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Workshop on Localization Phenomena in Novel Phases of Condensed Matter

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Experimental Observation of the Critical Regime of the Anderson Transaction with Cold Atoms

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Experimental observation of the critical regime of the Anderson transition with cold atoms

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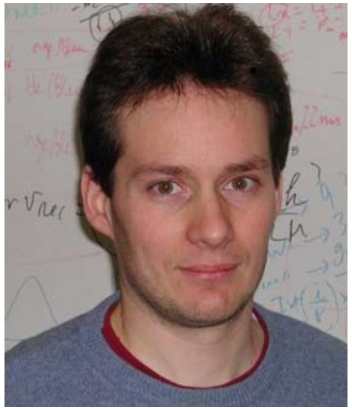

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Outline

- Anderson localization with cold atoms
- The periodically kicked rotor: dynamical localization and Anderson localization
- Experimental realization of the periodically kicked rotor
- Anderson localization and Anderson transition for the “three-color” kicked rotor
- Finite-time scaling
- Critical regime

Anderson localization with cold atoms

Essential features of Anderson localization:

- Inhibition of transport
  - direct measurement of the atomic wavefunction
- Due to quantum interference
  - preserve phase coherence of the atomic wavefunction
- Zero-temperature effect
  - cold atomic gas
- One-body physics
  - dilute gas (no Bose-Einstein condensate)
- Driven by the amount of disorder
  - use chaotic temporal dynamics instead of static disorder
- Depends on the dimension
  - effective dimension can be easily adjusted by varying the time sequence

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The (periodically) kicked rotor

Hamiltonian $$H = \frac{p^2}{2} + k\cos\theta \sum_{n=-\infty}^{+\infty} \delta(t - nT)$$

- Map from kick *n* to kick *n+1*:

$$\begin{cases} Tp_{n+1} = Tp_n + kT\sin\theta_n \\ \theta_{n+1} = \theta_n + Tp_{n+1} \end{cases}$$

Standard map

stochasticity parameter

$$K = kT$$

* The map is 2π periodic in θ and $I=Tp$,

* but the dynamics itself is not periodic in momentum space.

* The kicked rotor is periodic in θ, the spatially unfolded version is not => requires the inclusion of a (constant) quasi-momentum in the quantum treatment (Bloch theorem).

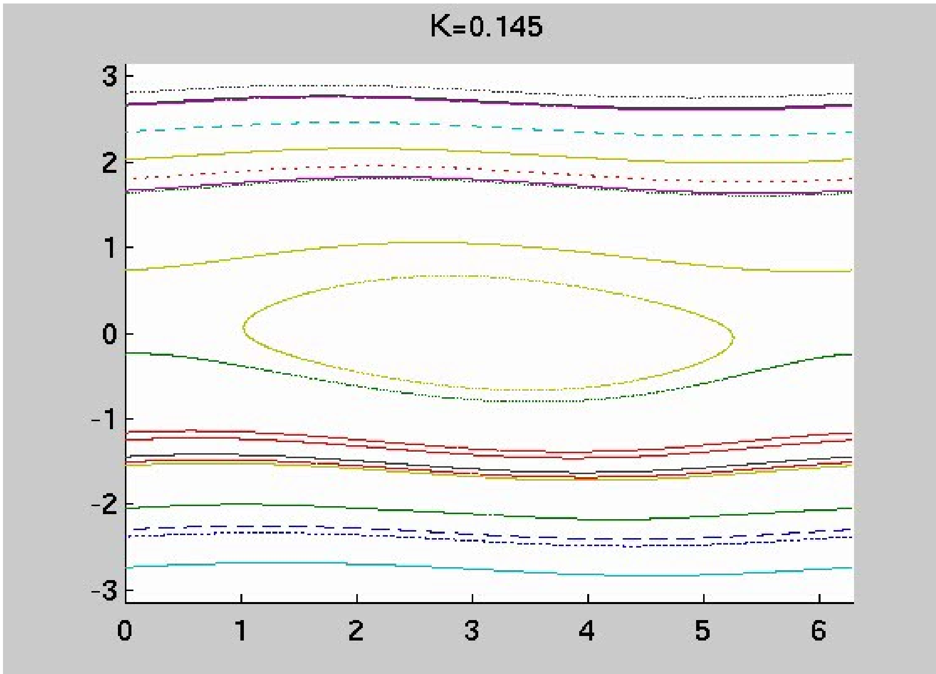
kicked rotor   spatially unfolded version

Classical dynamics of the kicked rotor

- Depends on a single parameter $K=kT$.
- Mainly regular for small K => momentum growth is bounded.

$$\begin{cases} I_{n+1} = I_n + K\sin\theta_n \\ \theta_{n+1} = \theta_n + I_{n+1} \end{cases}$$

$I=Tp$

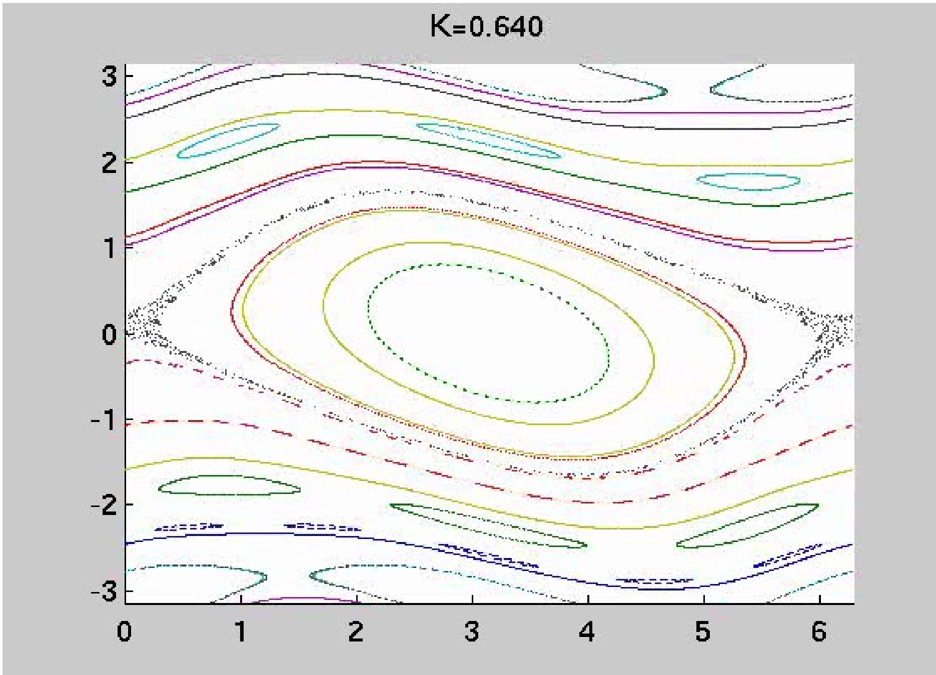


θ

Classical dynamics of the kicked rotor

- Depends on a single parameter $K=kT$.
- Mainly regular for small K => momentum growth is bounded.
- Chaos appears around K=0.5, then progressively invades the whole phase space.
- Unbounded momentum growth above K=1.

$I=Tp$



θ

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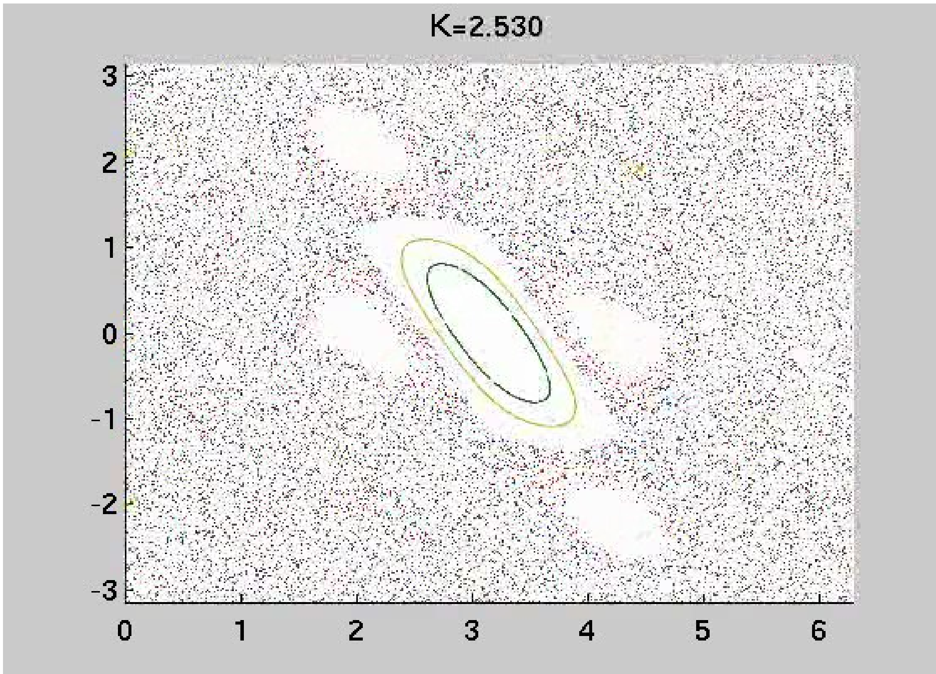
- Almost globally chaotic above K=4.

$$I=Tp$$



$$\theta$$

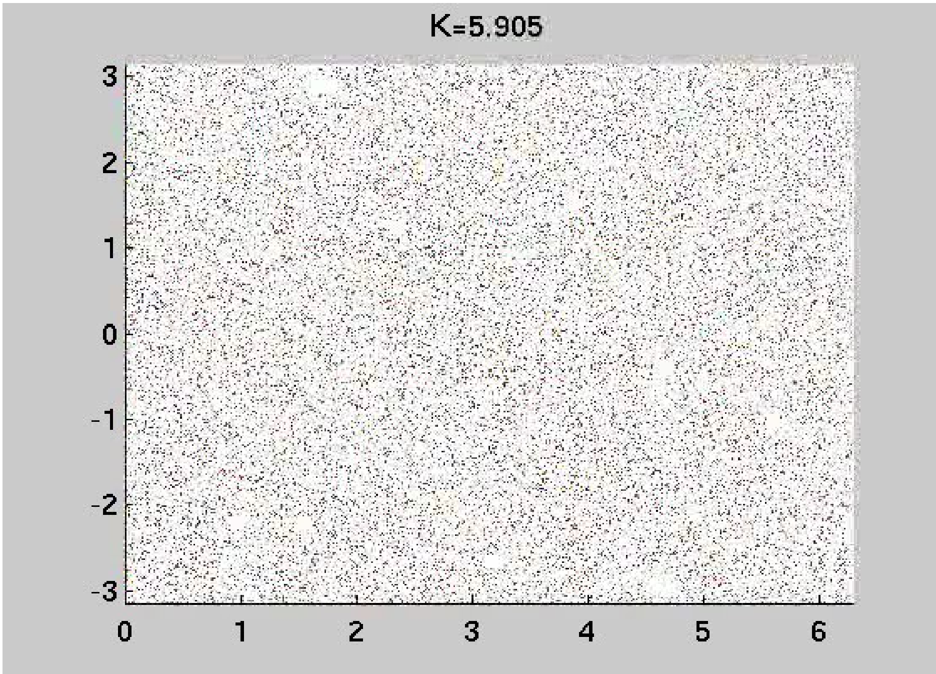
Classical dynamics of the kicked rotor

- Depends on a single parameter $K=kT$.
- Mainly regular for small K => momentum growth is bounded.
- Chaos appears around K=0.5, then progressively invades the whole phase space.
- Unbounded momentum growth above K=1.
- Almost globally chaotic above K=4.

$$p_{n+1} = p_n + k \sin \theta_n$$

- At large K, random walk in momentum space => chaotic diffusion
- Diffusion constant:

$$D = \frac{k^2}{2T}$$



θ

The quantum kicked rotor

- Hamiltonian $H = \frac{p^2}{2} + k \cos\theta \sum_{n=-\infty}^{+\infty} \delta(t - nT)$

