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# Spring School on Superstring Theory and Related Topics

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## Superstring multiloop amplitudes and non-renormalization theorems
## Lecture II
## - Basics of Pure Spinor Formalism -

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# II. Basics of Pure Spinor formalism

## A. (Linearized) sYM and sugra in $d=10$ superspace

-1: $\boxed{x^m, \theta^\alpha}$ $\quad m=0..9, \ \alpha=1..16$, $\quad \gamma^m_{\alpha\beta}=\gamma^m_{\beta\alpha}$, $\quad \gamma^{(m}_{\alpha\beta}\gamma^{n)\beta\gamma}=2\eta^{mn}\delta^\gamma_\alpha$

$$\eta_{mn}\gamma^m_{(\alpha\beta}\gamma^n_{\gamma)\delta}=0, \quad \gamma^{mnp}_{\alpha\beta}=-\gamma^{mnp}_{\beta\alpha}, \quad \gamma^{mnpqr}_{\alpha\beta}=\gamma^{mnpqr}_{\beta\alpha}$$

$$D_\alpha=\frac{\partial}{\partial\theta^\alpha}+(\gamma^m\theta)_\alpha\frac{\partial}{\partial x^m}, \quad \{D_\alpha, D_\beta\}=2\gamma^m_{\alpha\beta}\frac{\partial}{\partial x^m}$$

In ordinary Maxwell, introduce $A_m(x)$ and define $\nabla_m=\frac{\partial}{\partial x^m}+A_m(x)$

In super-Maxwell, introduce $A_\alpha(x,\theta)$, $A_m(x,\theta)$ and define

$$\nabla_\alpha=D_\alpha+A_\alpha(x,\theta), \quad \nabla_m=\frac{\partial}{\partial x^m}+A_m(x,\theta)$$

Defined up to gauge transf $\boxed{\delta A_\alpha=D_\alpha\Omega, \quad \delta A_m=\partial_m\Omega}$

nposing $\{\nabla_\alpha,\nabla_\beta\}=\gamma^m_{\alpha\beta}\nabla_m \Rightarrow$ eqns. of motion on $(A_\alpha, A_m)$

$$\boxed{\gamma_m^{\alpha\beta}(D_\alpha A_\beta+D_\beta A_\alpha)=A_m, \quad \gamma^{\alpha\beta}_{mnpqr}(D_\alpha A_\beta+D_\beta A_\alpha)=0}$$

auxiliary eqn. for $A_m$ $\qquad$ (Howe)

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$N=2$ : $\boxed{x^m, \theta^\alpha, \hat\theta^{\hat\alpha}}$ $\quad m=0..9, \ \alpha=1..16, \ \hat\alpha=1..16$

$D_{\hat\alpha}=\frac{\partial}{\partial\hat\theta^{\hat\alpha}}+(\gamma^m\hat\theta)_{\hat\alpha}\frac{\partial}{\partial x^m}$ $\qquad$ Same chirality $\Rightarrow$ IIB, Opposite chirality $\Rightarrow$ IIA

Linearized sugra described by $A_{\alpha\hat\alpha}(x,\theta,\hat\theta)$ satisfying eq. of motion $\gamma^{\alpha\beta}_{mnpqr}D_\alpha A_{\beta\hat\alpha}=\gamma^{\hat\alpha\hat\beta}_{mnpqr}D_{\hat\alpha}A_{\beta\hat\beta}=0$

In components, can gauge

$$A_\alpha = \underset{\substack{\uparrow\\ \text{photon}}}{(\gamma^m\theta)_\alpha\, a_m(x)} + (\gamma^m\theta)_\alpha(\gamma_m\theta)^\beta \underset{\substack{\uparrow\\ \text{photino}}}{\psi_\beta(x)} + \underset{\substack{\uparrow\\ \text{derivatives}\\ \text{of } a_m \text{ and } \psi_\beta}}{\ldots}$$

$$A_{\alpha\hat\beta} \simeq A_\alpha * A_{\hat\beta} = \left((\gamma^m\theta)_\alpha a_m(x) + (\gamma^n\theta)_\alpha(\gamma_m\theta)^\beta \psi_\beta(x) + \ldots\right)\left((\gamma^p\hat\theta)_{\hat\alpha}\bar a_p(x) + (\gamma^p\hat\theta)_{\hat\alpha}(\gamma_p\hat\theta)^{\hat\beta}\bar\psi_{\hat\beta} + \ldots\right)$$

$$= (\gamma^m\theta)_\alpha(\gamma^n\hat\theta)_{\hat\alpha}\left(g_{mn} + b_{mn} + \eta_{mn}\varphi\right) + (\gamma^m\theta)_\alpha(\gamma_m\theta)^\beta(\gamma^p\hat\theta)_{\hat\alpha}(\gamma_p\hat\theta)^{\hat\beta} F_{RR}^{\beta\hat\beta}$$

$$+ \ldots$$

B. Worldsheet action

$$S = \int d^2z \left(\tfrac{1}{2}\partial x^m \bar\partial x_m + p_\alpha \bar\partial\theta^\alpha + w_\alpha \bar\partial\lambda^\alpha + \hat p_{\hat\alpha}\partial\hat\theta^{\hat\alpha} + \hat w_{\hat\alpha}\partial\hat\lambda^{\hat\alpha}\right)$$

