8] All other fermionic interactions remain irrelevant. To treat the phonons, the most efficient approach seems to be to consider the effective four-Fermi interaction induced by phonon exchange, and then the same rule applies. We apportion the enhancement as a factor of $s^{-1/2}$ in each vertex, so the phonon-electron interaction scales as s^{-1} above E_1. At E_1, it is then of order $(m/M)^{1/2-1/2}$, unsuppressed in the $M \to \infty$ limit.

The existence of a marginal interaction only at special points in momentum space leaves the free-Fermi picture largely intact, but there are important changes. Consider the matrix element of some current between electrons of momenta $\mathbf{p}$ and $\mathbf{p}'$. The tree level graph is shown in figure 4a, and a one-loop correction in figure 4b. If $\mathbf{p}$ and $\mathbf{p}'$ are both near the Fermi surface but their difference is not small, then the interaction in figure 4b is irrelevant and the loop correction small. In an expectation value, however, where $\mathbf{p} = \mathbf{p}'$, the interaction is marginal and the loop graph is unsuppressed near the Fermi surface. Similarly, both interactions in the two-loop graph of figure 4c are marginal, and so on with any number of bubbles in the chain. The graphs with no irrelevant interactions thus form a geometric series. This is the Landau theory of Fermi liquids: expectation values of currents are modified from their free-field values by the interaction.

The same consideration applies to the electron-phonon vertex. Imagine

[7] Notice that we implicitly assume a discrete symmetry, namely parity invariance of the Fermi surface. Incidentally, one must be a bit careful. One would seem to find the same enhancement for $\mathbf{p}_1 = +\mathbf{p}_2$. In that case, however, the delta-function is degenerate only at one point on the Fermi surface, so that second order terms in $\delta\mathbf{k}$ are nonzero and the enhancement is only by $s^{-1/2}$.

[8] If $\mathbf{P}$ is not exactly zero, the interaction is marginal for $E > v_{\mathrm{F}} P$ and irrelevant for $E < v_{\mathrm{F}} P$.

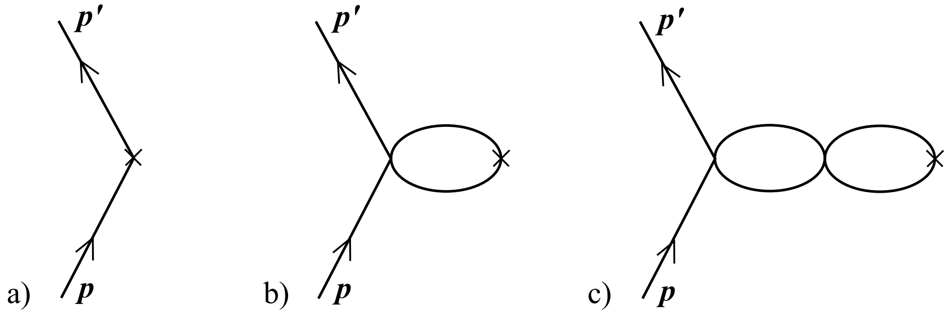

Figure 4: a) Tree-level matrix element of current. b) One-loop correction which is marginal at $\mathbf{p} = \mathbf{p}'$. c) Two-loop correction which is marginal at $\mathbf{p} = \mathbf{p}'$.

coupling in a phonon where the current appears in figure 4. As in the discussion of figure 2, the typical phonon momentum $\mathbf{q}$ is not small, so the interactions are irrelevant and only the tree level graph 4a contributes. This is Migdal's theorem.

Another way to think about the situation is that the interaction is always irrelevant and decreases with E, but for special kinematics an infrared divergence comes in to precisely offset this. We should emphasize the dependence on dimension. The analysis thus far is valid for all $d_{\mathrm{s}} \geq 2$. For one spatial dimension, however, there is no $\mathbf{k}$, only $\mathbf{l}$. The delta-function then always scales as s^{-1}, and the four-Fermi interaction is always marginal. In this case, there is no simplification of the theory—no irrelevant graphs and no Migdal's theorem—and there is more possibility of interesting dynamics.