- Evolution operator over one period:

$$U = \exp\left(\frac{-ip^2T}{2\hbar}\right) \times \exp\left(\frac{-ik\cos\theta}{\hbar}\right)$$

free evolution over one period

kick

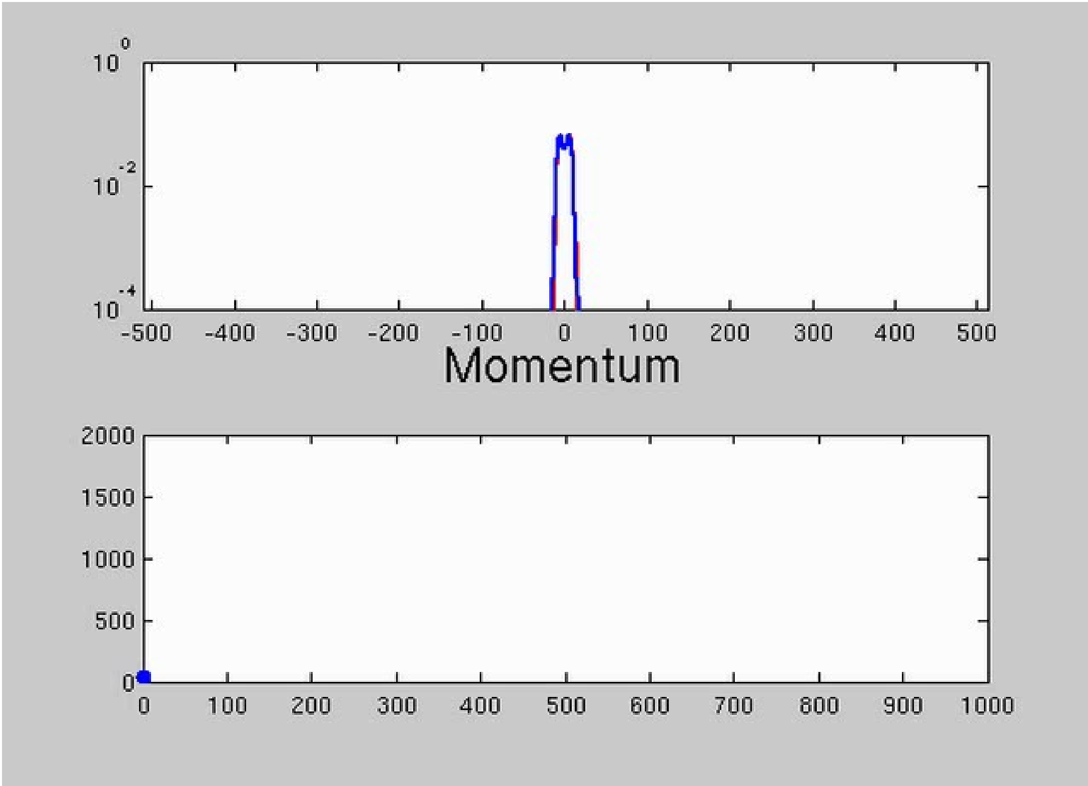
- Very easy to numerically generate the quantum evolution.

Quantum dynamics of the kicked rotor

- Numerical experiment: compare classical and quantum dynamics (averaged over an ensemble of initial conditions).
- Start from a state well localized in momentum space.

$|\psi(p)|^2$
(log scale)

$\langle p^2(t) \rangle$



Time t (number of kicks)

Quantum dynamics of the kicked rotor

- Numerical experiment: compare classical and quantum dynamics (averaged over an ensemble of initial conditions).
- Start from a state well localized in momentum space.
- At short time, the quantum and classical dynamics are equivalent.

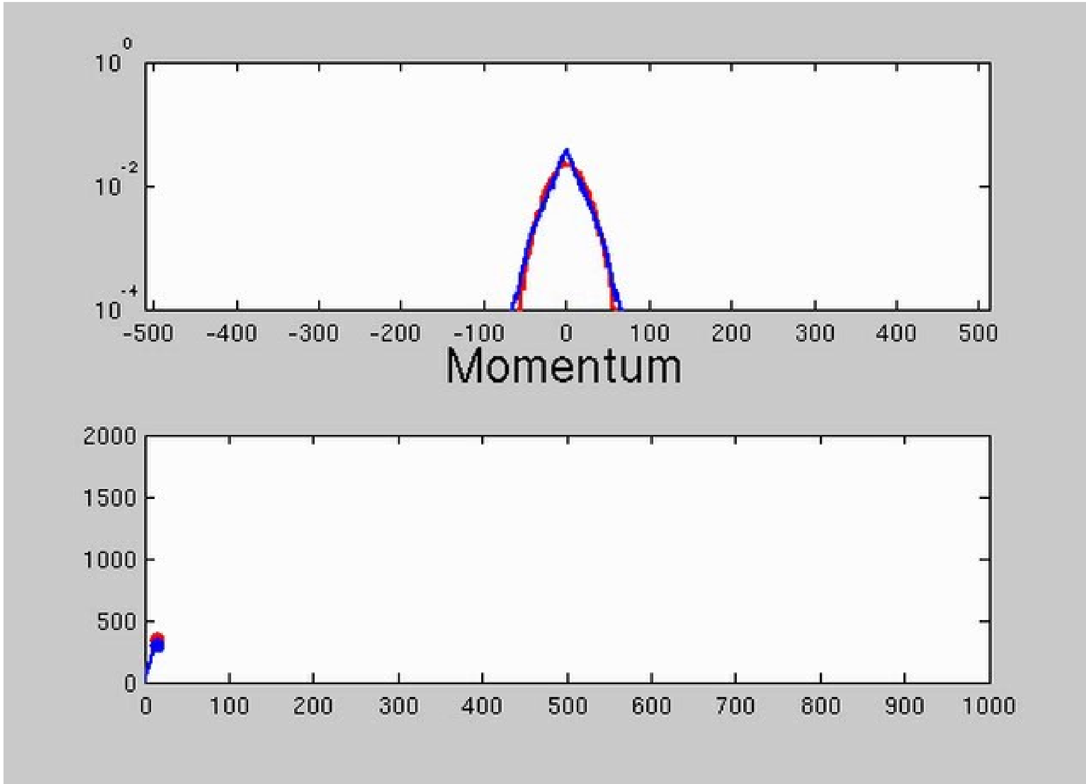
$$|\psi(p)|^2$$

(log scale)

Classical dynamics

Quantum dynamics

$$\langle p^2(t) \rangle$$



Time t (number of kicks)

Quantum dynamics of the kicked rotor

- Numerical experiment: compare classical and quantum dynamics (averaged over an ensemble of initial conditions).
- Start from a state well localized in momentum space.
- At long time, the quantum dynamics freeze.
- **Equivalent to Anderson localization in momentum space (Fishman et al, 1982)**

$$|\psi(p)|^2$$

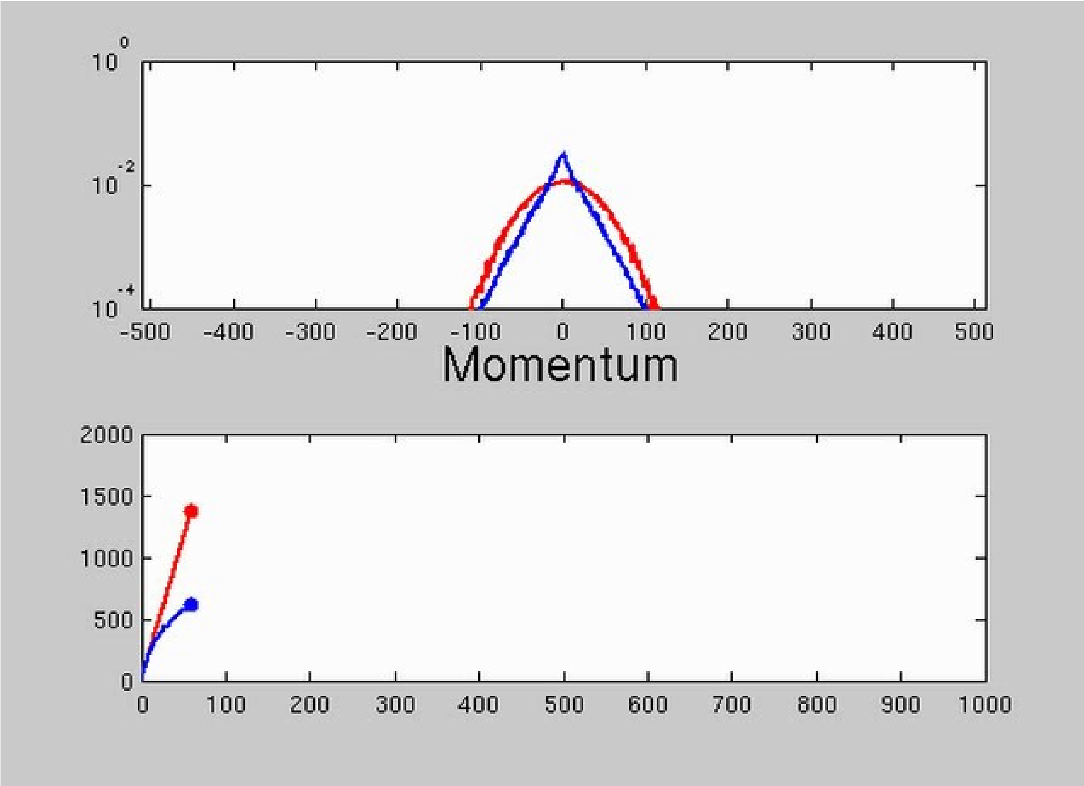
(log scale)

Quantum dynamics: dynamical localization

Casati et al. (1979)

$$\langle p^2(t)\rangle$$

Classical dynamics: chaotic diffusion



Time t (number of kicks)

Dynamical vs. Anderson localization

- The Hamiltonian is time-periodic => use Floquet theorem.
- Floquet states: eigenstates of the one-period evolution operator:

$$U|\phi_i\rangle = \exp\left(-\frac{iE_iT}{\hbar}\right)\ |\phi_i\rangle \qquad 0 \leq E_i < \frac{2\pi\hbar}{T}$$

- Expand the (unitary) kick operator:

$$\exp\left(-\frac{i}{\hbar}\ k\cos\theta\right) = \frac{1+iW(\theta)}{1-iW(\theta)}$$

with $W(\theta) = \sum_{r=-\infty}^{\infty} W_r\ \exp\left(ir\theta\right)$

Fishman, Grempel, Prange, PRL (1982)

- Similarly for the kinetic part:

$$\exp\left[-\frac{i}{\hbar}\left(\frac{p^2}{2}-E\right)\ T\right] = \frac{1+iV}{1-iV}$$

V is diagonal in the eigenbasis $|m\rangle$ of the momentum p.

- Expansion in the momentum eigenbasis:

$$\frac{1}{1-iW(\theta)}|\phi\rangle = \sum_m \chi_m\ |m\rangle$$ with $|\phi\rangle$ a Floquet state

Dynamical vs. Anderson localization (II)

- Then, the eigen-equation for the Floquet state can be rewritten:

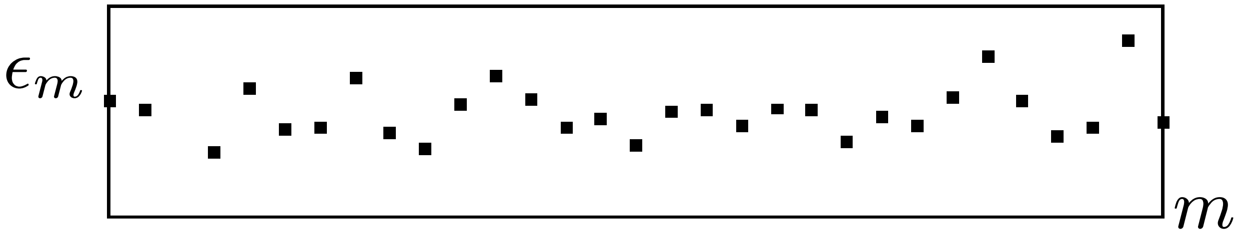
$$\epsilon_m \chi_m + \sum_{r \neq 0} W_r \chi_{m-r} = -W_0 \chi_m$$

on-site “random” energy hopping eigenenergy

with

$$\epsilon_m = \tan\left[\frac{\left(E - \frac{m^2\hbar^2}{2}\right)T}{2\hbar}\right]$$

- The ϵ_m are pseudo-random variables => similar to Anderson or Loyd model



- Various Floquet states correspond to various realizations of the disorder, but have the same localization length.
- The hopping integrals W_r increase with the kick strength K => K plays the role of a control parameter of the Anderson model.

