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Towards holographic duality for condensed matter
Lecture II

J.A. McGreevy

MIT (Massachusetts Institute of Technology)
Cambridge
USA

Towards
Physical Applications of Holographic Duality
Part 2

John McGreevy, MIT

Outline

1. Holographic duality with a view toward condensed matter
   [review: JM, 0909.0518]
2. Gravity duals of non-relativistic QFTs

   [Son, 0804.3972
   Koushik Balasubramanian, JM, 0804.4053
   Herzog, Rangamani, Ross, 0807.1099
   Maldacena, Martelli, Tachikawa, 0807.1100
   Allan Adams, KB, JM, 0807.1111
   KB, JM, to appear]
3. Non-Fermi liquids from non-holography
4. Non-Fermi liquids from holography

Recap

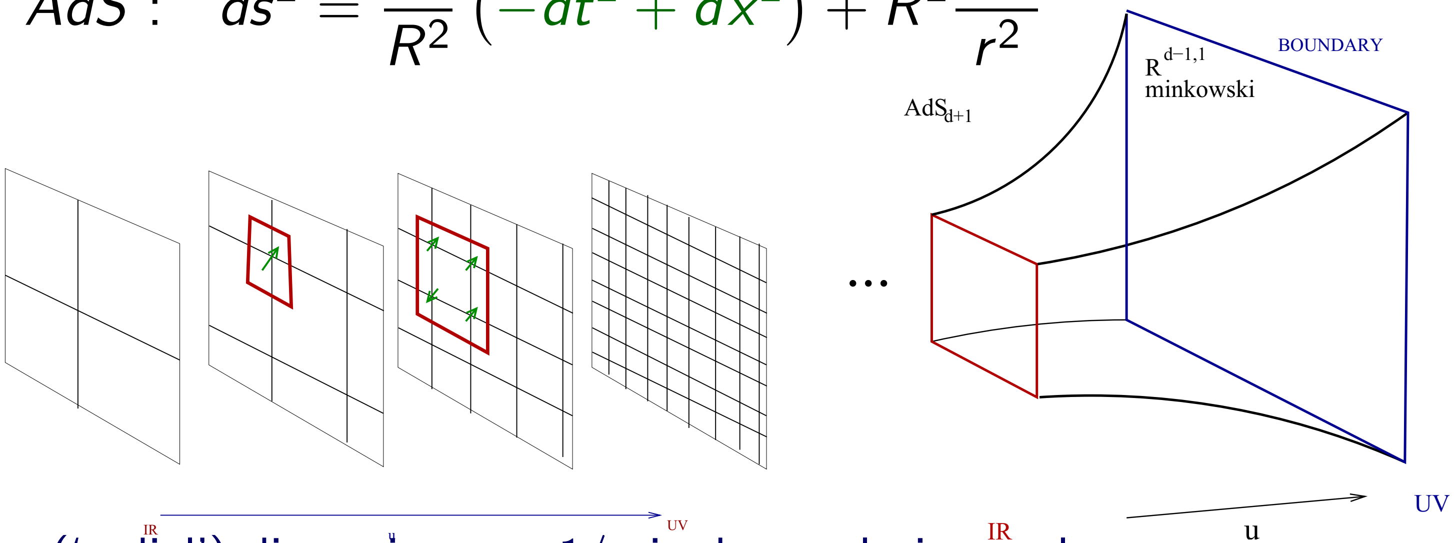
gravity in spacetimes$_{d+1}$
with timelike asymptotic boundaries $\quad \leftrightsquigarrow \quad QFT_d$

important special case:

gravity in AdS_{d+1} = d-dimensional conformal field theory (CFT)

isometries of AdS_{d+1} $\leftrightsquigarrow$ conformal symmetry

$$AdS: \quad ds^2 = \frac{r^2}{R^2}\left(-dt^2 + d\vec{x}^2\right) + R^2\frac{dr^2}{r^2}$$



The extra ('radial') dimension $u = 1/z$ is the resolution scale.