$(w_\alpha, p_\alpha) = \dim(1,0)$, $\begin{pmatrix}\hat p_{\hat\alpha}\\ \hat w_{\hat\alpha}\end{pmatrix} = \dim(0,1)$ $\qquad T = \frac{1}{2}\partial x^m \partial x_m + p_\alpha \partial\theta^\alpha + w_\alpha\partial\lambda^\alpha$

$\lambda^\alpha$ is a constrained bosonic spinor satisfying $\lambda^\alpha\gamma^m_{\alpha\beta}\lambda^\beta = 0$

"Cartan $d=10$ pure spinor"

$\lambda^\alpha$ has 11 indep. components : $\lambda^\alpha = \begin{cases} \lambda^{+++++} = \lambda^+ & 1\\ (\lambda^{+++--}, \ldots) = \lambda_{ab} & 10\\ (\lambda^{+----}, \ldots) = \lambda^a & 5 \end{cases}$

If $\lambda^+ \neq 0$, $\lambda^a = \frac{1}{8}\epsilon^{abcde}\dfrac{\lambda_{bc}\lambda_{de}}{\lambda^+}$ $\qquad \boxed{a = 1 \text{ to } 5}$

$\lambda_{ab}$ parameterizes $\dfrac{SO(10)}{U(5)}$ "projective pure spinor"

Constraint on $\lambda^\alpha \Rightarrow$ $w_\alpha$ defined up to gauge transf.

$$\delta w_\alpha = \Lambda_m \gamma^m_{\alpha\beta}\lambda^\beta$$

Gauge inv. combinations : $N^{mn} = \frac{1}{2}(\omega \gamma^{mn} \lambda)$, $J = \omega_\alpha \lambda^\alpha$, $T_\lambda = \omega_\alpha \partial \lambda^\alpha$

Similarly, $\hat\lambda \gamma^m \hat\lambda = 0$ and $\hat N^{mn} = \frac{1}{2}(\hat\omega \gamma^{mn} \hat\lambda)$, $\hat J = \hat\omega_{\hat\alpha} \hat\lambda^{\hat\alpha}$, $T_{\hat\lambda} = \hat\omega_{\hat\alpha} \partial \hat\lambda^{\hat\alpha}$

All bosons and fermions have either 0 or +1 conf. wt.

Conf. anomaly : $+10 - 32 + 22 = 0$ (like topological string)

Level of Lorentz generator is $\underset{(\theta,p)}{+4} \; \underset{(\omega,\lambda)}{-3} = \underset{\psi\text{'s in RNS}}{+1}$

OPE's : $p_\alpha(y)\, \theta^\beta(z) \to (y-z)^{-1} \delta_\alpha^\beta$, $N_{mn}(y) N_{pq}(z) \to (y-z)^{-1} \eta_{np} N_{qm} - 3 \frac{\eta_{np}\eta_{qm}}{(y-z)^2}$

$\lambda^\alpha(y) \lambda^\beta(z) \to$ regular , $\lambda^\alpha(y) N_{mn}(z) \to \frac{1}{2}(\gamma_{mn}\lambda)^\alpha (y-z)^{-1}$

## c. BRST operator

Physical states defined by cohomology of BRST operators

$\boxed{Q = \int dz\, \lambda^\alpha d_\alpha}$ and $\boxed{\bar Q = \int d\bar z\, \hat\lambda^{\hat\alpha} \hat d_{\hat\alpha}}$

$d_\alpha = p_\alpha + (\gamma^m \theta)_\alpha \partial x_m + \frac{1}{8}(\gamma^m\theta)_\alpha (\theta \gamma_m \partial\theta)$

satisfies OPE $d_\alpha(y)\, d_\beta(z) \to (y-z)^{-1}\, \Pi_m(z)\, \gamma^m_{\alpha\beta}$

$\Pi_m = \partial x_m + \theta \gamma_m \partial \theta$

$d_\alpha$ and $\Pi_m$ are spacetime supersymmetric (Siegel)

$d_\alpha(y) \Pi_m(z) \to (y-z)^{-1} \gamma_{m\alpha\beta}\, \partial\theta^\beta(z)$

Similarly, $\hat d_{\hat\alpha} = \hat p_{\hat\alpha} + (\gamma^m \hat\theta)_{\hat\alpha} \bar\partial x_m + \frac{1}{8}(\gamma^m \hat\theta)_{\hat\alpha}(\hat\theta \gamma_m \bar\partial \hat\theta)$, $\bar\Pi_m = \bar\partial x_m + \hat\theta \gamma_m \bar\partial \hat\theta$

## D. Vertex operators (massless)

Open string $\Rightarrow$ ghost number $+1$ $\Rightarrow$ $V = \lambda^\alpha A_\alpha(x,\theta)$

$d_\alpha(y) A_\beta(z) \rightarrow \underbrace{D_\alpha A_\beta(z)}_{y-z}$ $\Rightarrow$ $QV = \lambda^\alpha \lambda^\beta D_\alpha A_\beta$