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Quantum dynamics of the external motion of cold atoms

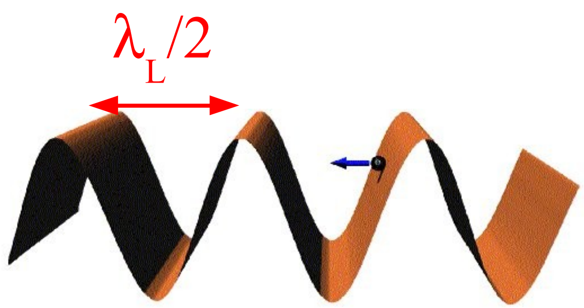
- Control of the dynamics with laser fields, magnetic fields, gravitational field.
- Orders of magnitude:
  - Velocity: cm/s
  - De Broglie wavelength: μm
  - Time: μs-ms

Very favorable!

- One-body (if sufficiently dilute, avoid BEC) zero-temperature quantum dynamics with small decoherence.
- The light-shift due to a quasi-resonant laser at frequency $\omega_{\mathrm{L}} = \omega_{\mathrm{at}} - \delta$ is proportional to I/δ, (with I the space-dependent laser intensity) and acts as an effective potential.
- With a standing wave, spatially periodic one-dimensional effective Hamiltonian (Ω:Rabi frequency):

$$H = \frac{p^2}{2M} + \frac{\hbar\Omega^2}{8\delta}\cos(2k_L x)f(t)$$

kicked rotor

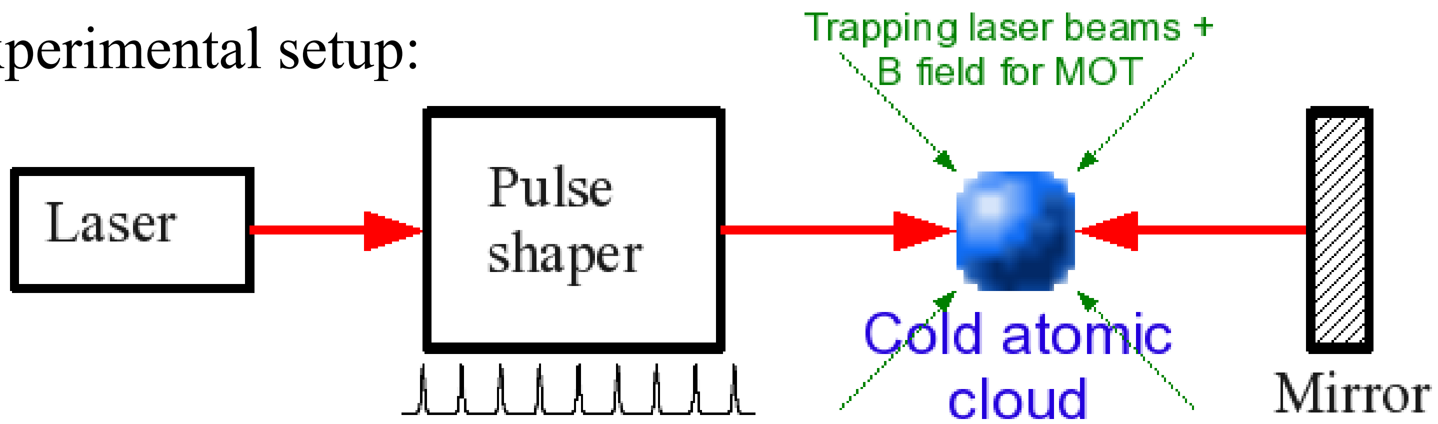


The kicked rotor using cold atoms

- Temporal modulation of the laser intensity (with δ-kicks) => kicked rotor with adjustable effective Plank constant (ω_r: atomic recoil frequency, T: kicking period).

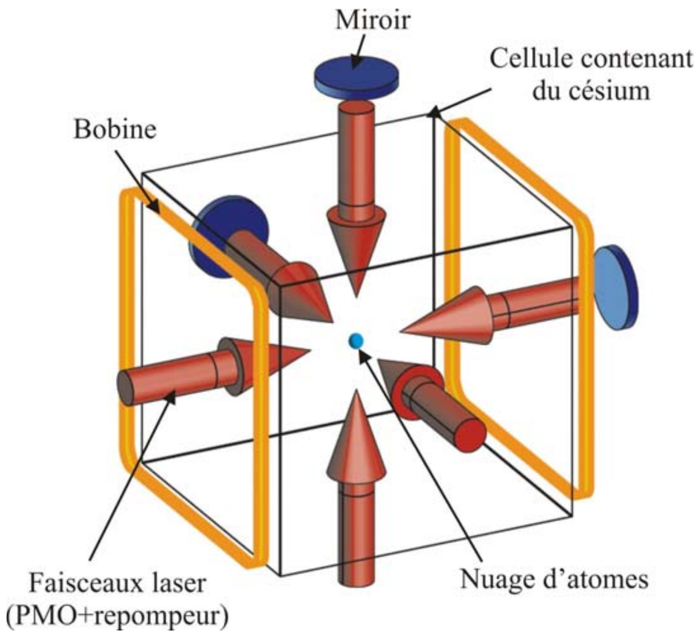
$$\hbar_{\text{eff}} = 8\omega_r T$$

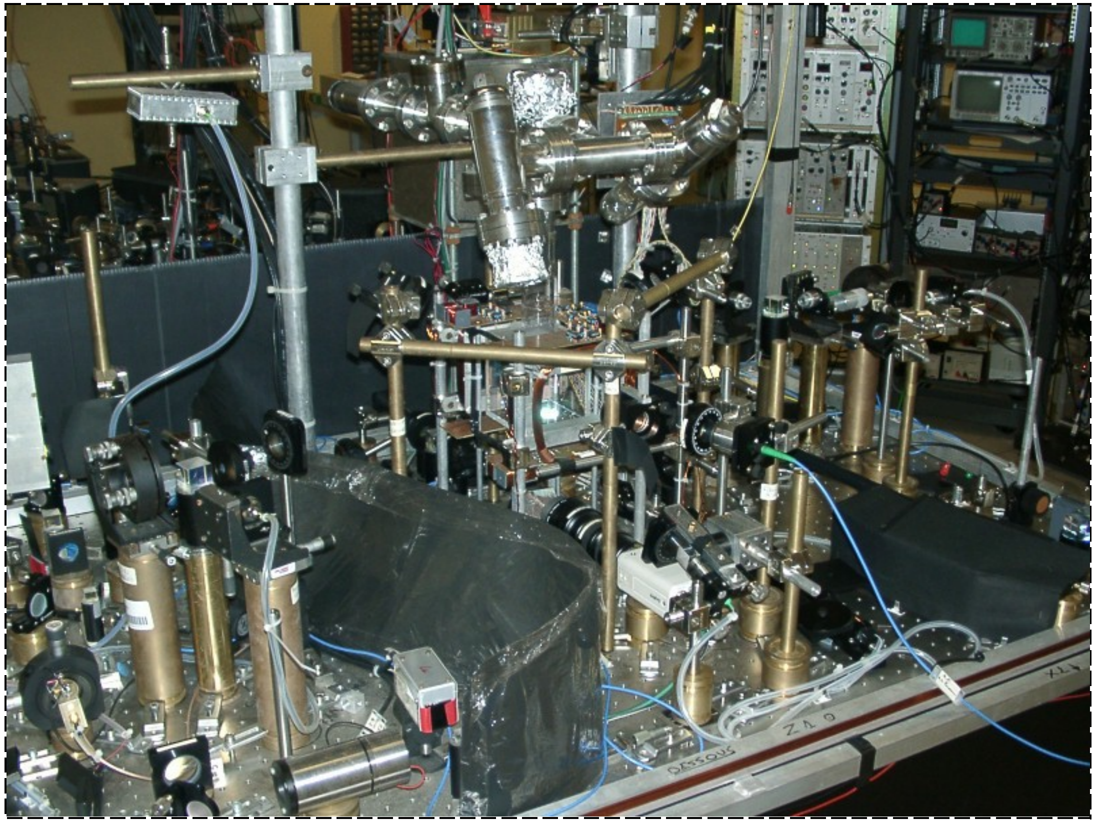
- Experimental setup:

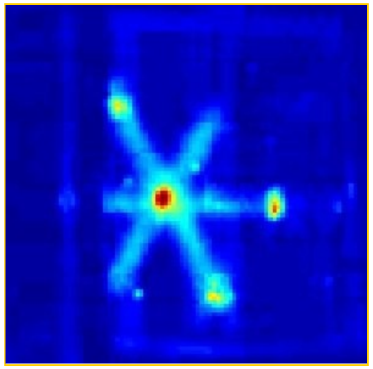


  - 1. Prepare a cold Cs cloud in a Magneto-Optical Trap;
  - 2. Switch off MOT and magnetic field;
  - 3. Apply the sequence of kicks (10-200 kicks);
  - 4. Analyze momentum distribution along the laser axis (using velocity selective Raman transitions);
- Atoms fall down because of gravitational field => 200 kicks maximum.
- Sources of decoherence: spontaneous emission, residual gravitational field along the laser axis are small enough over 200 kicks.

Experimental setup

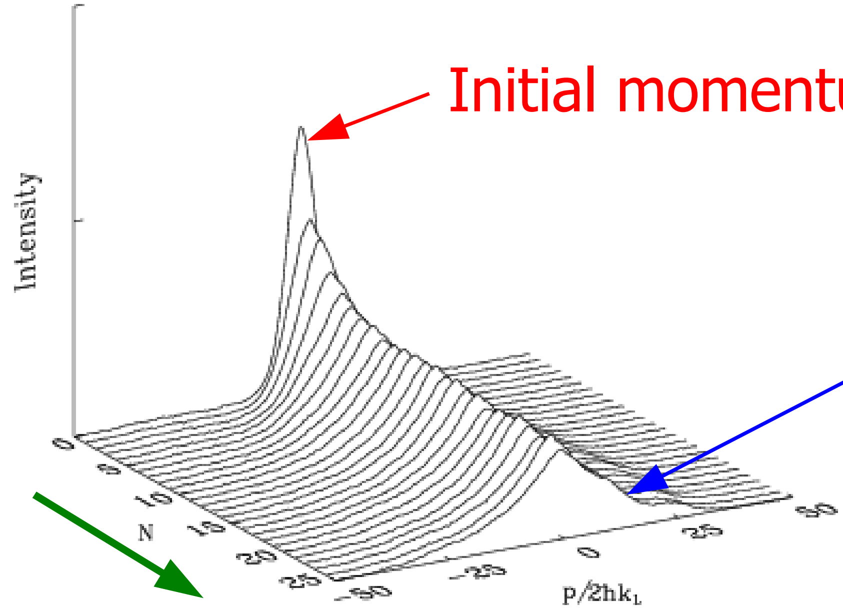






Standard Magneto-Optical Trap

Experimental observation of dynamical localization with cold atoms



Initial momentum distribution (Gaussian)

