The Abdus Salam
International Centre for Theoretical Physics

United Nations
Educational, Scientific and
Cultural Organization

IAEA
International Atomic Energy Agency

2134-19

Spring School on Superstring Theory and Related Topics

22 - 30 March 2010

Superstring multiloop amplitudes and non-renormalization theorems
Lecture III
- Multiloop Amplitude Prescription -

N.J. Berkovits
UESP (Universidade Estadual Paulista)
Sao Paulo
Brazil

III Multiloop Amplitude Prescription

A. Non-minimal variables

To obtain measure factor $\langle(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)\rangle$

~~from functional integration~~, introduce "non-minimal variables"

Left-moving: $(\bar\lambda_\alpha, \bar w^\alpha)$ $\quad (r_\alpha, s^\alpha)$, $\quad T_{norm} = \bar w^\alpha \partial\bar\lambda_\alpha + s^\alpha \partial r_\alpha$

bosons $\quad$ fermions $\quad$ (Nekrasov)

satisfying constraint $\bar\lambda\gamma^m\bar\lambda = 0$ and $\bar\lambda\gamma^m r = 0$

$\Rightarrow$ 11 indep $\bar\lambda$'s and 11 indep. r's.

Gauge inv $\Rightarrow$ 11 gauge inv. $\bar w$'s and s's.

Modify BRST operator to $Q = \int dz(\lambda^\alpha d_\alpha + \bar w^\alpha r_\alpha)$

$\Rightarrow$ non-min variables do not affect cohomology.

Vertex op's can be chosen indep. of non-min. variables

Tree amp: $\mathcal{A} = \langle V_1 V_2 V_3 \int dz\, U_4 \ldots \int dz\, U_N\rangle = \langle \lambda^\alpha\lambda^\beta\lambda^\gamma f_{\alpha\beta\gamma}(x,\theta)\rangle$

To regularize λ integration, insert $\mathcal{N} = \exp(\rho\{Q,\chi\})$ $\quad \rho$ constant

where $\chi = -\bar\lambda_\alpha\theta^\alpha \Rightarrow \mathcal{N} = \exp(-\rho(\bar\lambda_\alpha\lambda^\alpha + r_\alpha\theta^\alpha))$

Since $\mathcal{N} = 1 + \{Q,\Omega\}$, amplitude is independent of ρ

$\mathcal{A} = \langle \mathcal{N}\, V_1 V_2 V_3 \int dz\, U \ldots\rangle = \langle e^{-\rho(\bar\lambda\lambda + r\theta)}\lambda^\alpha\lambda^\beta\lambda^\gamma f_{\alpha\beta\gamma}(x,\theta)\rangle$

$\int d^{10}x \int d^{16}\theta \int d^{11}\lambda \int d^{11}\bar\lambda \int d^{11}r\; e^{-\rho(\bar\lambda\lambda + r\theta)}\lambda^\alpha\lambda^\beta\lambda^\gamma f_{\alpha\beta\gamma}(x,\theta)$

$$= \int d^{10}x \int d^{16}\theta \int \cancel{(\ldots)} \frac{d^{11}\lambda\, d^{11}\bar\lambda}{} (\rho^{11}\theta^{11})^{\rho\sigma\tau} \bar\lambda_\rho \bar\lambda_\sigma \bar\lambda_\tau\, \lambda^\alpha\lambda^\beta\lambda^\gamma f_{\alpha\beta\gamma}(x,\theta)\, e^{-\rho\lambda\bar\lambda}$$

$$= \int d^{10}x \int d^{16}\theta\, (\theta^{11})^{\rho\sigma\tau} f_{\rho\sigma\tau}(x,\theta) = \int d^{10}x \left(\frac{\partial}{\partial\theta}\right)^{5\,\rho\sigma\tau} f_{\rho\sigma\tau}(x,\theta)$$

B. b ghost

To compute loop amplitudes, need to include $(3g-3)$ b ghosts where b satisfies $\{Q, b\} = T$.

Using non-minimal variables, can construct

$$b = s^\alpha \partial\bar\lambda_\alpha + \frac{\bar\lambda_\alpha}{(\lambda\bar\lambda)} G^\alpha + \frac{\bar\lambda_\alpha r_\beta}{(\lambda\bar\lambda)^2} H^{\alpha\beta} + \frac{\bar\lambda_\alpha r_\beta r_\gamma}{(\lambda\bar\lambda)^3} K^{\alpha\beta\gamma} + \frac{\bar\lambda_\alpha r_\beta r_\gamma r_\delta}{(\lambda\bar\lambda)^4} L^{\alpha\beta\gamma\delta}$$

where $G^\alpha = \gamma_m^{\alpha\beta}\Pi^m d_\beta + N_{mn}(\gamma^{mn}\partial\theta)^\alpha$

$$H^{\alpha\beta} = \gamma^{\alpha\beta}_{mnp}\left(d\gamma^{mnp}d + N^{mn}\Pi^p\right)$$

$K^{\alpha\beta\gamma} = \gamma^{(\alpha\beta}_{mnp}(\gamma^m d)^{\gamma)} N^{np}$, $L^{\alpha\beta\gamma\delta} = (N^{mn}N^{qr})\, \gamma^{(\alpha\beta}_{mnp}\gamma^{p\gamma\delta)}_{qr}$