$QV = 0 \Rightarrow (\lambda \gamma_{mnpqr} \lambda)(\gamma^{mnpqr})^{\alpha\beta} D_\alpha A_\beta = 0$ $\Rightarrow$ eqn of motion for $A_\alpha$

Closed string $\Rightarrow$ ghost-number $(1,1)$ $\Rightarrow$ $V = \lambda^\alpha \hat{\lambda}^{\hat{\alpha}} A_{\alpha\hat{\alpha}}(x,\theta,\hat{\theta})$

$QV = \bar{Q}V = 0 \Rightarrow \lambda^\alpha \lambda^\beta \hat{\lambda}^{\hat{\alpha}} D_\alpha A_{\beta\hat{\alpha}} = \lambda^\alpha \hat{\lambda}^{\hat{\alpha}} \hat{\lambda}^{\hat{\beta}} D_{\hat{\alpha}} A_{\alpha\hat{\beta}} = 0$ $\Rightarrow$ eqns of motion for $A_{\alpha\hat{\alpha}}$.

Integrated vertex op's satisfy $QU = \partial V$ for open string

$\Rightarrow U = \partial\theta^\alpha A_\alpha + \Pi^m A_m + d_\alpha W^\alpha + N_{mn} F^{mn}$

where $A_m = \underbrace{\gamma_m^{\alpha\beta} D_\alpha A_\beta}_{= a_m + \ldots}$, $W^\alpha = \gamma_m^{\alpha\beta}(D_\alpha A^m - \partial^m A_\alpha) = \psi^\alpha + \ldots$

$F_{mn} = (\gamma_{mn})_\alpha{}^\beta D_\beta W^\alpha = \partial_{[m} a_{n]} + \ldots$

For closed string, $Q\bar{Q}U = \partial\bar{\partial}(\lambda\hat{\lambda}A_{\alpha\hat{\alpha}})$

$\Rightarrow U \simeq (\partial Y^M A_M + d_\alpha W^\alpha + N_{mn} F^{mn}) \times (\bar{\partial} Y^N \hat{A}_N + \hat{d}^{\hat{\beta}} \hat{W}_{\hat{\beta}} + \hat{N}_{mn} \hat{F}^{mn})$

$= \partial Y^M \bar{\partial} Y^N (G_{MN} + B_{MN}) + d_\alpha \hat{d}_{\hat{\beta}} F^{\alpha\hat{\beta}}_{RR} + \partial Y^M \hat{N}_{mn} \Omega_M{}^{mn}$

$+ N^{mn} \hat{N}^{pq} R_{mnpq} + \ldots$

## E. Tree amplitudes

$$\mathcal{A}_{open} = \left\langle V_1(z_1)\, V_2(z_2)\, V_3(z_3) \int dz_4\, U_4(z_4) \dots \int dz_N\, U_N(z_N) \right\rangle$$

Use OPE's to integrate over non-zero modes of worldsheet variables.

End up with $\mathcal{A} = \langle \lambda^\alpha \lambda^\beta \lambda^\gamma f_{\alpha\beta\gamma}(x,\theta) \rangle$

Naively, need to integrate over 11 $\lambda$'s $\Rightarrow (\infty)^{11}$
~~and integrate over 16 $\theta$'s.~~

As will be shown in next lecture, integration over $\lambda$'s cancels 11 $\theta$'s to give prescription $\boxed{1 = \langle (\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)\rangle}$

$$\mathcal{A} = \int d^{10}x \left(\frac{\partial}{\partial\theta}\right)^{5\,\alpha\beta\gamma} f_{\alpha\beta\gamma} \quad \text{where}$$

$$\left(\frac{\partial}{\partial\theta}\right)^{5\,\alpha\beta\gamma} = \left(\gamma^m \frac{\partial}{\partial\theta}\right)^\alpha \left(\gamma^n \frac{\partial}{\partial\theta}\right)^\beta \left(\gamma^p \frac{\partial}{\partial\theta}\right)^\gamma \left(\frac{\partial}{\partial\theta}\gamma_{mnp}\frac{\partial}{\partial\theta}\right)$$

For 3-point amp, $\mathcal{A} = \int d^{10}x \left(\frac{\partial}{\partial\theta}\right)^{5\,\alpha\beta\gamma} \left( A^{(1)}_\alpha A^{(2)}_\beta A^{(3)}_\gamma \right)$

$$= \int d^{10}x \left( \psi^{\alpha(1)} a^{(2)}_m \psi^{\beta(3)} \gamma^m_{\alpha\beta} + \partial_m a^{(1)}_n a^{m(2)} a^{n(3)} + \text{permutations}\right)$$

= cubic sYM vertex.

For closed string tree amplitude,

$$\mathcal{A} = \int d^{10}x \left(\frac{\partial}{\partial\theta}\right)^{5\,\alpha\beta\gamma} \left(\frac{\partial}{\partial\hat\theta}\right)^{5\,\hat\alpha\hat\beta\hat\gamma} f_{\alpha\beta\gamma\hat\alpha\hat\beta\hat\gamma}$$