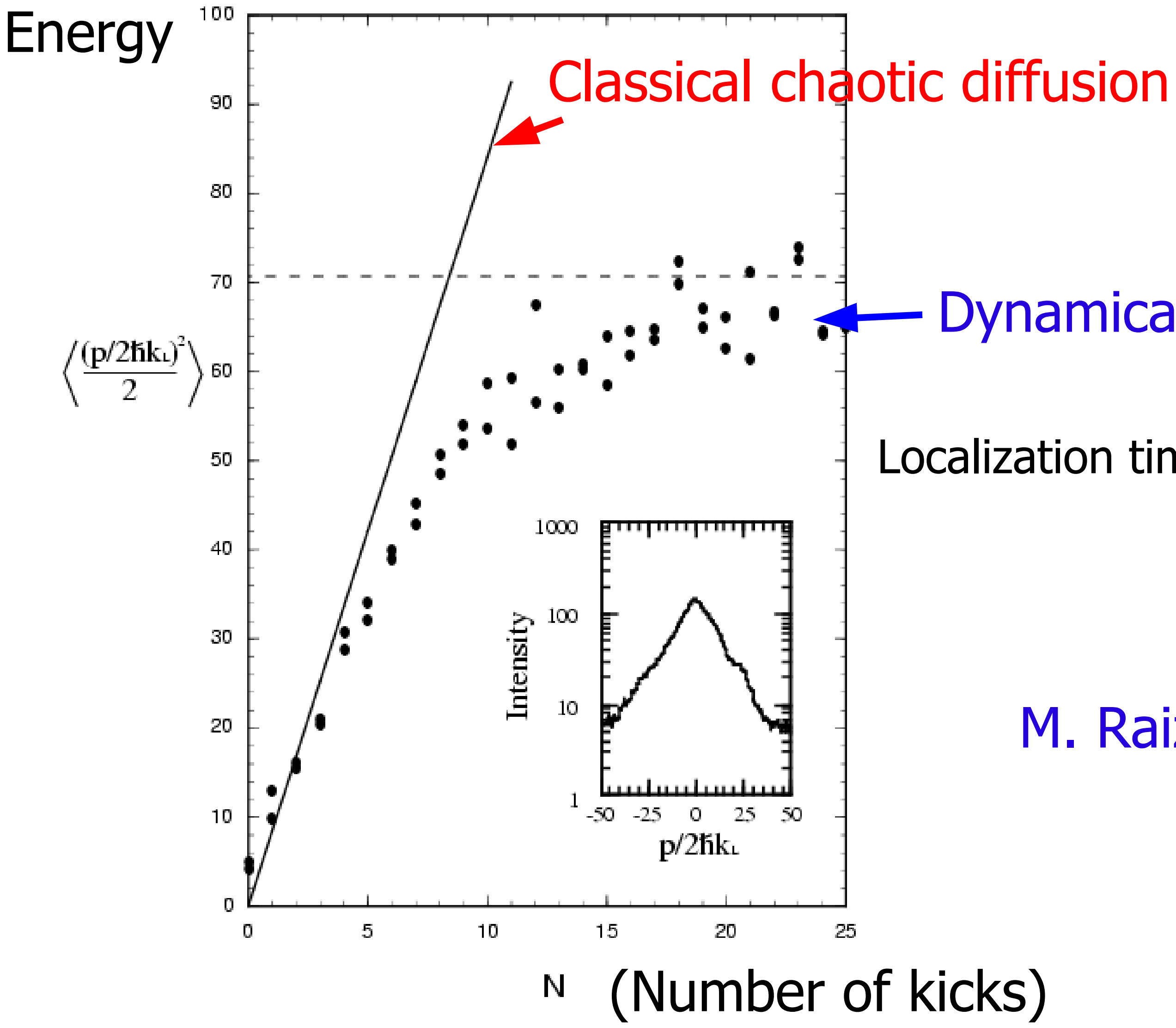
Final momentum distribution
(exponentially localized)

Time
(number of kicks)

Momentum (in units of
2 recoil momenta)

M. Raizen et al (1995)

Experimental observation of dynamical localization with cold atoms



Localization time of the order of 10 kicks

M. Raizen et al (1995)

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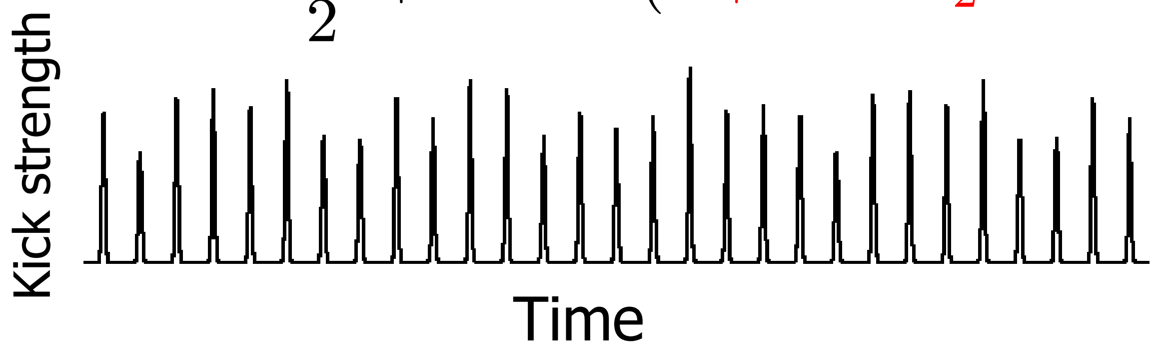
How to observe the Anderson transition in 3D?

- Dynamical vs. Anderson localization
  - Anderson localization: 1d disordered static x-space
  - Dynamical localization: 1d chaotic time-periodic p-space
- Simple idea: keep the spatial dynamics one-dimensional, but introduce one or several additional temporal dimensions.

$$H = \frac{p^2}{2} + k\cos\theta\ (1 + \epsilon\cos\omega_2 t\ \ \cos\omega_3 t) \sum_n \delta(t - nT)$$



- Substantially equivalent with the Anderson model in higher dimension (Casati, Guarneri and Shepelyansky, PRL, 1989).
- Prediction of a localized-delocalized Anderson transition when K=kT is increased in a 3-color system.

Mapping of the quasiperiodic kicked rotor on the 3D Anderson model

- The temporal evolution of the state $\psi(x)$ of the 1D quasiperiodic kicked rotor with Hamiltonian:

$$H = \frac{p^2}{2} + K\cos\theta\ (1 + \epsilon\cos\omega_2 t\ \cos\omega_3 t)\ \sum_n \delta(t-n)$$

is identical to the one of the state:

$$\Psi(x_1, x_2, x_3) = \psi(x_1)\delta(x_2)\delta(x_3)$$

of the 3D periodic kicked "rotor":

$$\mathcal{H} = \frac{p_1^2}{2} + \omega_2 p_2 + \omega_3 p_3 + K\cos x_1\,[1 + \varepsilon\cos x_2\cos x_3]\sum_n \delta(t-n)$$

Kick

Pseudo 3D "rotor"

- The initial state is completely delocalized in the transverse p_2 and p_3 directions.
- The classical dynamics of the 3D periodic kicked "rotor" is an anistropic chaotic diffusion **in momentum space**..

Mapping of the quasiperiodic kicked rotor on the 3D Anderson model

- Similarly to the 1D periodic kicked rotor, the Schroedinger equation for the Floquet states of the 3D periodic kicked "rotor" can be written as:

$$\epsilon_{\mathrm{m}}\,\Phi_{\mathrm{m}} + \sum_{\mathrm{r}\neq 0} W_{\mathrm{r}}\,\Phi_{\mathrm{m}-\mathrm{r}} = -W_0\,\Phi_{\mathrm{m}}$$

with the diagonal "disorder":

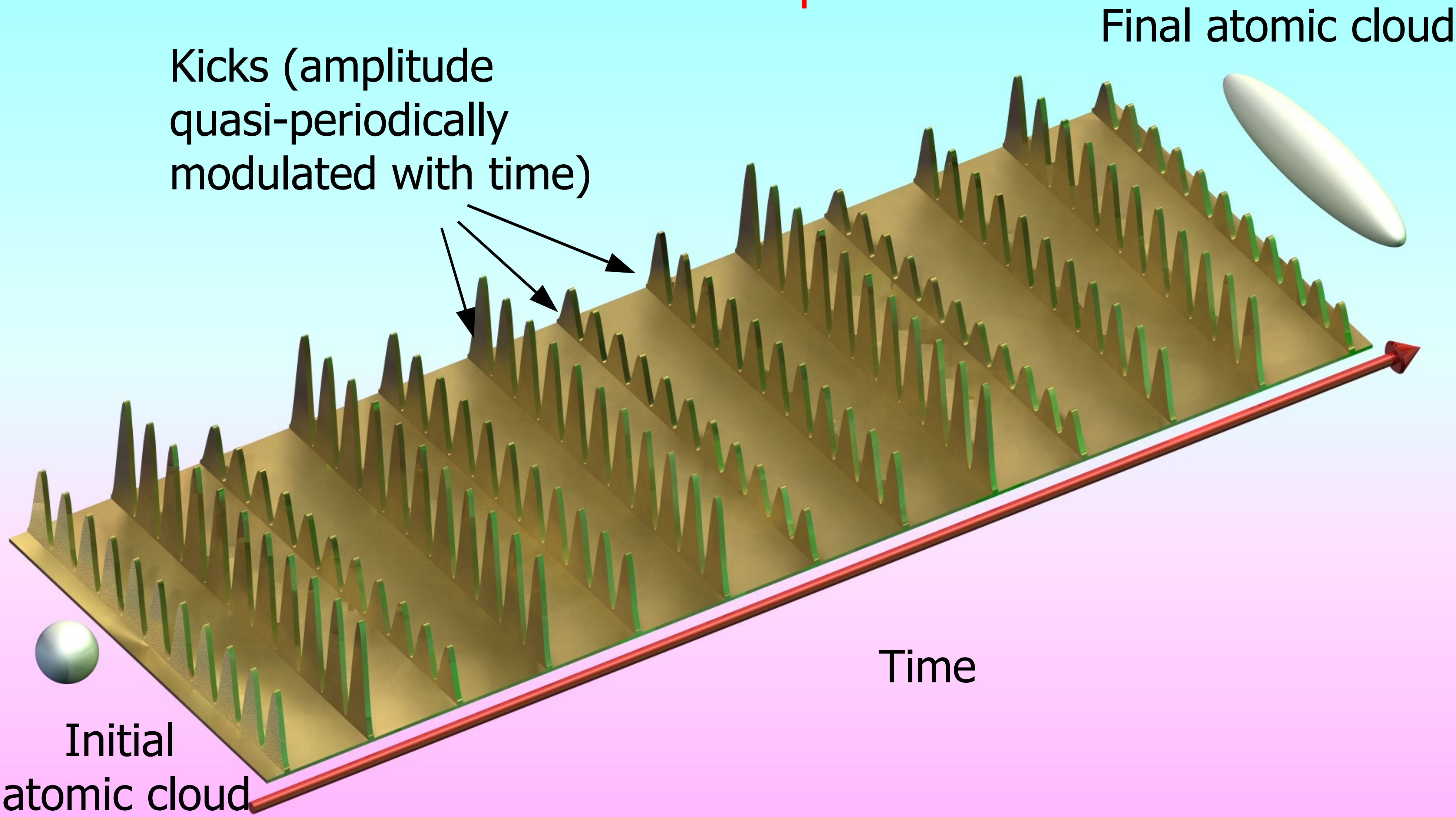
$$\epsilon_{\mathrm{m}} = \tan\left\{\frac{1}{2}\left[\omega - \left(\hbar\frac{{m_1}^2}{2} + \omega_2 m_2 + \omega_3 m_3\right)\right]\right\}$$

and the hopping coefficients W_m are the Fourier components of:

$$W(x_1, x_2, x_3) = \tan\left[K\cos x_1 (1+\varepsilon\cos x_2 \cos x_3)/\,2\hbar\right]$$

- Effective 3D Anderson model, with a metal-insulator transition. For details, see Lemarié et al, PRA, 80, 043626 (2009), arXiv:0907.3411.

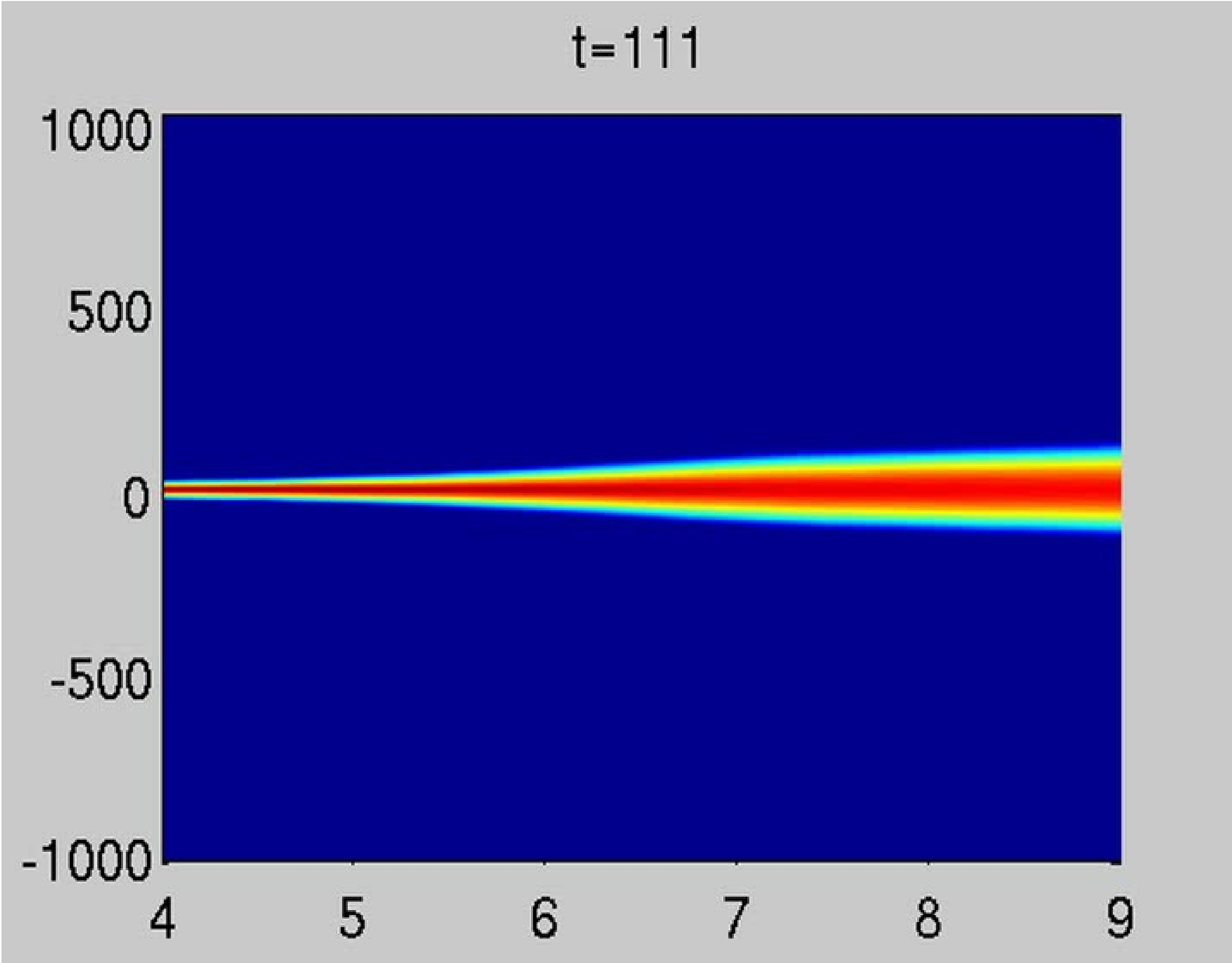
Schematic view of the experiment



$$H = \frac{p^2}{2} + k\cos\theta\ \left(1 + \epsilon\cos\omega_2 t\ \ \cos\omega_3 t\right)\ \sum_n \delta(t - nT)$$

