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Spring School on Superstring Theory and Related Topics

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Puzzles and Problems for Gravity and Glue
Lecture II

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The lecture only covered up-to page 12

Lecture 2

- last day saw potential confluence of AdS/CFT and QCD in finite temperature physics

- driven by two surprises

#1/ sQGP discovered @ RHIC

- behaves as a near ideal fluid with

$$\eta/s \simeq .08 - .16$$

#2/ using AdS/CFT, $N=4$ SYM plasma found to have

$$\eta/s = 1/4\pi \simeq .08$$

- would like to discuss above calculation but first want to consider η (and hydrodynamics in more detail

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(Baier et al, 0712.2451)

<u>Hydrodynamics</u> - an effective theory describing dynamics at long distance- and time-scales

- as in standard effective field theories, we are integrating out some microscopic or fast degrees of freedom

- unlike standard effective field theories, hydrodynamics is formulated in terms of equations of motion rather than an action principle ie, conservation of energy and momentum:

$$\nabla_\mu T^{\mu\nu} = 0$$

(simplest case with no (additional) conserved charges)

- to make progress, ~~one~~ assumes that locally the fluid is in <u>thermal equilibrium</u>

temperature $T(x^\mu)$
fluid velocity $u^\mu(x^\mu)$ $\leftarrow$ $u \cdot u = -1$

- expand in powers of derivatives

$$T^{\mu\nu} = T^{\mu\nu}_{(0)} + T^{\mu\nu}_{(1)} + \cdots$$

- to zeroth order, have familiar expression for an ideal fluid:

$$T^{\mu\nu}_{(0)} = \varepsilon\, u^\mu u^\nu + P\, \underbrace{\Delta^{\mu\nu}}_{g^{\mu\nu} + u^\mu u^\nu}$$

$\rightarrow$ spatial projection $u_\mu \Delta^{\mu\nu} = 0$

$$\left.\begin{array}{l} \varepsilon = \varepsilon(T) \\ P = P(T) \end{array}\right\} \text{constitutive eq's}$$

[or eq of state $P = P(\varepsilon)$]

- no dissipation!
  $\rightarrow$ go to next order in derivative expansion

$$T^{\mu\nu}_{(1)} = -\eta(\varepsilon)\, \sigma^{\mu\nu} - \zeta(\varepsilon)\, \Delta^{\mu\nu}(\nabla \cdot u)$$

$\uparrow$ shear viscosity $\qquad$ $\uparrow$ bulk viscosity

$$\sigma^{\mu\nu} = 2\, \nabla^{\langle\mu} u^{\nu\rangle}$$

$$= \Delta^{\mu\alpha} \Delta^{\nu\beta} (\nabla_\alpha u_\beta + \nabla_\beta u_\alpha) - \frac{2}{3} \Delta^{\mu\nu}\, \nabla \cdot u$$

$$(\Delta^{\alpha\beta} \nabla_\alpha u_\beta)$$

note: $\sigma^{\mu\nu} = \sigma^{\nu\mu}$ and $u_\mu \sigma^{\mu\nu} = 0$
$\Delta_{\mu\nu} \sigma^{\mu\nu} = 0$

hence by construction $u_\mu T_{(1)}^{\mu\nu} = 0$!

convention: go to local rest frame: $T^{0i} \equiv 0$

define $T^{00} \equiv \varepsilon$

$\rightarrow$ all higher order / dissipative contributions in spatial stresses $T_{(k)}^{ij}$

why no $\nabla_\mu \varepsilon$?

- we are ~~assuming~~ assuming derivative expansion is a perturbative expansion and so can substitute lower order eq's in higher order terms
- allows us to replace such derivatives in terms of derivatives of u^μ

eg. $u^\mu \nabla_\mu (\varepsilon + P) = -(\varepsilon + P) \nabla_\alpha u^\alpha$