As was discussed in lecture 1, when we have an interaction which is classically marginal it is important to look at the quantum corrections. Figure 5 shows the four-Fermi vertex and the one-loop correction. The Feynman rules are easily worked out. It is convenient to focus on the case that $V(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4)$ is a constant, which is an approximation often made in prac-

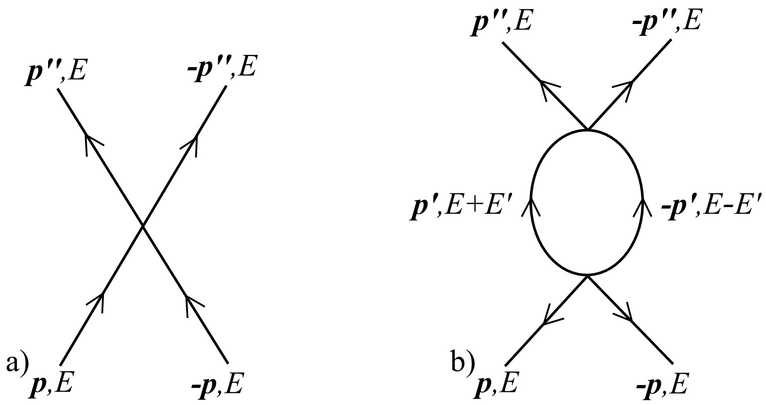

Figure 5: Scattering of electrons $(\mathbf{p}, E)$ and $(-\mathbf{p}, E)$ to $(\mathbf{p}'', E)$ and $(-\mathbf{p}'', E)$. a) Tree level. b) One loop.

tice. Then the one loop four-Fermi amplitude of figure 5b is

$$V^2 \int \frac{dE'\, d^2\mathbf{k}'\, dl'}{(2\pi)^4} \frac{1}{\Big[(1+i\epsilon)(E+E') - v_{\mathrm{F}}(\mathbf{k}')l'\Big]\Big[(1+i\epsilon)(E-E') - v_{\mathrm{F}}(\mathbf{k}')l'\Big]}. \tag{25}$$

We are only interested in the large logarithm, and so do not need to know the details of how the upper cutoff at E_0 is implemented. Evaluating the integral to this accuracy gives

$$V(E) = V - V^2 N\Big\{\ln(E_0/E) + O(1)\Big\} + O(V^3), \tag{26}$$

where

$$N = \int \frac{d^2\mathbf{k}'}{(2\pi)^3} \frac{1}{v_{\mathrm{F}}(\mathbf{k}')} \tag{27}$$

is the density of states at the Fermi energy. Inserting into the renormalization group (10) for V, one determines $b = N$:

$$E\partial_E V(E) = NV^2(E) + O(V^3), \tag{28}$$

with solution

$$V(E) = \frac{V}{1 + NV\ln(E_0/E)}. \tag{29}$$

A repulsive interaction ($V > 0$) thus grows weaker at low energy, while an attractive interaction ($V < 0$) grows stronger.

Now we are in a position to learn something interesting. We start at E_0 with a four-Fermi coupling $V_{\rm C}$ and the phonon-electron coupling g, where we again for simplicity ignore the $\mathbf{k}$ dependences. The subscript C is for Coulomb, since this is some sort of screened Coulomb interaction. Defining $\mu = NV_{\rm C}$ and $\mu^* = NV_{\rm C}(E_1)$, the coupling is renormalized as in eq. (29),

$$\mu^* = \frac{\mu}{1 + \frac{\mu}{2}\ln(M/m)}. \tag{30}$$

The coupling g is not renormalized, by Migdal's theorem. At E_1, scaling has brought the dimensionless magnitude of g to order 1. Integrating out the phonons produces a new $O(1)$ contribution $V_{\rm p}$ to the four-Fermi interaction. It is conventional to define $NV_{\rm p} = -\lambda$, so the total four-Fermi interaction just below E_1 is

$$NV(E_1^-) = \mu^* - \lambda. \tag{31}$$

What happens next depends on the sign of $\mu^* - \lambda$. If it is positive, then $V(E)$ below E_1 just grows weaker and weaker—not very exciting. If, however, it is negative, then the coupling grows and becomes strong at a scale

$$E_c = E_1 e^{-1/(\lambda-\mu^*)} = E_0 \left(\frac{m}{M}\right)^{1/2} e^{-1/(\lambda-\mu^*)}. \tag{32}$$