Numerical results for the three-color kicked rotor

Momentum distribution

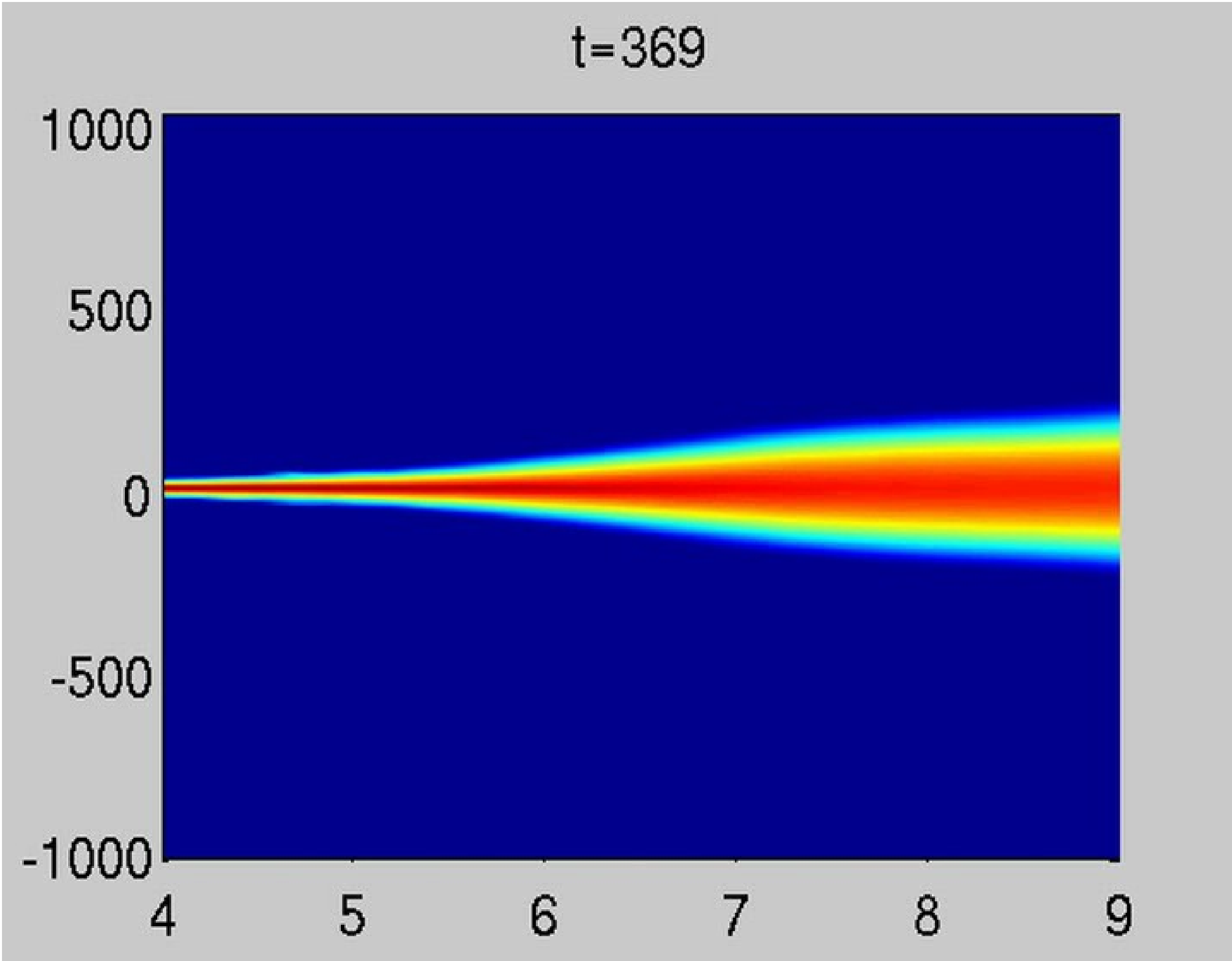


K (kick strength)

Numerical results for the three-color kicked rotor

Momentum distribution

Localized regime

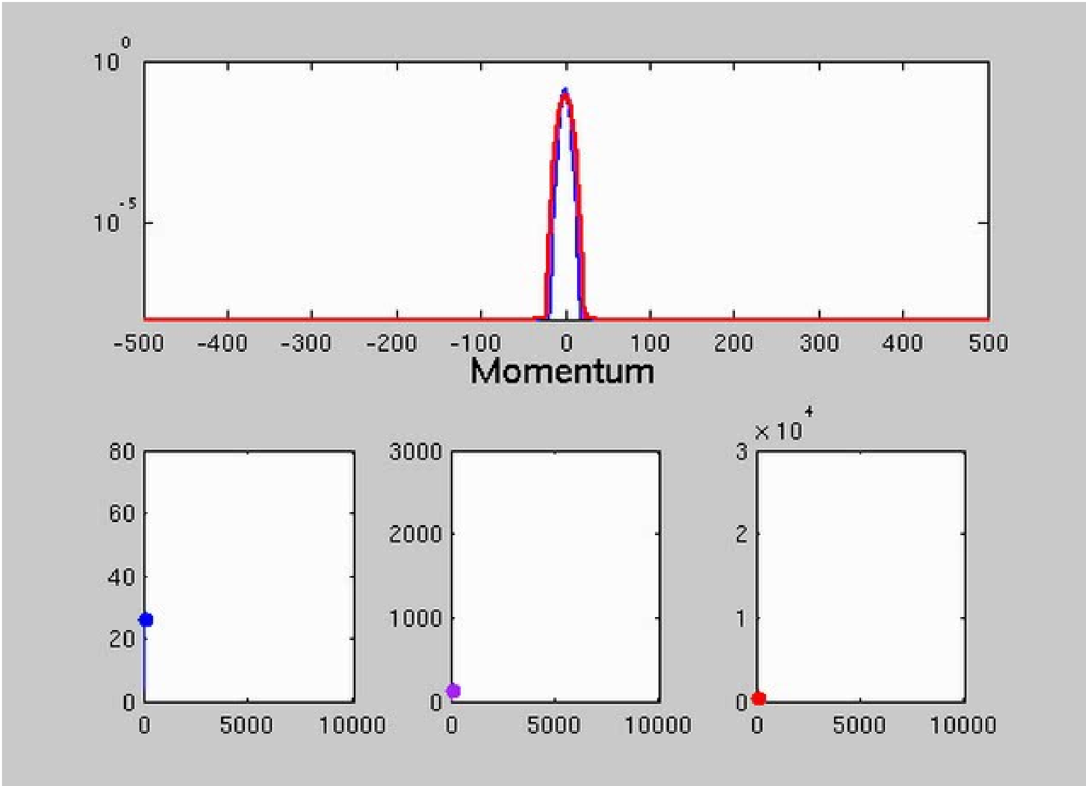


Diffusive regime

K (kick strength)

How to identify unambiguously the transition?

$|\psi(p)|^2$



$\langle p^2(t) \rangle$

3 increasing K values

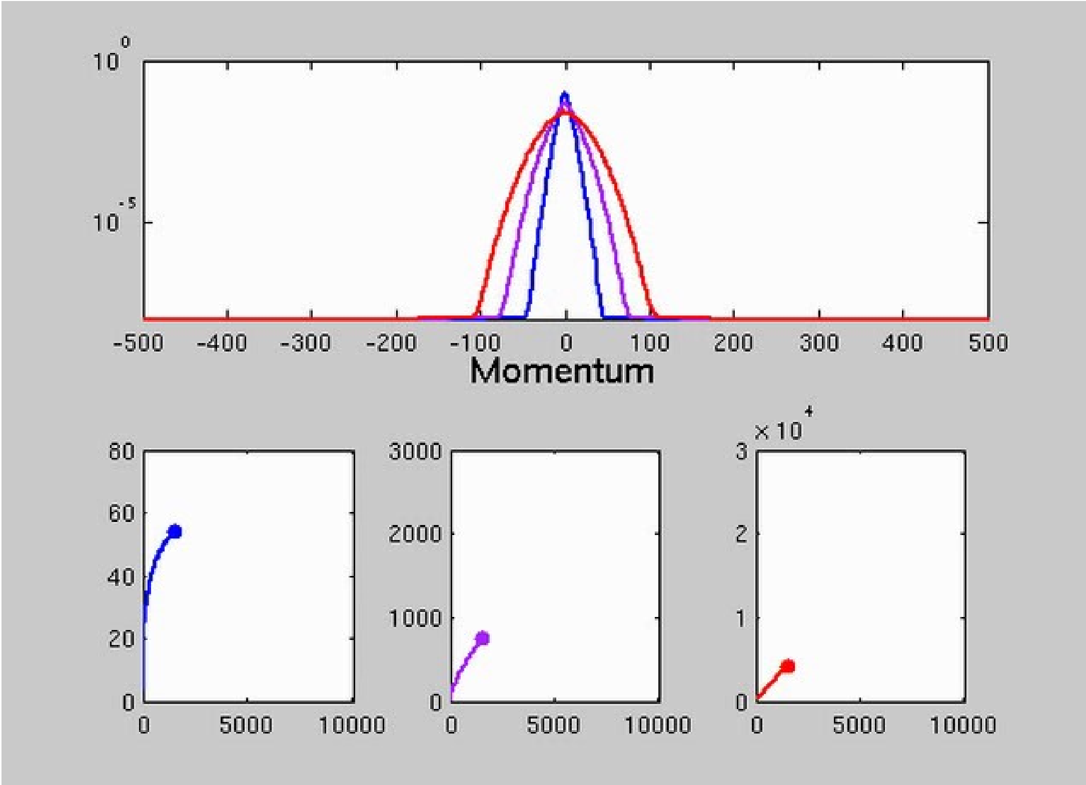
Time (number of kicks)

- At criticality, one expects an anomalous diffusion with (see below)

$$\gamma = \frac{2}{3}$$

How to identify unambiguously the transition?