fields in bulk $\leftrightsquigarrow$ (possibly-) running couplings

$$Z_{QFT}[\text{sources}, \phi_0] \approx e^{-S_{\text{bulk}}[\text{boundary conditions at } u\to\infty]}\Big|_{\text{saddle of } S_{\text{bulk}}}$$

Vacuum of CFT, euclidean case

Return to the scalar wave equation in momentum space:

$$0 = [z^{d+1}\partial_z(z^{-d+1}\partial_z) - m^2L^2 - z^2k^2]f_k(z)$$

If $k^2 > 0$ (spacelike or Euclidean) the general solution is
(a_K, a_I, integration consts):

$$f_k(z) = a_K z^{d/2}K_\nu(kz) + a_I z^{d/2}I_\nu(kz), \quad \nu = \Delta - \frac{d}{2} = \sqrt{(d/2)^2 + m^2L^2}.$$

In the interior of AdS ($z \to \infty$), the Bessel functions behave as

$$K_\nu(kz) \overset{z\to\infty}{\approx} e^{-kz} \qquad I_\nu(kz) \overset{z\to\infty}{\approx} e^{kz} \ .$$

regularity in the interior uniquely fixes $\underline{f}_k \propto K_\nu$.
Plugging this into the action S gives $\langle\mathcal{O}(x)\mathcal{O}(0)\rangle \sim \frac{1}{|x|^{2\Delta}}$
note: $\exists$ nonlinear uniqueness statement, 'Graham-Lee theorem'

Real-time

In Lorentzian signature with timelike k^2 $(\omega^2 > \vec{k}^2)$,
$\exists$ many solutions with the same UV behavior ($z \to 0$), different IR behavior:

$$z^{d/2} K_{\pm\nu}(iqz) \overset{z\to\infty}{\approx} e^{\pm iqz} \qquad q \equiv \sqrt{\omega^2 - \vec{k}^2}$$

these modes oscillate near the Poincaré horizon.
this ambiguity reflects the multiplicity of real-time Green's f'ns.

Important example: retarded Green's function, describes causal response of the system to a perturbation.

Linear response: nothing fancy, just QM

The retarded Green's function for two observables $\mathcal{O}_A$ and $\mathcal{O}_B$ is

$$G^R_{\mathcal{O}_A\mathcal{O}_B}(\omega, k) = -i\int d^{d-1}x dt\ e^{i\omega t - ik\cdot x}\theta(t)\langle[\mathcal{O}_A(t,x), \mathcal{O}_B(0,0)]\rangle$$

$\theta(t) = 1$ for $t > 0$, else zero.

(We care about this because it determines what $\langle\mathcal{O}_A\rangle$ does if we kick the system via $\mathcal{O}_B$.)

the source is a time dependent perturbation to the Hamiltonian:

$$\delta H(t) = \int d^{d-1}x \phi_{B(0)}(t,x)\mathcal{O}_B(x)\,.$$

$$\begin{aligned}\langle\mathcal{O}_A\rangle(t,x) &\equiv \mathrm{Tr}\ \rho(t)\,\mathcal{O}_A(x) \\ &= \mathrm{Tr}\ \rho_0\, U^{-1}(t)\mathcal{O}_A(t,x)U(t)\end{aligned}$$

in interaction picture: $U(t) = Te^{-i\int^t \delta H(t')dt'}$ (e.g. $\rho_0 = e^{-\beta H_0}$)

Linear response, cont'd

linearize in small perturbation:

$$
\begin{aligned}
\delta\langle\mathcal{O}_A\rangle(t,x) &= -i\mathrm{Tr}\ \rho_0 \int^t dt'[\mathcal{O}_A(t,x), \delta H(t')] \\
&= -i \int^t d^{d-1}x' dt' \langle[\mathcal{O}_A(t,x), \mathcal{O}_B(t',x')]\rangle \phi_{B(0)}(t',x') \\
&= \int dx' G_R(x,x')\phi_B(x')
\end{aligned}
$$

fourier transform:

$$
\delta\langle\mathcal{O}_A\rangle(\omega,k) = G^R_{\mathcal{O}_A\mathcal{O}_B}(\omega,k)\delta\phi_{B(0)}(\omega,k)
$$

gives expression above.