What happens at strong coupling? It seems to be a fairly general rule of nature that gapless fermions with a strongly attractive interaction are unstable, so that a fermion bilinear condenses, breaks symmetries, and produces a gap. In QCD this breaks the chiral symmetry. Here, the attractive channel involves two electrons, so the condensate breaks the electromagnetic $U(1)$ and produces superconductivity: this is the BCS theory. Because of the simplicity of Fermi liquid theory, it is not necessary to guess about the condensation. Calculating the quantum effective potential for the electron-electron condensate, interactions where a pair of electrons vanish into the vacuum are marginal because the pair has zero momentum. The one-loop graph is thus marginal, but all higher graphs are irrelevant. In other words, the effective potential sums up the 'cactus' graphs, the same as in large-N $O(N)$ models and mean field theory. The gap and the critical temperature are indeed of the form (32), with calculable numerical coefficients.

So BCS superconductivity is another example of 'a marginal coupling grows strong and something interesting happens.' The simple renormalization group analysis gives a great deal of information. It does not get the $O(1)$ coefficient in the critical temperature, but it gets something else which is often omitted in simple treatments of BCS: the renormalized Coulomb repulsion. This correction is significant for at least two reasons. The first is the likelihood of superconductivity, which depends on $\mu^* - \lambda$ being negative. The initial four-Fermi interaction, being a screened Coulomb interaction, is most likely to be repulsive, positive. The phonon contribution $V_{\rm p}$ is attractive, negative, because it arises from second order perturbation theory (hence the sign in the definition of λ). Now, μ^* and λ are both of order 1; since the only small parameter is m/M, this just means that they do not go as a power of M in the $M \to \infty$ limit. In fact, they are both generally within an order of magnitude of unity, and there is a simple model of solids in which they are equal.[9] This model, however, does not take into account the renormalization (30). The renormalization is substantial because the logarithm is approximately 10, and one sees that μ^* cannot exceed 0.2 no matter how large μ is. Thus, superconductivity is more common than it would otherwise be.

The renormalization of the Coulomb correction is also important to the isotope effect, the variation of the critical temperature with ion mass M.

[9] See chapter 26 of Ashcroft and Mermin. It might seem surprising that the phonon interaction, which vanishes when $M \to \infty$, can compete with the Coulomb interaction, which does not. The point is that this is only true at energies below the Debye scale, which also vanishes as $M \to \infty$. At these low energies, the M^{-1} from the interaction cancels against an M^{-1} in the denominator of the phonon propagator.

As M is varied, there is an overall $M^{-1/2}$ in the critical scale (32), coming from the change in the Debye scale. There is also an implicit dependence on M in μ^*. When the Debye scale is lowered, the Coulomb interaction suffers more renormalization and so is reduced; this goes in the opposite direction, favoring superconductivity. The exponent

$$\alpha = -M\partial_M E_c = \frac{1}{2}\left\{1 - \frac{\mu^{*2}}{(\lambda - \mu^*)^2}\right\} \tag{33}$$

is the naive $\frac{1}{2}$ when μ^* is much less than λ, but as μ^*/λ increases α can be substantially smaller, as is found in some materials.

When the coupling $\lambda - \mu^*$ is large, the renormalization group analysis does not give a large ratio between E_1 and E_c. It is then not possible simply to integrate the phonons out; the full phonon propagator must be retained. This leads to the Eliashberg theory, which is still solvable in the sense that it can be reduced to an integral equation. Because of Migdal's theorem, the Schwinger-Dyson equation for the two-point function closes. This resolves the last of the puzzles with which this lecture began. The Eliashberg theory involves several phenomenological functions, which are precisely those appearing in the effective action.

Now, what about high T_c? Figure 6 shows a graph of resistivity versus temperature for a typical high-T_c material. One sees the expected drop to zero at low temperature, but there is also something very puzzling: the resistivity is linear to good accuracy above T_c,

$$\rho(T) \sim A + BT. \tag{34}$$

By comparison, the resistivity above T_c in an ordinary metallic superconductor goes as

$$\rho(T) \sim A + CT^5. \tag{35}$$

How does this relate to what we have learned? We know that conductors are very simple, nearly free, and that any physical effect will have some definite