$|\psi(p)|^2$



$\langle p^2(t) \rangle$

3 increasing K values

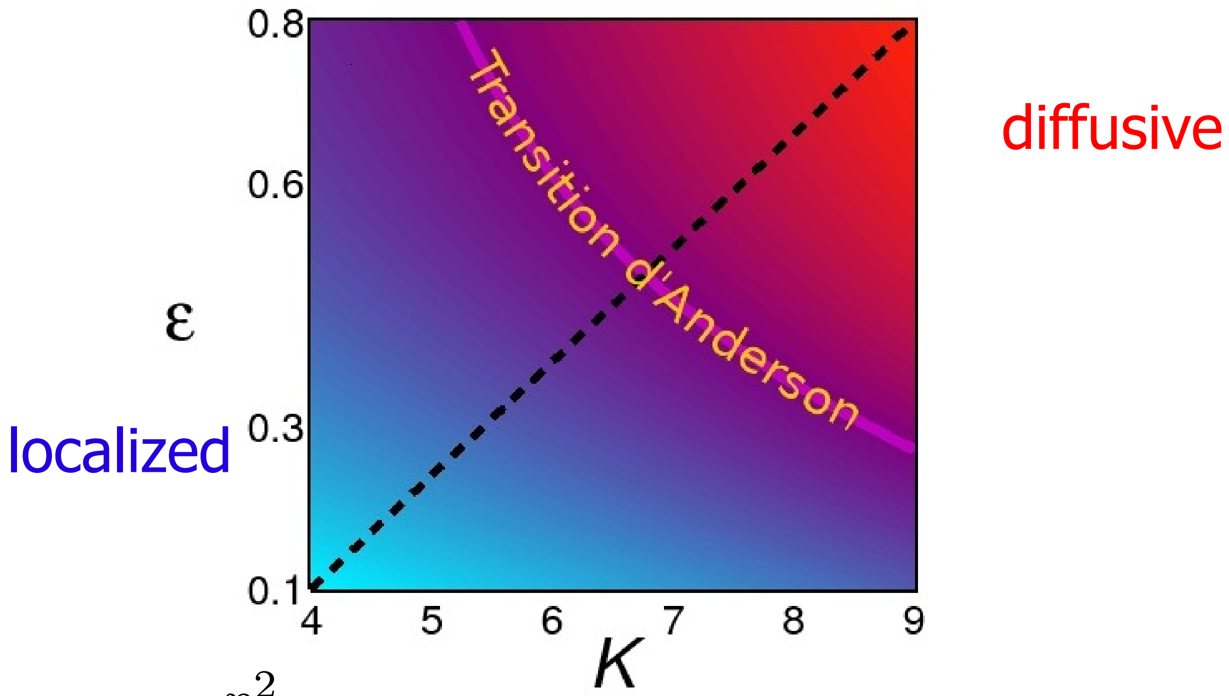
Time (number of kicks)

- At criticality, one expects an anomalous diffusion with (see below)

$$\langle p^2(t) \rangle \simeq t^{\gamma} \quad \text{with} \quad \gamma = \frac{2}{3}$$

Choice of parameters

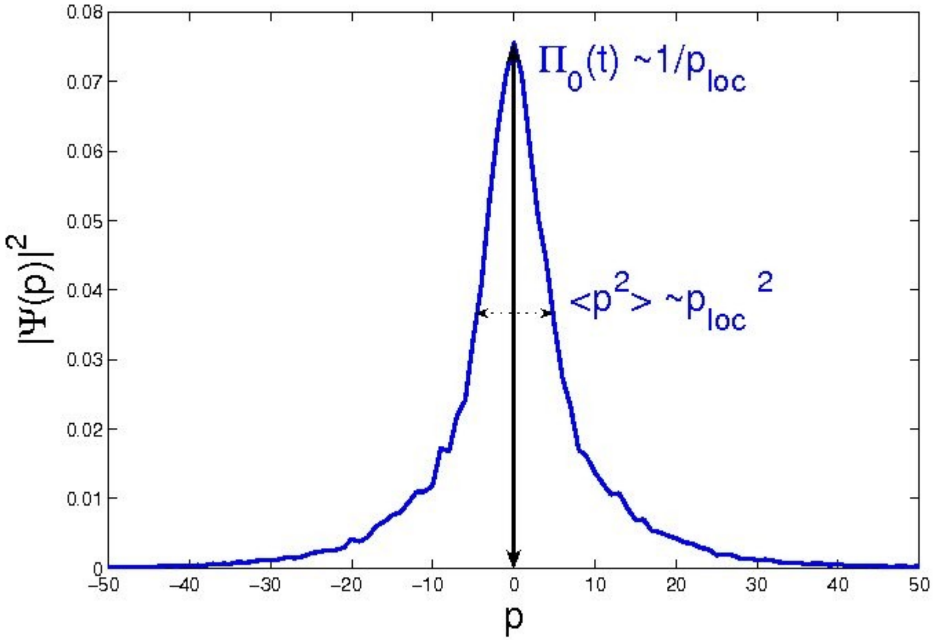
- Frequencies are chosen as incommensurate as possible (ratio $\sqrt{5}$ and $\sqrt{13}$).
- Effective Planck's constant far from rational multiples of π (avoid quantum resonances).
- Cross the transition line in (K,ε) plane "at right angle".

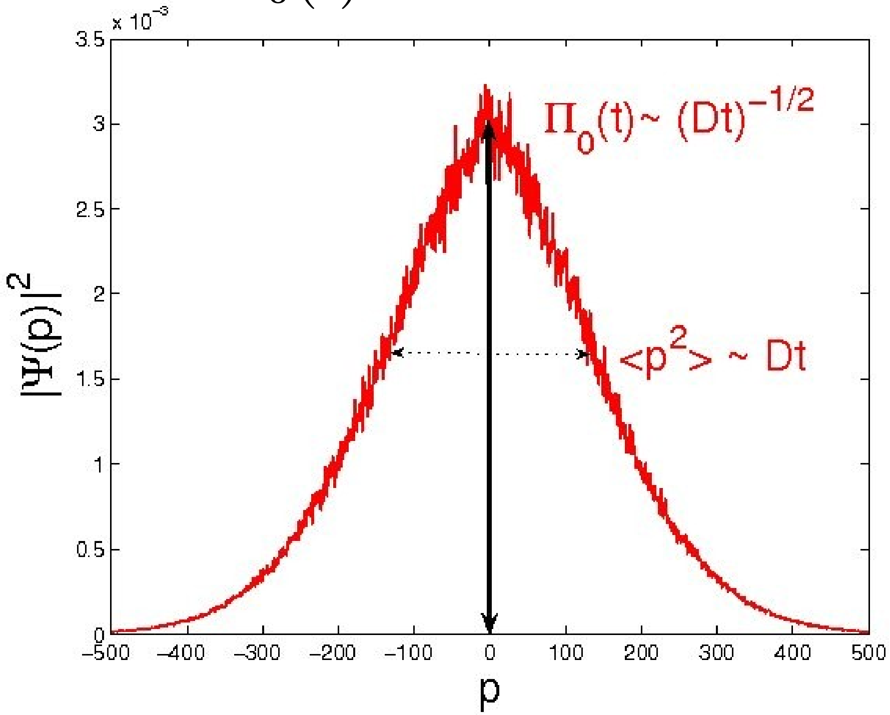


$$H = \frac{p^2}{2} + k\cos\theta\ (1 + \epsilon\cos\omega_2 t\ \ \cos\omega_3 t)\ \sum_n \delta(t - nT)$$

Localized or diffusive?

- Instead of measuring $\langle p^2(t) \rangle$, it is simpler to measure the population in the zero-momentum class $\Pi_0(t)$