Linear response, an example

perturbation: an external electric field, $E_x = i\omega A_x$
couples via $\delta H = A_x J^x$ where J is the electric current $(\mathcal{O}_B = J_x)$
response: the electric current $(\mathcal{O}_A = J_x)$

$$\delta\langle\mathcal{O}_A\rangle(\omega,k) = G^R_{\mathcal{O}_A\mathcal{O}_B}(\omega,k)\delta\phi_{B(0)}(\omega,k)$$

it's safe to assume $\langle J\rangle_{E=0} = 0$:

$$\langle\mathcal{O}_J\rangle(\omega,k) = G^R_{JJ}(\omega,k)A_x = G^R_{JJ}(\omega,k)\frac{E_x}{i\omega}$$

Ohm's law: $J = \sigma E$

$$\implies \textit{Kubo formula}: \quad \boxed{\sigma(\omega,k) = \frac{G^R_{JJ}(\omega,k)}{i\omega}}$$

Holographic real-time prescription is easy

Claim [Son-Starinets 2002]: corresponds to the solution which at $z \to \infty$ describes stuff falling into the horizon

- Both the retarded response and stuff falling through the horizon describe things that *happen*, rather than *unhappen*.
- You can check that this prescription gives the correct analytic structure of $G_R(\omega)$ ([Son-Starinets] and all the hundreds of papers that have used this prescription).
- It has been derived from a holographic version of the Schwinger-Keldysh prescription [Herzog-Son, Maldacena, Skenderis-van Rees].

The fact that stuff goes past the horizon and doesn't come out is what breaks time-reversal invariance in the holographic computation of G^R.

Here, the ingoing choice is $\phi(t, z) \sim e^{-i\omega t + iqz}$:

as t grows, the wavefront moves to larger z.

(the solution which computes causal response is $z^{d/2} K_{+\nu}(iqz)$.)

The same prescription, adapted to the black hole horizon, works in the finite temperature case.

What to do with the solution

determining $\langle \mathcal{O}\mathcal{O} \rangle$ is like a scattering problem in QM

The solution of the equations of motion, satisfying the desired IR bc, behaves near the boundary as

$$\underline{\phi}(z,x) \approx \left(\frac{z}{L}\right)^{\Delta_-} \phi_0(x) \left(1 + \mathcal{O}(z^2)\right) + \left(\frac{z}{L}\right)^{\Delta_+} \phi_1(x) \left(1 + \mathcal{O}(z^2)\right);$$

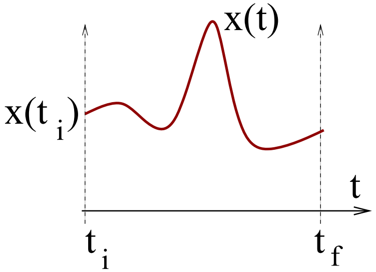
this formula defines the coefficient ϕ_1 of the subleading behavior of the solution.

All the information about G is in ϕ_0, ϕ_1.

recall: $Z[\phi_0] \equiv e^{-W[\phi_0]} \simeq e^{-S_{\rm bulk}[\underline{\phi}]}\big|_{\phi \overset{z\to 0}{\to} z^{\Delta_-}\phi_0}$

confession: this is a euclidean eqn. next: a nice general trick. [Iqbal-Liu]

classical mechanics interlude: consider a particle in 1d with action $S[x] = \int_{t_i}^{t_f} dtL$. The variation of the action with respect to the initial value of the coordinate is the initial momentum:



$$\Pi(t_i) = \frac{\delta S}{\delta x(t_i)}, \quad \Pi(t) \equiv \frac{\partial L}{\partial \dot{x}} \quad . \tag{1}$$

Thinking of the radial direction of AdS as time, a mild generalization of (1): [Iqbal-Liu]

$$\langle \mathcal{O}(x) \rangle = \frac{\delta W[\phi_0]}{\delta \phi_0(x)} = \lim_{z \to 0} \left(\frac{z}{L} \right)^{\Delta_-} \Pi(z, x)|_{\text{finite}},$$

where $\Pi \equiv \frac{\partial \mathcal{L}}{\partial(\partial_z \phi)}$ is the bulk field-momentum with z treated as time.

two minor subtleties:

(1) the factor of z^{Δ}_{-} arises because of our renormalization of ϕ: $\phi \sim z^{\Delta_-} \phi_0$, so $\frac{\partial}{\partial \phi_0} = z^{-\Delta_-} \frac{\partial}{\partial \phi(z=\epsilon)}$.