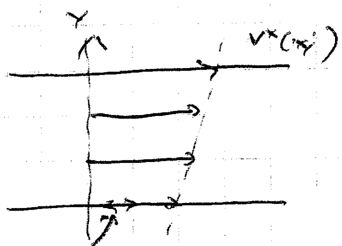
$$= -(\varepsilon + P) \Delta^{\alpha\beta} \nabla_\alpha u_\beta$$

consider a simple case with shear flow

$$g_{\alpha\beta} = \eta_{\alpha\beta}$$

$$u^\alpha = (1, v^x(y), 0, 0)$$

$$T^{xy} = T_{(1)}^{xy} = -\eta\, \partial_y v^x$$



stress induced in flow (as desired)

similarly bulk viscosity term $\propto \nabla \cdot u$
and so ζ induces stress to resist
expansion in fluid flow

in present case, we are interested
in underlying conformal theory and
so we impose an extra constraint,
namely

$$T^{\mu}{}_{\mu} = 0$$

from $T_{(0)}{}^{\mu}{}_{\mu} = 0 \longrightarrow p = \varepsilon/3$

from $T_{(1)}{}^{\mu}{}_{\mu} = 0 \longrightarrow \zeta = 0$

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in present case, first order term
is sufficient for our discussion,
but to illustrate continuation of
expansion to second order

$$T_{(2)}^{\mu\nu} = \eta\, \tau_{\Pi} \left[(u^{\alpha} \nabla_{\alpha}\, \sigma)^{\langle\mu\nu\rangle} + \tfrac{1}{3}\, \sigma^{\mu\nu}\, \nabla \cdot u \right]$$

$$+ \lambda_{1}\, \sigma^{\langle\mu}{}_{\alpha}\, \sigma^{\nu\rangle\alpha} + \cdots$$

- new coefficient, τ_{Π}, is relaxation time
  which we may refer to later

- "$\cdots$" involves 3 more terms
  involving background curvature or
  vorticity (ie $\nabla_{[\mu} u_{\nu]}$)

- this description with 5 new 2nd order
  coefficients applies for a conformal
  fluid – in general case, 13 new
  coefficients appear at second order
  (Romatschke, 0906.4787)

Exercise:

In a conformal fluid, the temperature sets the only scale. Hence one must have

$$\varepsilon = a\, T^4 \qquad P = \tfrac{1}{3} a\, T^4$$

a) using the zero'th order eq's show that:

$$\frac{1}{T} u^\alpha \nabla_\alpha T = -\tfrac{1}{3} \nabla \cdot u$$

$$\frac{1}{T} \Delta^{\mu\alpha} \nabla_\alpha T = - u^\alpha \nabla_\alpha u^\mu$$

which allows one to eliminate any T derivatives in the 1st order stress tensor $T^{\mu\nu}_{(1)}$

b) what is the T dependence of η in a conformal fluid?

(Ans. $\eta = b\, T^3$)

Exercise:

Convince yourself that in a nonrelativistic limit, one recovers the expected Navier-Stokes equations from

$$\nabla_\nu T^{\nu\mu} = 0 .$$

Kubo's formula for η

- couple some set of operators $\mathcal{O}_a(x)$ to sources $J_a(x)$

$$S = S_0 + \int d^4x \; J_a(x)\, \mathcal{O}_a(x)$$

- $J_a(x)$ shift $\langle \mathcal{O}_a \rangle$ away from equilibrium values (which we can assume vanish)

- if J_a is small, linear response theory gives

$$\langle \mathcal{O}_a(x) \rangle = -\int d^4y \; G^R_{ab}(x-y)\, J_b(y)$$

using retarded Green's function:

$$i\, G^R_{ab}(x-y) = \Theta(x^0 - y^0) \langle [\mathcal{O}_a(x), \mathcal{O}_b(y)] \rangle$$

- in present case, operators of interest are components of stress-energy and the standard source is the metric $g_{\mu\nu}$ (ie, consider small fluctuations from flat space to determine two point correlators)

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

- as a special case, consider perturbation with only nonvanishing component $h_{xy}(t)$

- if ~~fluid~~ begins at rest $u^\mu = (1,0,0,0)$ it remains at rest because the special perturbation chosen is homogeneous

- hence only effect is to disturb the metric, Christoffel symbols, ... in stress tensor
- only component:

$$T^{xy} = -P\, h_{xy} + \eta\, \dot{h}_{xy} + \cdots \quad \leftarrow \ddot{h}_{xy} \text{ from } T^{\mu\nu}_{(2)}$$

(applies even if $S \neq 0$)

- compare to linear response theory

→ in low frequency limit and with zero spatial momentum, retarded Green's function

$$G^R_{xy,xy} = -i \int d^4x\, e^{i\omega t}\, \Theta(t) \langle [T_{xy}(t,x), T_{xy}(0,0)] \rangle$$

$$= P - i\eta\omega + O(\omega^2)$$

Kubo's formula now relates η to Green's function

$$\eta = -\lim_{\omega \to 0} \frac{1}{\omega} \operatorname{Im} G^R_{xy,xy}(\omega, 0)$$

- we can determine η from $G^R_{xy,xy}$ but this must be determined at finite temperature and strong coupling for sQGP or $N=4$ SYM (really hard!!)