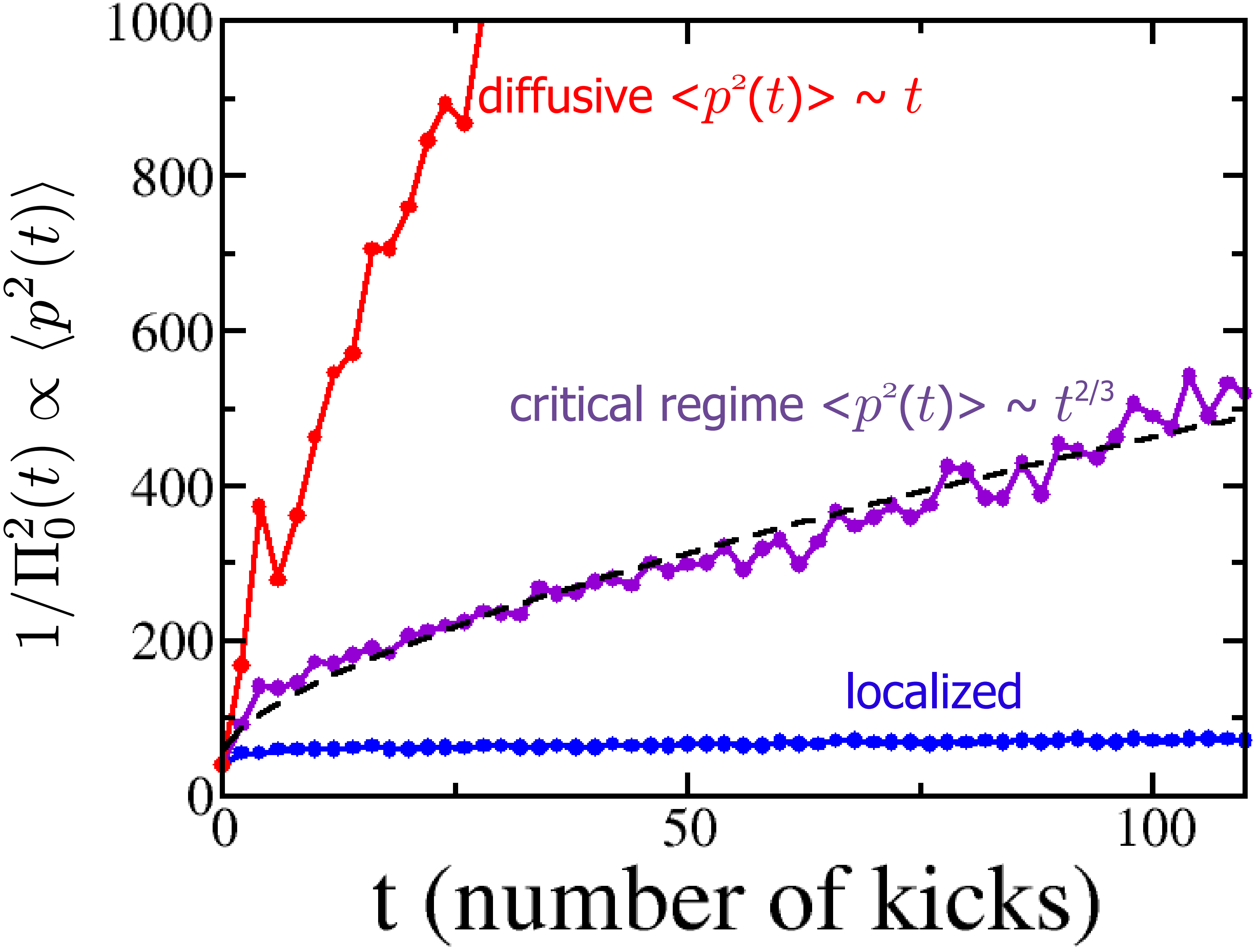


- Both in the localized and the diffusive regime, one has

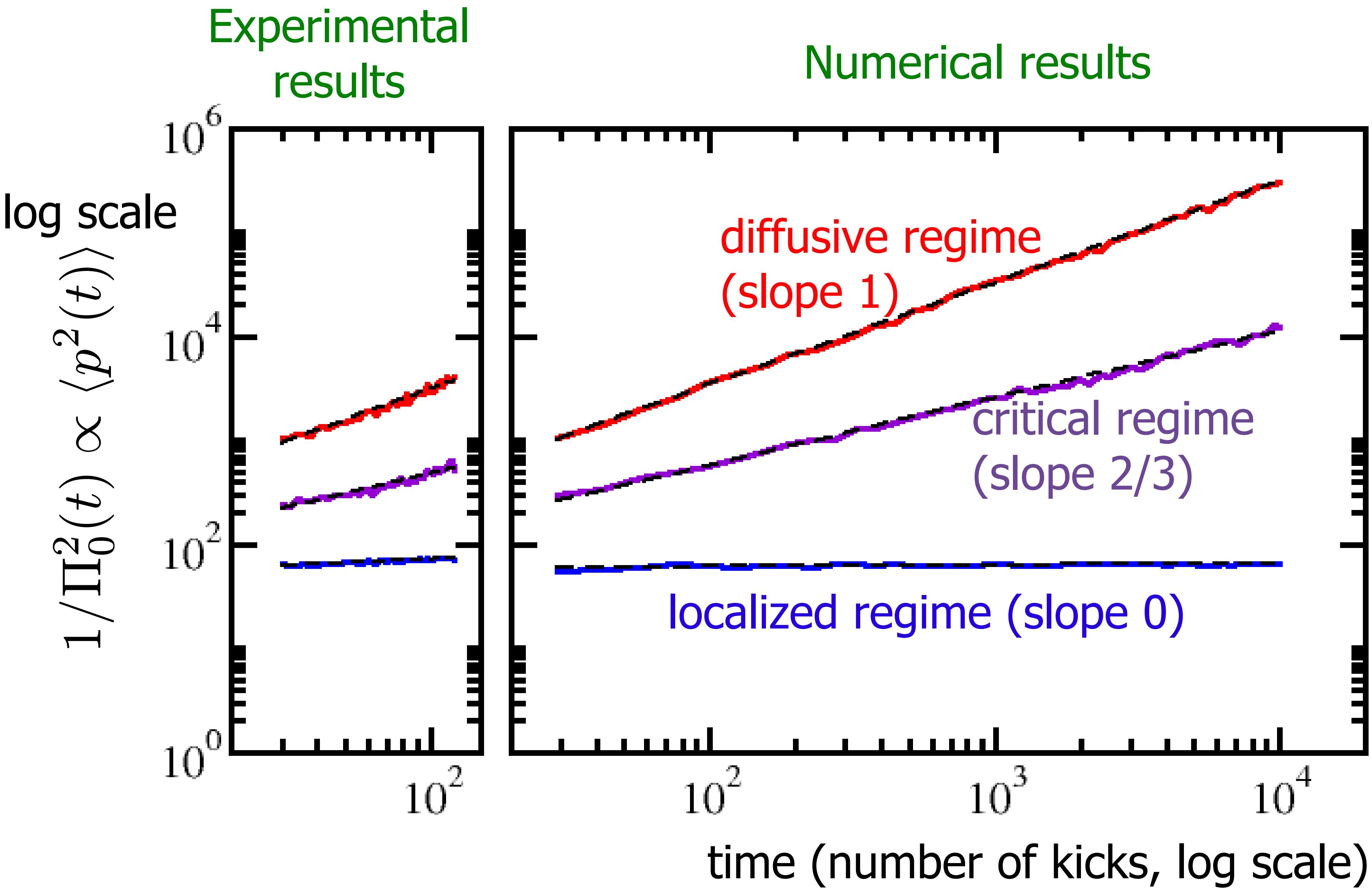
$$\Pi_0^2(t) \propto \frac{1}{\langle p^2(t) \rangle}$$

- In the experiments, one uses $\Pi_0(t)$ for detecting transition from localized to diffusive.

From localization to diffusive regime: experimental results



From localized to diffusive regime



Scaling laws in the vicinity of the Anderson transition

- Characteristic lengths
  - Localization length $p_{\mathrm{loc}} \sim (K_c - K)^{-\nu}$
  - Diffusion constant $D \sim (K - K_c)^s$
- Unified description of the localized and diffusive regimes (one-parameter scaling law):

$$\langle p^2(t)\rangle \sim t^{k_1} F[(K - K_c)t^{k_2}]$$

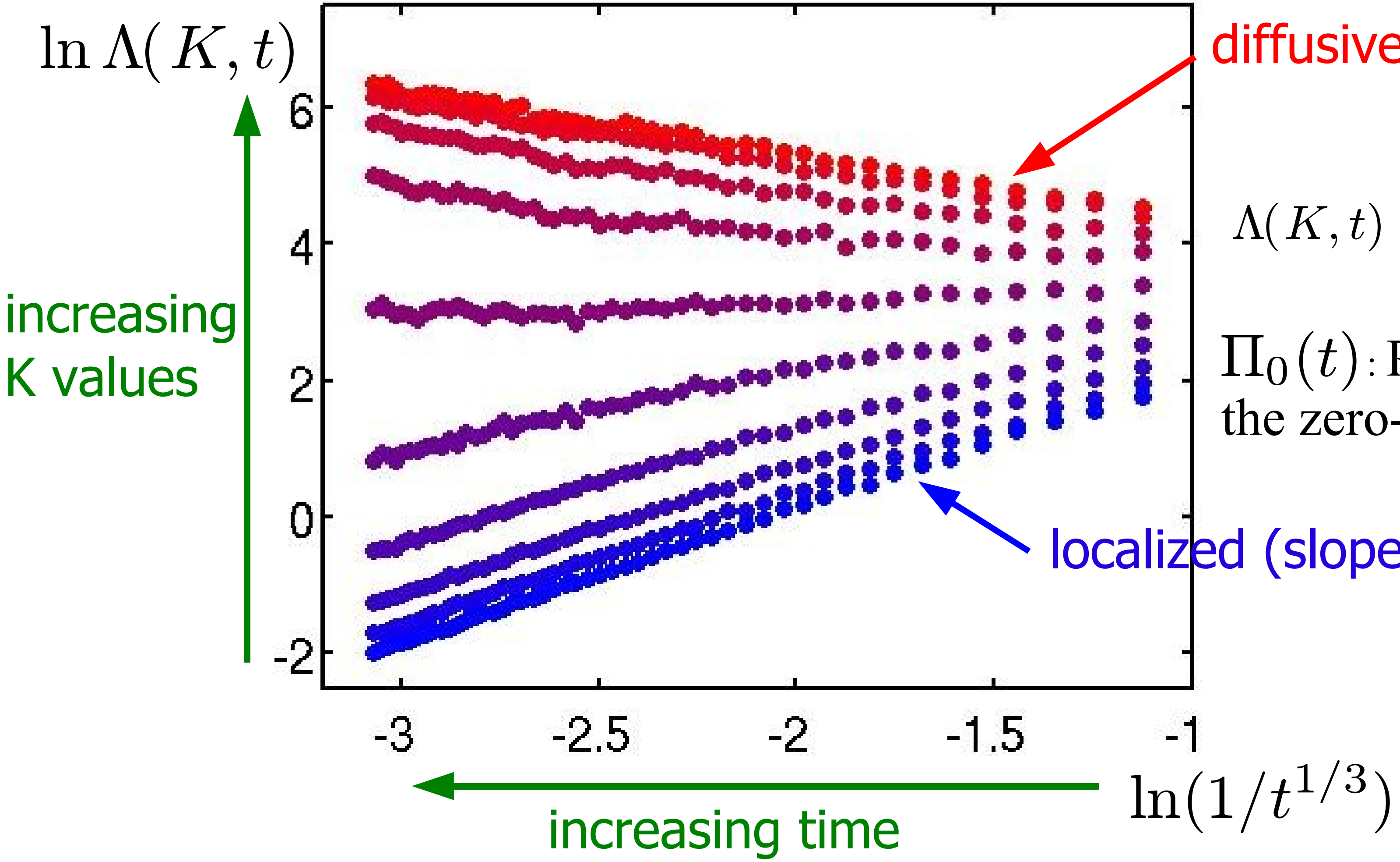
- Asymptotic regimes for F:
  - Localized: $k_1 - 2\nu k_2 = 0$
  - Diffusive: $k_1 + s k_2 = 1$
- Together with Wegner's law $s = (d-2)\nu$, it gives

$$k_1 = 2/3 \qquad k_2 = 1/3\nu$$

Numerical and experimental data agree perfectly well with the 2/3 exponent.

- Thus, we study $\Lambda(K,t) = \dfrac{1}{\Pi_0^2(t)\ t^{2/3}}$ versus t and K.

Rescaled numerical results



$$\Lambda(K,t) = \frac{1}{\Pi_0^2(t)\ t^{2/3}}$$

$\Pi_0(t)$: Population in the zero-velocity class

- The critical regime is the horizontal line.
- Problem: it requires very long times to accurately measure the position of the transition as well as the critical exponent.

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Finite-time scaling

- Directly transpose the ideas of finite size scaling (in usual configuration space) to the temporal dynamics of the kicked rotor => finite time scaling.
- Assume the one-parameter scaling law

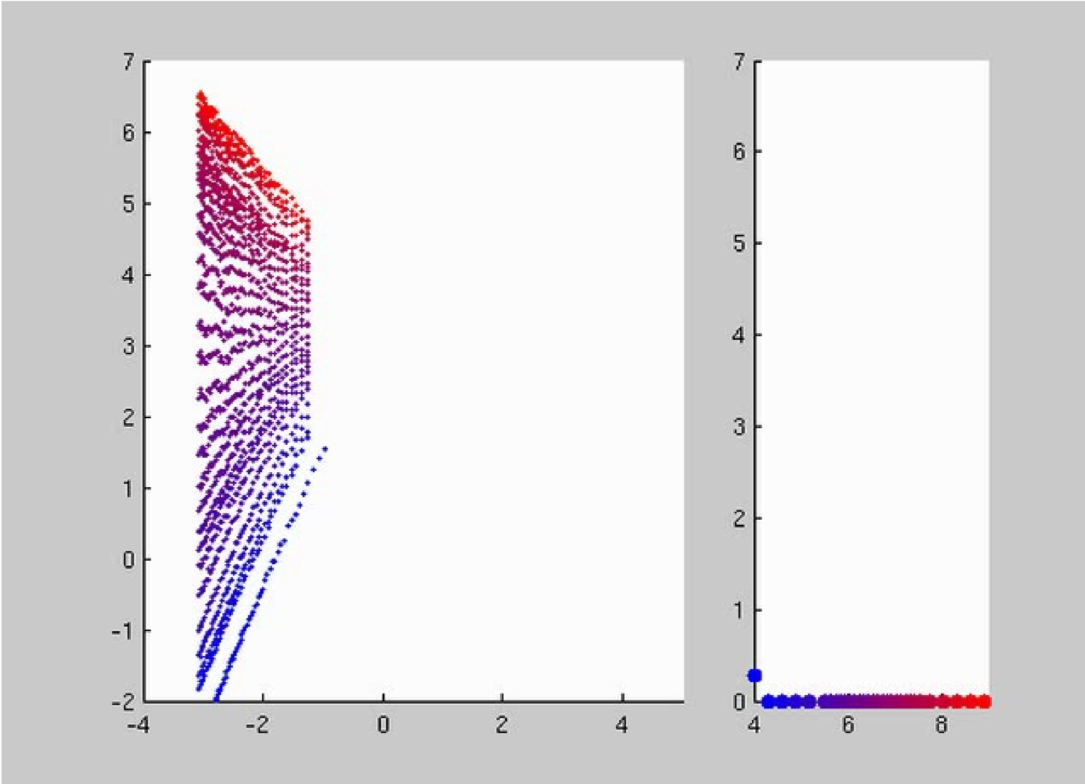
$$\Lambda(t) = \frac{1}{\Pi_0^2(t)\ t^{2/3}} = \mathcal{F}\left(\frac{\xi(K)}{t^{1/3}}\right)$$

  where the localization "length" (actually localization time) ξ depends only on the kick strength K.
- By simultaneously fitting various values of K and t, one can extract both the localization length $\xi(K)$ and the scaling function.

Finite time scaling

$$\Lambda(t) = \frac{1}{\Pi_0^2(t)\ t^{2/3}} = \mathcal{F}\left(\frac{\xi(K)}{t^{1/3}}\right)$$

$\ln(\Lambda) = \ln(1/\Pi_0^2(t)t^{2/3})$



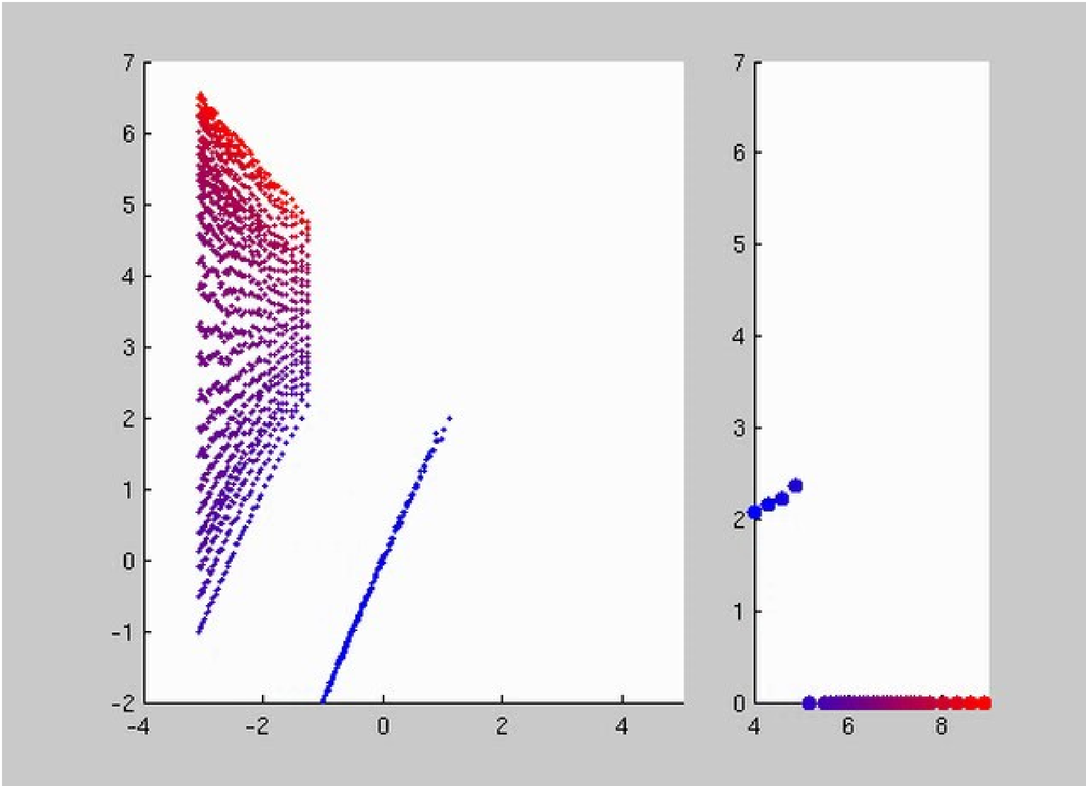
$\ln(1/t^{1/3})$

Finite time scaling

$$\Lambda(t) = \frac{1}{\Pi_0^2(t)\ t^{2/3}} = \mathcal{F}\left(\frac{\xi(K)}{t^{1/3}}\right)$$