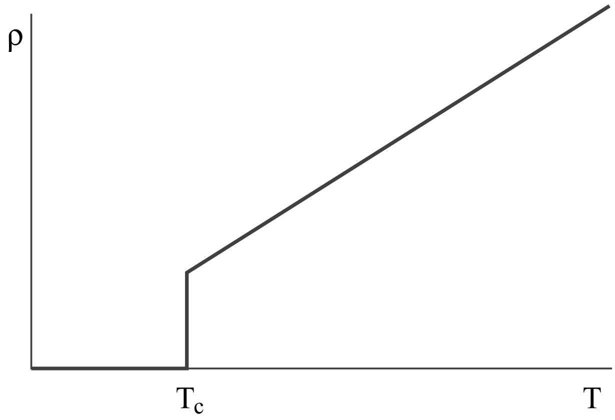

Figure 6: Resistivity versus temperature in a typical high-T_c material: zero below T_c, and linear above.

energy dependence governed by the lowest dimension operator that could be responsible. For example the T^0 resistivity is from impurity scattering.[10] The T^5 resistivity is from phonon scattering; the high power of temperature is because we are below the Debye temperature, so only the long-wavelength phonons remain, their contribution suppressed by phase space and the $\mathbf{q}$ in the vertex. What can give T^1? *Nothing.* Write down the most general possible effective Lagrangian and there is no operator or process that would this power of the temperature. This is one of several related anomalies in these materials. To steal a phrase from Mike Turner, figure 6 shows the conductor from Hell.

To be precise, there is nothing of this magnitude in the generic Fermi liquid theory, but in special cases the infrared divergences are enhanced and new effects are possible. For example, consider free electrons on a square lattice of side a, with amplitude t per unit time to hop to one of the nearest

[10]Incidentally, there is perhaps some indication that A is anomalously small, even zero, in the best-prepared high-T_c materials.

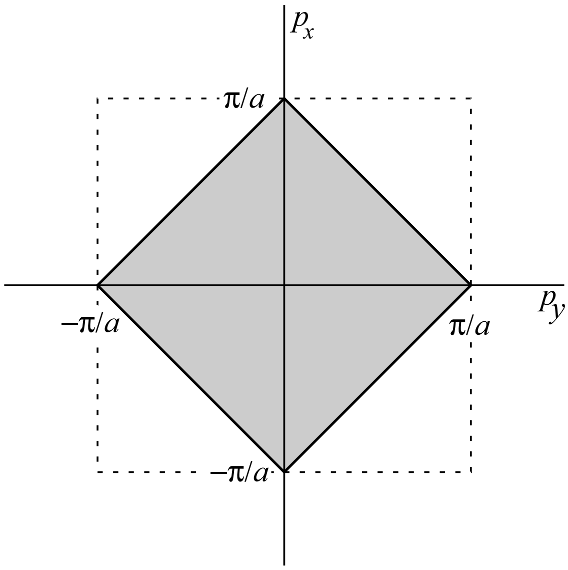

Figure 7: Diamond-shaped Fermi surface for the half-filled squared lattice. The dashed lines at $|P_{x,y}| = \pi/a$ bound the Brillouin zone and are periodically identified.

neighbor sites. This models the CuO planes of the high-T_c materials. Going to momentum space, the one-electron energy works out to

$$\varepsilon(\mathbf{p}) = -t(\cos p_x a + \cos p_y a). \tag{36}$$

For half-filling, $\varepsilon_{\mathrm{F}} = 0$, the Fermi sea is as shown in figure 7. There are two special features. The first is the presence of *van Hove* singularities, the corners of the diamond where $\partial_{\mathbf{p}}\varepsilon = \mathbf{v}_{\mathrm{F}}$ vanishes. At a van Hove singularity the density of states N diverges logarithmically, enhancing the interactions. The second special feature is *nesting*, which means that the opposite edges of the Fermi surface differ by a fixed translation $(\pi/a, \pi/a)$. Because of nesting, the interaction between an electron-hole pair with total momentum $(p_x, p_y) = (\pi/a, \pi/a)$ becomes marginal just as for an electron pair of zero momentum. For positive V this is attractive and favors condensation, producing either a position-dependent charge density $\psi^\dagger_\sigma \psi_\sigma$ (a charge density wave), or a position-dependent spin density $\psi^\dagger_\sigma \, \boldsymbol{\sigma}_{\sigma\sigma'} \psi_{\sigma'}$ (antiferromagnetism). So here are two more phenomena that can arise from the growth of a marginal coupling. It has been proposed that a Fermi surface which has van Hove singularities, or which is nested, or which sits near the antiferromagnetic transition, would have sufficiently enhanced infrared fluctations to account for the anomalous behavior of the high-T_c materials.