(2) Π itself in general has a term proportional to the source ϕ_0

Linear response from holography

with these caveats, away from the support of the source:

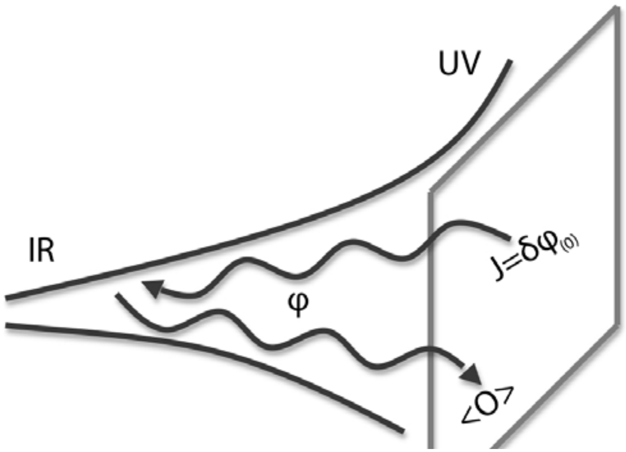
$$\langle \mathcal{O}(x) \rangle = \mathfrak{K} \frac{2\Delta - d}{L} \phi_1(x).$$

linearize in the size of the perturbing source:

$$\langle \mathcal{O}(x) \rangle = G_R \cdot \delta\phi_0$$

summary: the leading behavior of the solution encodes the source *i.e.* the perturbation of the *action* of the QFT.
the coefficient of the subleading falloff encodes the response [Balasubramanian et al, 1996].

$$\boxed{G \propto \frac{\delta\phi_1}{\delta\phi_0}}$$



[figure: Hartnoll, 0909.3553]

(Quasi)normal modes

determining $\langle \mathcal{O}\mathcal{O} \rangle$ is like a scattering problem in QM

The solution of the equations of motion, satisfying the the desired IR bc, behaves near the boundary as

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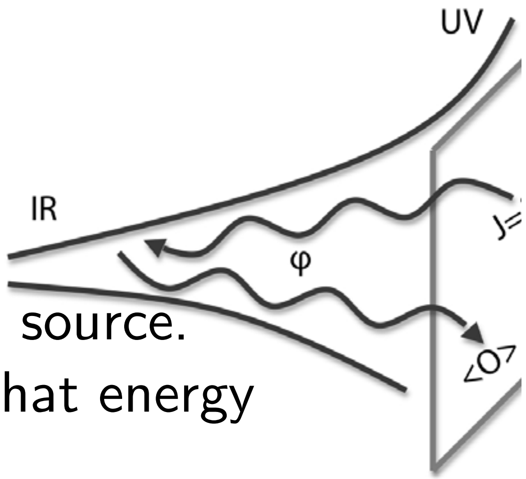
summary: the leading behavior of the solution encodes the source *i.e.* the perturbation of the *action* of the QFT.
the coefficient of the subleading falloff encodes the response

$$\boxed{G \propto \frac{\phi_1}{\phi_0}}$$



[figure: Hartnoll, 0909.3553]

G has poles when $\phi_1 \neq 0, \phi_0 = 0$: response without source.
this means that the system has an actual mode at that energy
(if $\omega \in \mathbb{C}$, 'quasinormal mode')

A useful visualization: 'Witten diagrams'