- AdS / CFT turns calculation for $N=4$ SYM to simple weakly coupled calculation in dual gravity theory

Exercise

Verify the expression for T^{xy} at the top of the page

AdS / CFT Thermodynamics

gauge theory thermodynamics
= black hole thermodynamics

Witten, hep-th/9803131
(and many more)

hence first step is to replace AdS_5 metric with (planar) AdS_5 black hole:

$$ds^2 = \frac{r^2}{L^2}\left(-f(r)\,dt^2 + d\vec{x}^2\right) + \frac{L^2}{r^2}\frac{dr^2}{f(r)}$$

where $f(r) = 1 - r_0^4/r^4$

simplest approach is to Wick rotate $t \to i\tau$ and apply semi-classical techniques developed ~~with~~ Euclidean path integral

eg given

$$ds_E^2 = r^2/L^2\left(+\ f(r)\,d\tau^2 + d\vec{x}^2\right) + L^2/r^2\ \frac{dr^2}{f(r)}$$

demanding geometry is non singular @ $r = r_0$ requires τ is periodic with period: $\Delta\tau = \beta = \pi L^2/r_0$

$\longrightarrow$ temperature: $T = r_0/\pi L^2$

horizon entropy: $S = \dfrac{A}{4G_5} = \dfrac{1}{4G_5}\dfrac{r_0^3}{L^3}\int d^3x$

$\longrightarrow$ entropy density:

$$s = \frac{S}{\int d^3x} = \frac{1}{4G_5}\frac{r_0^3}{L^3} = \frac{V_{S^5}}{4G_{10}}\ r_0^3/L^3$$

$$s = \frac{\pi^2}{2} N_c^2 T^3$$

Exercise:

Verify the above result using

$$\frac{1}{16\pi G_{10}} = \frac{2\pi}{(2\pi l_s)^8 g_s^2} \qquad V_{S^5} = \pi^3 L^5$$

Exercise

Calculate the Euclidean action for the black hole metric

$$I_E = -\frac{1}{16\pi G_5}\int d^5x \sqrt{g}\left(\frac{12}{L^2} + R\right) + \text{boundary terms}$$

and apply background subtraction (ie, subtract the action for empty AdS_5 space – boundary terms are irrelevant in difference!!)

Verify the following:

free energy density:

$$f = (I_E - I_E^0)/\beta\int d^3x = -\frac{\pi^2}{8} N_c^2 T^4$$

energy density:

$$\rho = -T^2 \frac{\partial}{\partial T}(f/T) = \frac{3\pi^2}{8} N_c^2 T^4$$

entropy density:

$$s = -\frac{\partial f}{\partial T} = \frac{\pi^2}{2} N_c^2 T^3$$

Holographic calculation of η

recall Kubo's formula:

$$\eta = - \lim_{\omega \to 0} \operatorname{Im} G^R_{xy,xy}(\omega, 0)$$

- need retarded Green's function for T^{xy} in thermal ~~state~~ ensemble

- AdS/CFT translates this calculation to calculating graviton correlator for mode h_{xy} in black hole bkgd.

- as warm-up, consider correlator for a massless scalar

- convenient to work with new radial coordinate $u = r_0^2/r^2$

$$ds^2 = \frac{r_0^2}{L^2} \frac{1}{u} \left(- f(u)\, dt^2 + d\vec{x}^2 \right) + \frac{L^2}{4u^2} \frac{du^2}{f(u)}$$

with $f(u) = 1 - u^2$

horizon : $r = r_0 \longrightarrow u = 1$
asymp. boundary: $r = \infty \longrightarrow u = 0$

considering asymptotic solutions of scalar wave eq $\Box \phi = 0$, one finds two indep. solutions

$$\phi = \phi_0(x) + \phi_1(x)\, u^2$$

AdS/CFT gives us the interpretation that ϕ_0 acts as a source / current in the CFT coupled to a dim-4 operator