Is this plausible? Recall our earlier observation that the shape of the Fermi surface is a relevant parameter—a shift in the Fermi surface acts like a mass, cutting off the enhanced infrared fluctuations. For these to persist down to low energy the Fermi surface must be highly fine-tuned.[11] This might occur in a few very particular substances, but can it be happening here? The linear resistivity is present in many different high-T_c materials (though not all), and it is stable against changes in the doping (filling fraction) of order 5 to 10 percent. The width of the electron band ($E_0 \sim 4t$) is roughly 2 eV, so this represents a shift in the Fermi surface of order 0.1 eV. On the other hand, the anomalous behavior persists below 100 K (0.01 eV). In one substance, Bi2201, which is in the high-T_c class although its transition temperature is rather low, the resistivity is beautifully linear from 700 K down to 7 K ($<$ 0.001 eV), and is stable against changes in doping. So most of the high-T_c materials must be fine-tuned to accuracy 10^{-2} and Bi2201 must be fine-tuned to accuracy 10^{-3}. This is not obviously bad, since there are thousands of substances from which to choose. But if fine tuning is the answer, one would expect it to be spoiled by a small change in the doping, which will shift all the relevant parameters, and this is not the case. In Bi2201, in particular, a fine tuning to accuracy better than 10^{-3} would have to survive shifts of order $10^{-1}E_0$ in the relevant parameters. This seems

[11] Notice that the van Hove singularity is less unnatural than nesting, since the former requires only a single parameter to be tuned (the level, which must pass through the point where $\mathbf{v}_{\mathrm{F}} = 0$), while a very large (in principle infinite) number must be tuned for nesting.

inconceivable unless there are no relevant parameters at all. The special Fermi surface cannot account for the anomalous behavior. We must find a low-energy effective theory which is *natural*, in the same sense as used in particle physics: there are no relevant terms allowed by the symmetries.

We should mention one other possibility. At temperatures above their frequency, phonons do give a linear resistivity. In the high-T_c materials, the phonon spectrum runs up to 0.05 eV, with only 5 to 10 per cent of the density of states below 0.01 eV. To account for the resistivity, even excluding 2201, one would have to suppose that this small fraction accounts for almost all the resistivity, which is implausible. And only the very longest-wavelength phonons, which give the T^5 behavior, survive down to the 0.001 eV of 2201.

It appears that the low energy excitations are not described by the effective field theory that we have described, but by something different. Perhaps we should not be surprised by this, since we began with a guess about the spectrum. Rather, it is surprising that the guess is correct in so many cases.[12] From studies of strongly interacting electron systems one can motivate several other guesses.[13] Typically the low energy theory includes fermions and also gauge fields (which seem like a good thing from the point of view of naturalness) and/or scalars (which do not). For example, anyon theories can be regarded as fermions interacting with a gauge field which has a T-violating Chern-Simons action. Another possibility is fermions with a T-preserving Maxwell action. The normal-state properties of the anyon theory do not seem to have been studied extensively. The T-preserving theory has been argued to give the right behavior, but it is strongly coupled and not well understood. Anderson has proposed what is apparently the Fermi-liquid theory but with the four-Fermi interaction singular in momentum space. The effect of the singularity is to enhance the infrared behavior so that the system behaves as though it were one-dimensional (which, as we have noted, is always

[12] Though there are some other examples of apparent non-Fermi liquid behavior.
[13] The book by Fradkin gives a review of recent ideas.

marginal). It is difficult to understand the origin of this interaction; in particular, it is long-ranged in position space but is not mediated by any field in the low energy theory, which seems to violate locality. Finally, there is a semi-phenomenological idea known as the 'marginal Fermi liquid,' which I have not been able to translate into effective field theory language.

Notice that we have not discussed the mechanism for superconductivity itself; the normal state is puzzling enough. If one can figure out what the low energy theory is, the mechanism of condensation will presumably be evident.

In terms of sheer numbers, there seems to be a move away from exotic field ideas and back to more conventional ones in this subject. This is largely because none of the new theories has made the sort of clear-cut and testable predictions that the BCS theory does. From my discussions with various people and reading of the literature, however, it seems that attempts to explain the normal state properties in a conventional way always require the extreme fine-tunings described above. This seems to be a subject where particle theorists can contribute: the basic issue is one of field theory, where many of the unfamiliar details of solid-state physics are irrelevant (in the technical and colloquial senses).