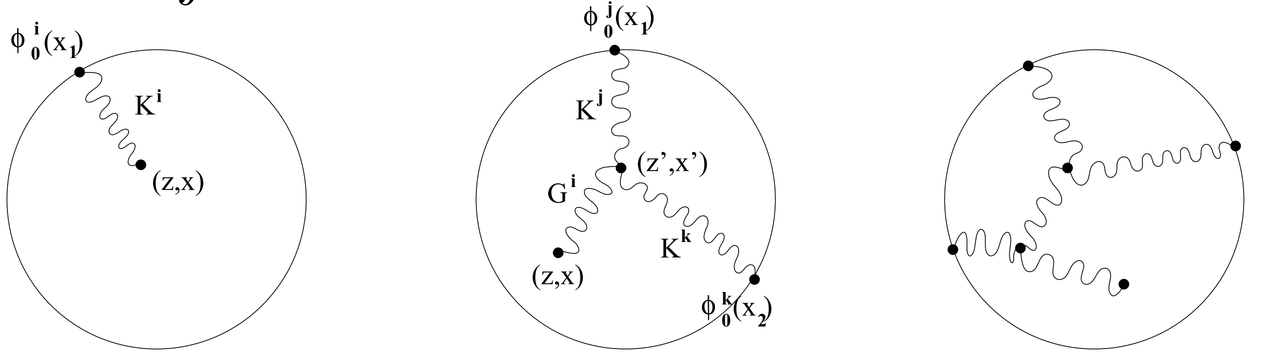
e.g. consider 3-point function, $\langle \mathcal{O}\mathcal{O}\mathcal{O} \rangle = \left(\frac{\delta}{\delta \phi_0}\right)^3 \ln Z|_{\phi_0=0}$.

cubic coupling matters:

$$(\Box - m_i^2)\phi_i(z,x) = b\phi_j\phi_k\epsilon^{ijk}$$

Solve perturbatively in ϕ_0: (K, G are Green's f'ns for $\Box - m_i^2$)

$$\underline{\phi}^i(z,x) = \int d^d x_1 K^{\Delta_i}(z,x;x_1)\phi_0^i(x_1)$$
$$+ \quad b\epsilon^{ijk} \int d^d x' dz' \sqrt{g} G^{\Delta_i}(z,x;z',x')$$
$$\times \quad \int d^d x_1 \int d^d x_2 K^{\Delta_j}(z',x';x_1)\phi_0^j(x_1) K^{\Delta_k}(z',x';x_2)\phi_0^k(x_2) + o(b^2\phi_0^3$$



external legs $\leftrightarrow$ sources ϕ_0, vertices $\leftrightarrow$ bulk interactions

Finite temperature

AdS was scale invariant. sol'n dual to *vacuum* of CFT.
saddle point for CFT in an ensemble with a scale (some relevant perturbation) is a geometry which approaches AdS near the bdy:

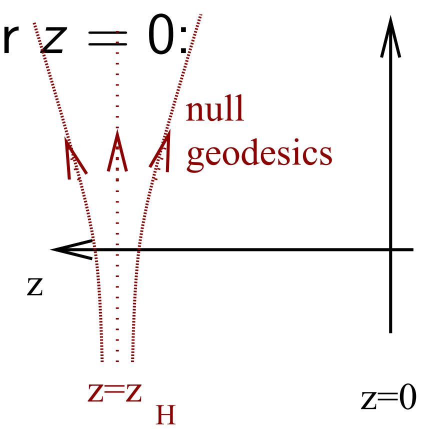
$$ds^2 = \frac{L^2}{z^2}\left(-f\,dt^2 + d\vec{x}^2 + \frac{dz^2}{f}\right) \qquad f = 1 - \frac{z^d}{z_H^d}$$

When the emblackening factor $f \overset{z\to 0}{\to} 1$ this is the Poincaré AdS metric.

[exercise: check that this solves the same EOM as AdS.]

It has a horizon at $z = z_H$, where the emblackening factor $f \propto z - z_H$

Events at $z > z_H$ can't influence the boundary near $z = 0$:



Physics of horizons

Claim: geometries with horizons describe thermally mixed states.