$$\int d^4x \; \phi_0(x)\, \mathcal{O}(x)$$

(if ϕ is dilaton, $\mathcal{O}$ is $N=4$ SYM lagrangian density)

so we have

$$Z_{CFT}(\phi_0) = Z_{string}(\phi_0)$$
$$\simeq e^{i I_{grav}(\phi_0)}$$

using a saddle-point approximation

in this approx, the connected Green's function comes from

$$\frac{\delta I_{grav}}{\delta\phi_0(x)\,\delta\phi_0(y)}\bigg|_{\phi_0=0} = \langle \mathcal{O}(x)\,\mathcal{O}(y)\rangle$$

saddle-point approx means we evaluate I_{grav} "on-shell" and so we need to solve for ϕ in BH background

the appropriate boundary condition on interior (ie @ horizon) will then fix ϕ_1 in terms of ϕ_0

near horizon, writing

$$\phi = e^{-i\omega t + i\vec{k}\cdot\vec{x}}\,\phi_k(u)$$

the wave equation becomes

$$0 = \partial_u^2\phi_k + \frac{1}{1-u}\partial_u\phi_k + \left(\frac{\omega}{4\pi T}\right)\phi_k$$

with solutions:

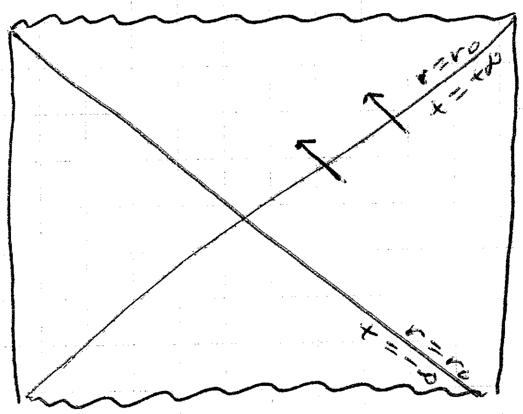
$$\phi_k^{\pm} \sim (1-u)^{\pm i\frac{\omega}{4\pi T}} \leftarrow \left(\text{recall } T = \frac{r_0}{\pi L^2}\right)$$

- solutions oscillate (not damped) near horizon

- appropriate b.c. comes from consideration of particular correlator one needs

- in Kubo formula, consider retarded Green's function and appropriate b.c. is that we want infalling waves

(Herzog + Son, hep-th/0212072)



- for retarded Green's function, natural that sources only cause disturbances to the future

- this picks out the solutions

$$\phi_k^- \sim (1-u)^{-i\frac{\omega}{4\pi T}}$$

Exercise

a) Show the near horizon metric (with $u \sim 1$) becomes

$$ds^2 = -\frac{2r_0^2}{L^2}(1-u)\,dt^2 + \frac{r_0^2}{L^2}d\vec{x}^2 + \frac{L^2 du^2}{8(1-u)}$$

b) use this to verify the form of the near horizon eq. and solutions on the previous page

c) show the coord. singularity in the near-horizon metric can be resolved by replacing t with one of

$$dv^{\pm} = dt \pm \frac{1}{4\pi T} \frac{du}{1-u}$$

integrate these expressions to show the outgoing coord v^+ is finite on the past event horizon and the ingoing coord v^- is finite on the future event horizon

hence show an infalling wave takes the form

$$e^{-i\omega v^-} = e^{-i\omega t} (1-u)^{-i\frac{\omega}{4\pi T}}$$

- prescription to evaluate the two-pt correlator now requires we evaluate the quadratic action on-shell

- upon integrating by parts and using the eom, one is left with a boundary term

$$I = -\tfrac{1}{2} \int d^5x \sqrt{-g}\, g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$$

$$= \tfrac{1}{2} \int d^4x\, \phi \underbrace{\left(-\sqrt{-g}\, g^{uu} \partial_u \phi\right)}\Big|_{u=0}$$

following Iqbal + Liu, 0809.3808 we introduce a "radial momentum" $\Pi(u) \equiv \frac{\delta \mathcal{L}}{\delta \partial_u \phi}$

$\Pi_k \equiv \frac{\delta \mathcal{L}}{\delta \partial_u \phi_{-k}}$

$$= -\frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \phi_{-k} \Pi_k \Big|_{u=0}$$