Can verify $\{Q, G^\alpha\} = T_{min}\lambda^\alpha$, $[Q, H^{\alpha\beta}] = \lambda^{[\alpha}G^{\beta]}$, $\{Q, K^{\alpha\beta\gamma}\} = \lambda^{(\alpha}H^{\beta\gamma)}$

$[Q, L^{\alpha\beta\gamma\delta}] = \lambda^{(\alpha}K^{\beta\gamma\delta)}$, $\lambda^{(\alpha}L^{\beta\gamma\delta\rho)} = 0$

C. Multiloop prescription

$$\mathcal{A}_g = \int d^{3g-3}\tau \left\langle \int dz_1 U_1 \ldots \int dz_N U_N \left(\prod_{I=1}^{3g-3}\int \mu_I b(y_I)\right) \mathcal{N}_g \right\rangle$$

Functional integration over worldsheet non-zero modes is trivial (OPE's, cancellation of partition fncn's)

Need to regularize integration over zero modes

$$\int d^{10}x \int d^{16}\theta \int d^{11}\lambda \prod_{I=1}^{8} \int d^{11} w_I \int d^{16} p_I \int d^{11}\bar{\lambda} \int d^{11} r \prod_{I=1}^{8} \int d^{11}\bar{w}_I \int d^{11} s_I \quad \mathcal{N}_0 \, \mathcal{N}_R$$

$$\mathcal{N}_0 = \exp\left(-\rho\left(\bar{\lambda}_\alpha \lambda^\alpha + r_\alpha \theta^\alpha + \sum_{I=1}^{8}\left(w_{I\alpha}\bar{w}_I^\alpha + d_{I\alpha} s_I^\alpha\right)\right)\right)$$

$$= \exp\left(-\rho \left\{Q, \bar{\lambda}_\alpha \theta^\alpha + \sum_{I=1}^{8} w_{I\alpha} s_I^\alpha \right\}\right)$$

$\mathcal{N}_0$ regularizes divergence when $\lambda\bar{\lambda} \to \infty$ or $w\bar{w} \to \infty$.

Because of poles in b ghost when $\lambda\bar{\lambda} \to 0$,

need to regularize divergence if b ghosts contribute $(\lambda\bar{\lambda})^{-11}$.

May occur at 3-loops from 6 b ghosts

(each b ghost contributes $\sum_{n=0}^{3} \left(\frac{r}{\lambda\bar{\lambda}}\right)^n b_n$).

Regularization is unnecessary if at least one r zero mode comes from $\mathcal{N}_0$ ($\Rightarrow$ ~~at least one~~ less than 16 θ^α's ~~comes~~ come from external vertex operators

$\Rightarrow$ Amplitude is superspace F-term)

$$\mathcal{N}_R = \exp\left(-\sum\left(w_\alpha(y_R)\bar{w}^\alpha(y_R) + d_\alpha(y_R) s^\alpha(y_R)\right)\right) = \exp\left(-\sum_{R=1}^{N}\{Q, w_\alpha s^\alpha(y_R)\}\right)$$

$\prod_{R=1}^{N} \mathcal{N}_R$ regularizes divergence if b ghosts contribute $(\lambda\bar{\lambda})^{-\ell}$ where $\ell < 11(N+1)$

$$\mathcal{N}_R = \prod_{k=1}^{N} \int d^N f_R \, d^N \bar{f}_R \; \exp\left(-\sum_{k=1}^{N} f_R^\alpha \bar{f}_{R\alpha}\right) \exp\left(-\sum_{k=1}^{N} f_R^\alpha w_\alpha(y_R)\right)$$

$$\exp\left(-\sum_{k=1}^{N} \bar{f}_{R\alpha} \bar{w}^\alpha(y_R)\right) \exp\left(-\sum^{N} d_\alpha s^\alpha(y_R)\right)$$

$\exp\left(-f_R^\alpha w_\alpha(y_R)\right)$ shifts $\lambda^\alpha(z) \to \lambda^\alpha(z) - f_R^\alpha \frac{1}{z-y_R}$

$\exp\left(-f_{R\alpha} \bar{w}^\alpha(y_R)\right)$ shifts $\bar{\lambda}_\alpha(z) \to \bar{\lambda}_\alpha(z) - \bar{f}_{R\alpha} \frac{1}{z-y_R}$

$$b(z) \to b(z) = \sum_{n=0}^{3} \frac{1}{\sqrt{\left(\lambda(z) - \frac{f_R^\alpha}{z-y_R}\right)\left(\bar{\lambda}(z) - \frac{\bar{f}_R}{z-y_R}\right)}} \, b_n(z)$$

So $b(z)$ has a pole at $\left|\lambda(z) - \frac{f_R^\alpha}{z-y_R}\right| = 0$.

$\Rightarrow$ Never have divergences since b's ~~have poles~~ do not ~~at different locations~~ have coincident poles which diverge faster than $|\lambda - c|^{-N}$.

However, it is difficult to do explicit computations when $\mathcal{N}_R$ is necessary.

(NB + N. Nekrasov)
(NB + Y. Aisaka)