Why: Near the horizon ($z \sim z_H$),

$$ds^2 \sim -\kappa^2\rho^2 dt^2 + d\rho^2 + \frac{L^2}{z_H^2} d\vec{x}^2 \quad \rho^2 \equiv \frac{2}{\kappa z_H^2}(z - z_H) + o(z - z_H)^2$$

$\kappa \equiv \frac{4}{|f'(z_H)|} = d/2z_H$ is called the 'surface gravity'

Ccontinue this geometry to euclidean time, $t \to i\tau$:

$$ds^2 \sim \kappa^2\rho^2 d\tau^2 + d\rho^2 + \frac{L^2}{z_H^2} d\vec{x}^2 \tag{2}$$

which looks like $\mathbb{R}^{d-1} \times \mathbb{R}^2_{\rho,\kappa\tau}$ with polar coordinates $\rho, \kappa\tau$.

There is a deficit angle in this plane unless we identify

$$\kappa\tau \simeq \kappa\tau + 2\pi.$$

A deficit angle would mean nonzero Ricci scalar curvature, which would mean that the geometry is *not* a saddle point of our bulk path integral.

So: $T = \kappa/(2\pi) = 1/(\pi z_H)$.

(Note: this is the temperature of the Hawking radiation.)

Static BH describes thermal equilibrium

This identification on τ also applies at the boundary. If

$$ds^2_{bulk} \overset{z \to 0}{\approx} \frac{dz^2}{z^2} + \frac{L^2}{z^2} g^{(0)}_{\mu\nu} dx^\mu dx^\nu$$

then, up to a factor, the boundary metric is $g^{(0)}_{\mu\nu}$.
This includes making the euclidean time periodic.

$$A = \int_{z=z_H, \text{fixed } t} \sqrt{g}\, d^{d-1}x = \left(\frac{L}{z_H}\right)^{d-1} V$$

The Bekenstein-Hawking entropy is

$$S = \frac{A}{4G_N} = \frac{L^{d-1}}{4G_N} \frac{V}{z_H^{d-1}} = \frac{N^2}{2\pi} (\pi T)^{d-1} V = \frac{\pi^2}{2} N^2 V T^{d-1} \quad . \qquad (3)$$

The Bekenstein-Hawking entropy *density* is

$$s_{BH} = \frac{S_{BH}}{V} = \frac{a_{BH}}{4G_N}.$$

where $a_{BH} \equiv \frac{A}{V}$ is the 'area density' of the black hole.

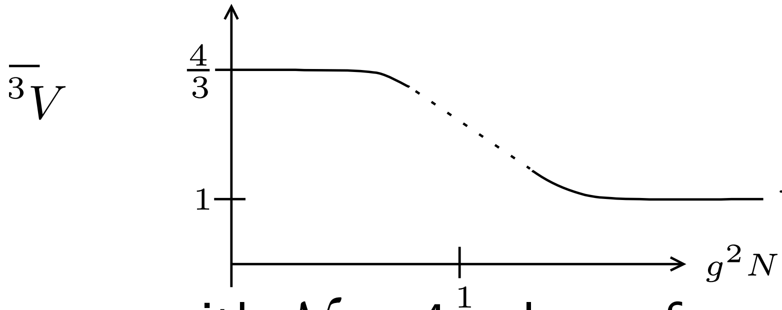
QFT thermo from black holes cont'd

how to think about this:

$$Z_{CFT}(T) \approx e^{-S^{\text{eucl}}_{\text{bulk}}[\underline{g}]}$$

$\underline{g}$ is the saddle with the correct periodicity of eucl time at the bdy.

(warning: boundary terms in action are important)



$$Z_{CFT}(T) = e^{-\beta F}$$

$$-\frac{F}{V} = \frac{L^2}{16\pi G_N}\frac{1}{z_H^4} = \frac{\pi^2}{8}N^2T^4.$$

with $\mathcal{N}=4$ values of parameters, $F(\lambda=\infty) = \frac{3}{4}F(\lambda=0)$.

checks:

- S_{BH} (horizon) $=$ $-\frac{\partial F}{\partial T}$ (integral over all spacetime)

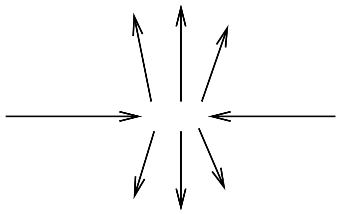
(relatedly: first law of thermo holds)