Q: So 'quasiparticle' means the quantum of an effective field?

A: More-or-less. As used in condensed matter physics, the term has one additional implication that will not always hold in effective field theory: that the decay rate of the quasiparticle vanishes faster than the energy E as E goes to zero. There may be systems for which this is not true (a nonrelativistic system at a nontrivial fixed point being the obvious case), but where one still expects the low energy fluctuations to be represented by some field theory.

Exercise: Consider the term

$$\int dt\, d^3\mathbf{p}_1 d^3\mathbf{p}_2\, U(\mathbf{p}_1, \mathbf{p}_2)\psi^\dagger_\sigma(\mathbf{p}_1)\psi_\sigma(\mathbf{p}_2). \tag{37}$$

This is impurity scattering: notice the lack of momentum conservation. Show that this is marginal, and that its beta-function vanishes.

Exercise: Now consider an impurity of spin s, which can exchange spin with the electron:

$$\int dt\, d^3\mathbf{p}_1 d^3\mathbf{p}_2\, J(\mathbf{p}_1,\mathbf{p}_2)\psi^\dagger_\sigma(\mathbf{p}_1)\ \boldsymbol{\sigma}_{\sigma\sigma'}\psi_{\sigma'}(\mathbf{p}_2)\cdot\mathbf{S}, \tag{38}$$

where $\mathbf{S}$ are the spin-s matrices for the impurity. Show that this is marginal and that the beta-function is negative, taking J to be a constant for simplicity. This is the Kondo problem. The nonvanishing beta-function means that the coupling grows with decreasing energy (for J positive). This is vividly seen in measurements of resistivity as a function of temperature, which increases as T decreases rather than showing the simple constant behavior of potential scattering. When the coupling gets strong, a number of behaviors are possible, depending on the value of s, sign of J, and various generalizations. In particular, in some cases one finds fixed points with critical behavior given by rather nontrivial conformal field theories: more examples of the interesting things that can happen when a marginal coupling gets strong!

Exercise: Show that if the Fermi surface is right at a van Hove singularity, then under scaling of the energy to zero and of the momenta *toward the singular point,* the four-Fermi interaction is marginal in *two* space dimensions. In other words, if all electron momenta in a graph lie near the singularity, the graph is marginal: one does not have the usual simplifications of Landau theory.

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Bibliography

Lecture 1

Wilson's approach to the effective action is developed in

K. G. Wilson, Phys. Rev. **B4** (1971) 3174, 3184.

For further developments see

K. G. Wilson and J. G. Kogut, Phys. Rep. **12** (1974) 75;

F. J. Wegner, in *Phase Transitions and Critical Phenomena, Vol. 6,* ed. C. Domb and M. S. Green, Academic Press, London, 1976;

L. P. Kadanoff, Rev. Mod. Phys. **49** (1977) 267;

K. G. Wilson, Rev. Mod. Phys. **55** (1983) 583.

For the treatment of perturbative renormalization from Wilson's point of view see

J. Polchinski, Nucl. Phys. **B231** (1984) 269;

G. Gallavotti, Rev. Mod. Phys. **57** (1985) 471.

As I hope is clear from the discussion, these ideas do not depend on perturbation theory, and have been used to prove the existence of the continuum limit nonpertubatively in asymptotically free theories. This is done for the Gross-Neveu model in

K. Gawedzki and A. Kupiainen, Comm. Math. Phys. **102** (1985) 1,

and for $D = 4$ non-Abelian gauge theories in a series of papers culminating in

T. Balaban, Comm. Math. Phys. **122** (1989) 355.

The idea that pion physics can be encoded in a Lagrangian appeared in

S. Weinberg, Phys. Rev. Lett. **18** (1967) 188; Phys. Rev. **166** (1968) 1568,

and was developed further in

S. Coleman, J. Wess, and B. Zumino, Phys. Rev. **177** (1969) 2239;

C. G. Callan, S. Coleman, J. Wess, and B. Zumino, Phys. Rev. **177** (1969) 2247;

S. Weinberg, Physica **96A** (1979) 327;

J. Gasser and H. Leutwyler, Ann. Phys. (N.Y.) **158** (1984) 142;

H. Georgi, *Weak Interactions and Modern Particle Theory,* Benjamin/Cummings, Menlo Park, 1984.

Two other classic papers are

T. Appelquist and J. Carazzone, Phys. Rev. **D11** (1975) 2856;

S. Weinberg, Phys. Lett. **B91** (1980) 51.

A recent application, to the 'oblique' corrections in weak interaction theory, is