- here we Fourier transform for convenience

- note in principle, one might have included an additional boundary term at the horizon. however the correct prescription is only to keep the asymptotic boundary term at $u = 0$

- in more covariant discussion, we would note the horizon is not a boundary and this term gets pushed to the other asymptotic boundary

- instructed to make the 2nd variation

$$G_R(\omega, \vec{q}) = \frac{\delta^2 I}{\delta \phi_0 \; \delta \phi_0}$$

but simpler to evaluate

$$G_R(\omega, \vec{q}) = \left. \frac{\phi_{-k}(u) \; \Pi_k(u)}{\phi_{-k}(u) \; \phi_k(u)} \right|_{u \to 0}$$

where factors are added in the denominator to properly normalize the Green's function

- now in analogy to the Kubo formula let us evaluate the "transport coefficient"

$$\begin{aligned} \xi &= - \lim_{\omega \to 0} \frac{1}{\omega} \mathrm{Im}\, G_R(\omega, \vec{0}) \\ &= - \lim_{\omega \to 0} \left. \mathrm{Im} \frac{\Pi(u)}{\omega \phi(u)} \right|_{u \to 0} \end{aligned}$$

(for notational convenience, $\Pi(u) = \Pi_{\{\omega, \vec{q}=0\}}(u)$)

- now by definition at $\vec{q} = 0$, eom is

$$\begin{aligned} \partial_u \Pi(u) &= A(u)\, \omega^2 \; \phi(u) \\ &= O(\omega^2) \end{aligned}$$

$\therefore$ if the low frequency limit, we can treat $\Pi(u)$ as a constant.

further for a massless scalar, ϕ = constant is an exact solution at $\omega = 0 = \vec{q}$

hence for small ω

$$\omega \, \partial_u \phi(u) = O(\omega^2)$$

and in same low frequency limit, we can also treat the denominator as constant

that is, we can evaluate the RHS above at any radius u

$$\xi = - \lim_{\omega \to 0} \operatorname{Im} \frac{\Pi(u)}{\omega \phi(u)}$$

ie, in low frequency limit, bulk flow of this expression is trivial

simplest to evaluate at the horizon $(u \to 1)$ where b.c. gave

$$\phi(u) \simeq (1-u)^{-i\omega/4\pi T}$$

hence

$$\Pi(u) = -\sqrt{-g}\, g^{uu} \partial_u \phi(u)$$

$$= -i \, \omega \, \phi(u) \, r_0^3/L^3$$

$$\therefore \quad \xi = r_0^3/L^3$$

Exercise

Verify the evaluation of ξ given above.

for viscosity, we need to repeat the same calculation for perturbations of the metric: $h_{xy}\ (u, x^\mu)$

Note however that if we write

$$h_{xy}\ (u, x^\mu) = \frac{r_0^2}{L^2 u}\ \phi(u, x^\mu)$$

then the quadratic action for ϕ is precisely that of the massless scalar considered above — up to an extra overall factor of $\frac{1}{16\pi G_5}$

hence without any further effort we can write

$$\eta = \frac{1}{16\pi G_5}\ r_0^3/L^3 = \frac{\pi}{8}\ N_c^2\ T^3$$

recall we found

$$s = \frac{1}{4 G_5}\ r_0^3/L^3 = \frac{\pi^2}{2}\ N_c^2\ T^3$$

and hence

$$\eta/s = 1/4\pi$$

Exercise:

Verify this result for η/s.

<u>Note</u> ~~[crossed out]~~ our expectation is that "the viscosity is small" but here we found $\eta \propto N_c^2$ and so in fact is extremely large

when we say the sQGP or N=4 SYM plasma has a low viscosity it is in comparison to the entropy density ie, η/s is remarkably small in comparison to normal materials

the result $\eta/s = 1/4\pi$ has been shown to apply for all gauge theories (in large N_c and large λ limit) that allow for a super gravity dual

eg Buchel, hep-th/0311175

Benincasa et al, hep-th/0610145

it has been tested for various gauge groups, matter content, chemical potentials, non-commutative (spatial) directions, external background fields, as well as Dp-branes and M-branes

these observations (and others) lead Kovtun, Son + Starinets (hep-th/0309213) to conjecture that this represented a lower bound on η/s that all materials should respect, ie,

$$\eta/s \geq 1/4\pi$$

$\longrightarrow$ more on conjectured KSS bound next lecture