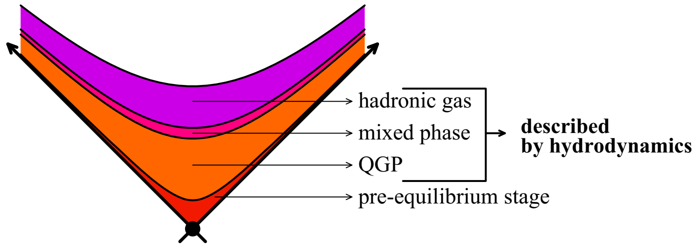
- $c_V > 0$ for AdS BH. (unlike schwarzchild in asymptotically flat space!)
- uniqueness of stationary BH ('no hair') $\leftrightsquigarrow$ few state variables in eq thermo

Sample application: approach to equilibrium

Important question for interpreting RHIC data: how long does it take before hydro sets in?

initially in gold-gold collision: anisotropic momentum-space distribution





[Heller-Janik-Peschanski]

after time τ_{th}: locally thermal distribution and hydrodynamics.

At RHIC: τ_{th} much smaller than perturbation theory answer.
(τ_{th} affects measurement of viscosity:
good elliptic flow requires both low η and early applicability of hydro)

Thermal equilibrium of CFT stuff $\leftrightsquigarrow$ AdS black hole
$T \leftrightsquigarrow$ location of horizon $r = r_H$

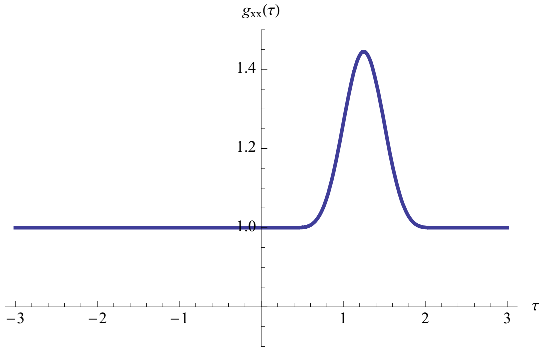
Local thermal equilbrium (hydro) $\leftrightsquigarrow$ slowly-varying deformations of AdS BH: $r = r_H(\vec{x}, t)$. [Janik-Peschanski, Bhattacharyya et al]

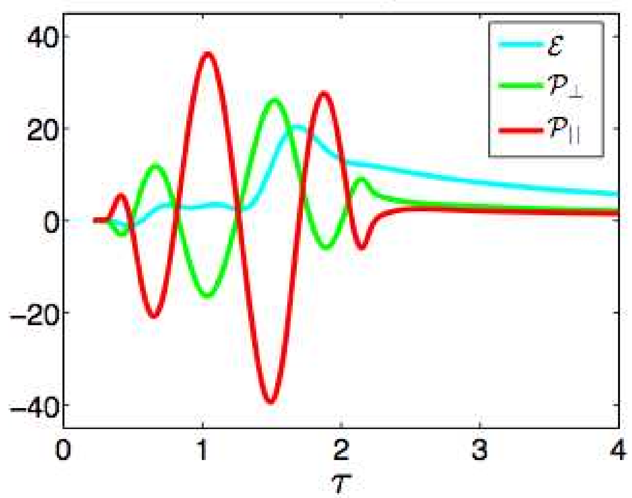
Approach to equilibrium

bulk picture: dynamics of gravitational collapse.

dissipation: energy falls into BH [Horowitz-Hubeny, 99]

- quasinormal modes of a small BH [Freiss et al, 06] $\tau_{th} \sim \frac{1}{8T_{peak}}$.
- far-from equilibrium processes: [Chesler-Yaffe, 08, 09] (PDEs!)

input: 

output: 

black hole forms from vacuum initial conditions.

brutally brief summary: all relaxation timescales $\tau_{th} \sim T^{-1}$.

- Lesson: In these models, breakdown of hydro in this model is not set by higher-derivative terms, but from non-hydrodynamic modes.

far-reaching consequence: gravity as an entropic force. [E. Verlinde, 1001....]

This is the end of part 2.