H. Georgi, Nucl. Phys. **B361** (1991) 339.

An interesting situation arises when heavy particles are in interaction with light, with all kinetic energies small compared to the heavy particle rest masses. By assumption, the number of heavy particles is then fixed, and the low energy degrees of freedom consist of the light fields plus the positions and the spin and internal quantum numbers of the heavy particles. The case of heavy and light quarks has recently been studied extensively; for some discussions see

H. D. Politzer and M. B. Wise, Phys. Lett. **B208** (1988) 504;

H. Georgi, Phys. Lett. **B240** (1990) 447.

The case of pions in interaction with slow nuclei has been studied in

S. Weinberg, Nucl. Phys. **B363** (1991) 3.

For another introduction to effective field theory, see

G. P. Lepage, in Proceedings of TASI 1989.

An interesting history of the changing philosophy of renormalization is

T. Y. Cao and S. S. Schweber, *The Conceptual Foundations and Philosophical Aspects of Renormalization Theory,* 1991 (unpublished).

This is a rare instance where the historians are ahead of most textbooks!

Obviously this is a very selective list. In particular, I have chosen papers based on pedagogic value rather than priority, taking those that apply the effective Lagrangian language in a general way.

Lecture 2

The Wilsonian approach to Fermi liquid theory is developed in

G. Benfatto and G. Gallavotti, J. Stat. Phys. **59** (1990) 541; Phys. Rev. **B42** (1990) 9967;

R. Shankar, Physica **A177** (1991) 530.

It is developed further in unpublished work of Shankar, who likens the Landau theory to a large-N expansion, with N corresponding to the area of the Fermi surface (in units of E). See also

P. W. Anderson, *Basic Notions of Condensed Matter Physics,* Benjamin/Cummings, Menlo Park, 1984.

Standard treatments of Fermi-liquid theory and superconductivity can be found in

A. A. Abrikosov, L. P. Gorkov, and I. E. Dzyaloshinski, *Methods of Quantum Field Theory in Statistical Mechanics,* Dover, New York, 1963;

P. Noziéres, *Theory of Interacting Fermi Systems,* Benjamin, New York, 1964;

J. R. Schrieffer, *Theory of Superconductivity,* Benjamin/Cummings, Menlo Park, 1964;

G. Baym and C. Pethick, *Landau Fermi-Liquid Theory,* Wiley, New York, 1991.

Baym and Pethick is primarily concerned with liquid ^{3}He, another example of a Fermi liquid. Superconductivity at strong coupling is discussed in Abrikosov, et. al., section 35, and Schrieffer, section 7-3. The renormalization of the Coulomb interaction is discussed in

P. Morel and P. W. Anderson, Phys. Rev. **125** (1962) 1263.

A thorough introduction to solid state physics can be found in

N. W. Ashcroft and N. D. Mermin, *Solid State Physics,* Holt, Rinehart and Winston, New York, 1976.

The anomalous properties of the normal state in the high T_c materials are discussed by Philip Anderson and Patrick Lee in

High Temperature Superconductivity, Proceedings of the 1989 Los Alamos Symposium, ed. K. S. Bedell, et. al., Addison-Wesley, Redwood City CA, 1990.

See also the interesting discussion between Anderson and Schrieffer in

Physics Today (June 1991) 55.

The proceedings of the meeting High Temperature Superconductors III, Kanazawa, Japan, 1991, give a comprehensive overview of recent theoretical ideas and experimental results. They are published as

Physica, **C185-C189**.

For see a nice plot of the linear resistivity in Ba2201, see

S. Marten, A. T. Fiory, R. M. Fleming, L. F. Schneemeyer, and J. V. Waszczak, Phys. Rev. **B41**, (1990) 846.

For some Ba2201 data which shows the stability of the linear resistivity against changes in doping, see

P. V. Sastry, J. V. Yakhmi, R. M. Iyer, C. K. Subramanian, and R. Srinivasan, Physica **C178** (1991) 110.

A review of recent work on strongly coupled electron systems is

E. Fradkin, *Field Theories of Condensed Matter Systems,* Addison-Wesley, Redwood City, 1991.