Diffusive

$\ln(\Lambda) = \ln(1/\Pi_0^2(t)t^{2/3})$



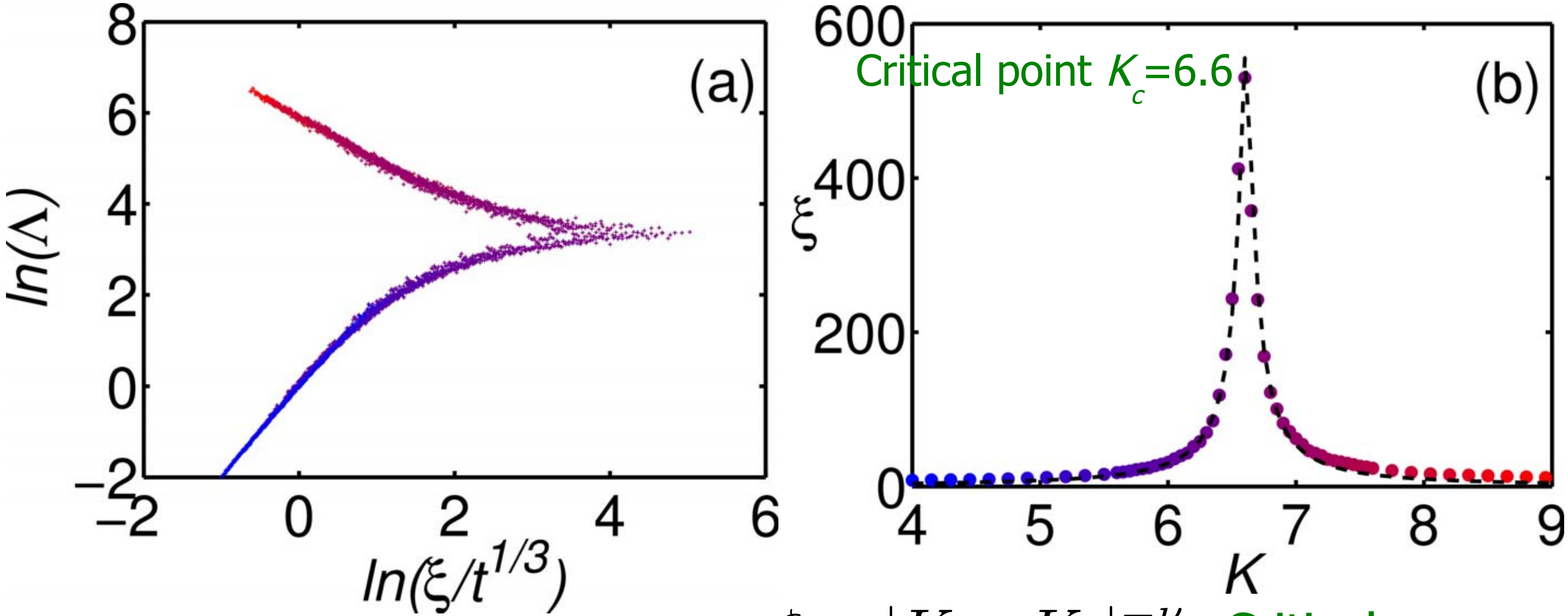
$\ln\xi(K)$

Localized $\ln(1/t^{1/3})$ $\ln(\xi(K)/t^{1/3})$ K

The "displacement" is proportional to ξ(K)

Finite time scaling analysis of numerical results

Scaling function $\Lambda(t) = \dfrac{1}{\Pi_0^2(t)\ t^{2/3}}$ Localization length



$\xi \sim |K - K_c|^{-\nu}$ Critical exponent

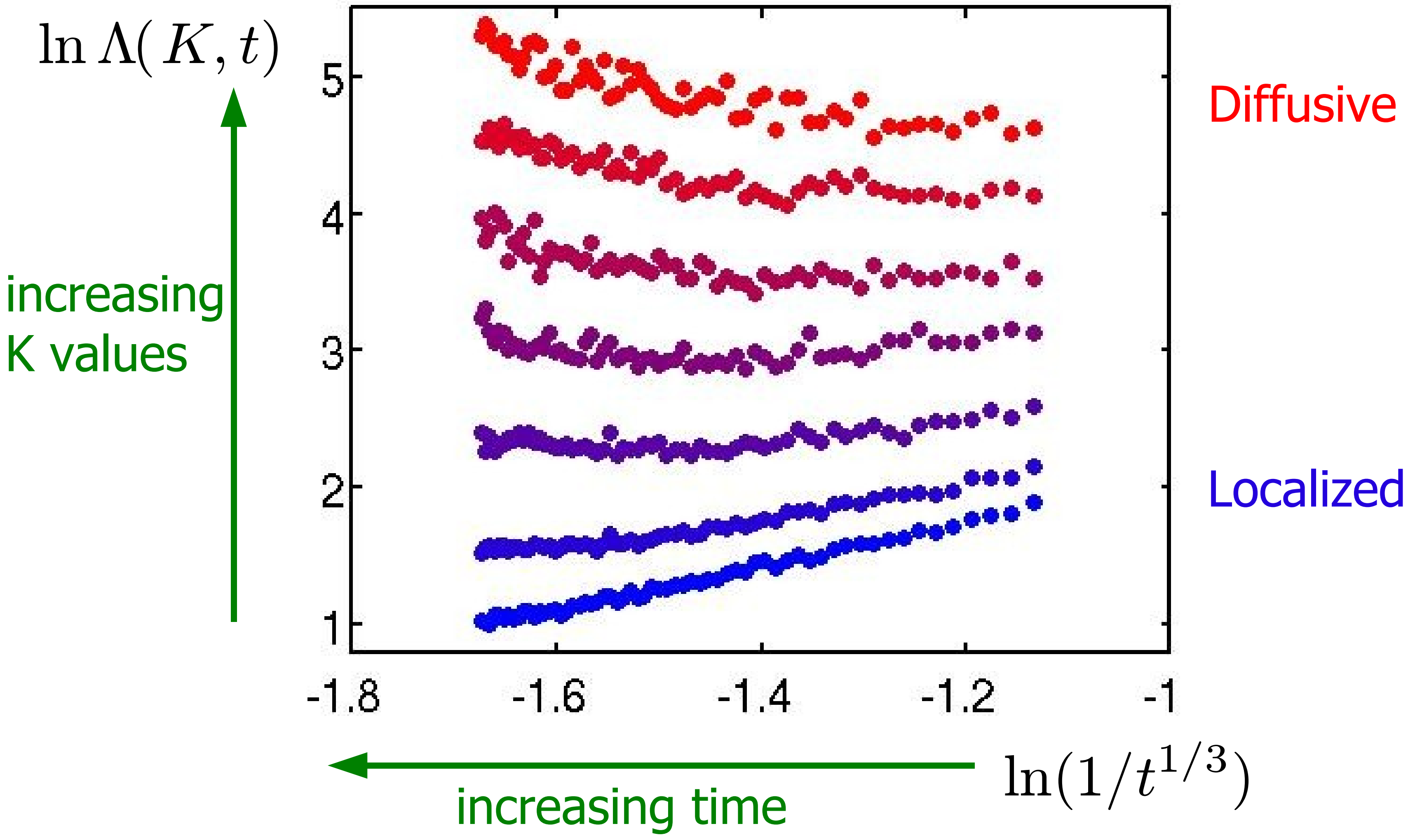
$\nu = 1.60 \pm 0.05$

Chabé et al, PRL, **101**, 255702 (2008)

Numerical data up 10^6 kicks, latest result: $\nu = 1.58 \pm 0.02$

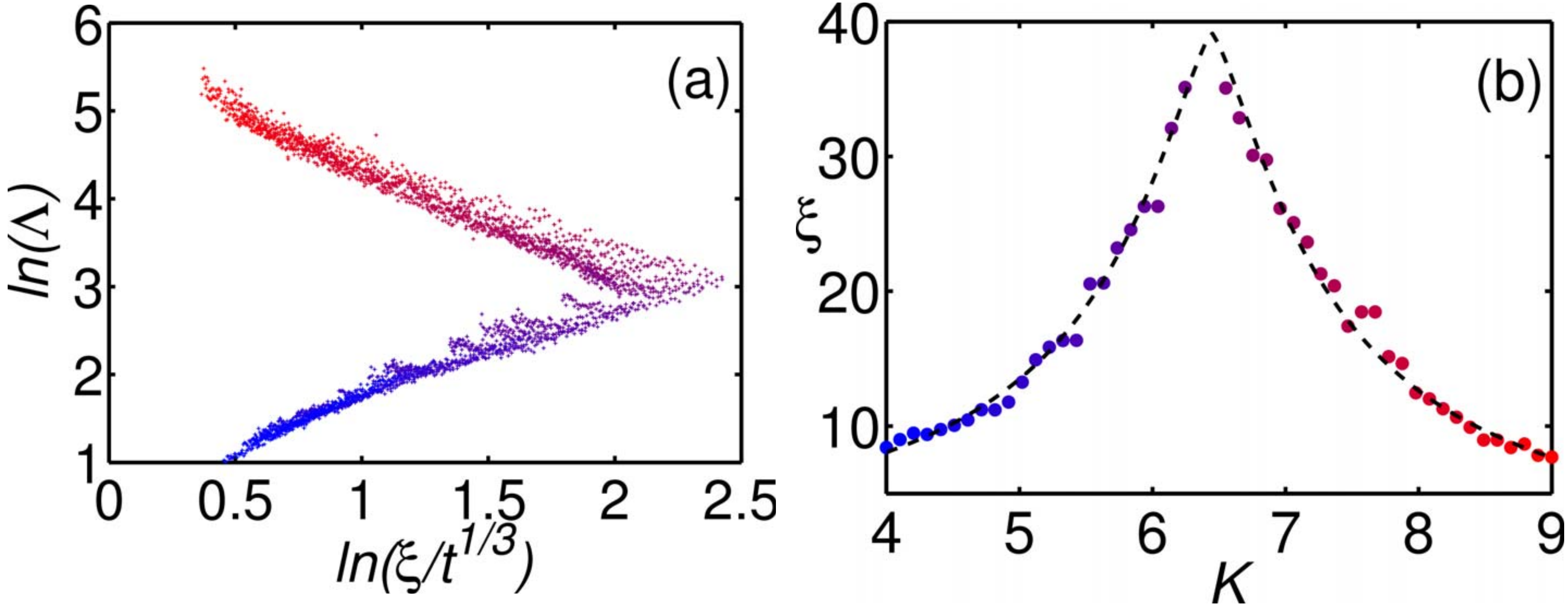
Experimental results

- The temporal dynamics is more restricted => usage of finite time scaling is mandatory



Finite time scaling analysis of experimental results

Scaling function $\Lambda(t) = \frac{1}{\Pi_0^2(t)\ t^{2/3}}$ Localization length



Critical point K_c=6.4

$\xi \sim |K - K_c|^{-\nu}$

Critical exponent $\nu = 1.4 \pm 0.3$

Excellent agreement with numerics without any adjustable parameter

Chabé et al, PRL, **101**, 255702 (2008)

Universality of the Anderson transition

- The Anderson transition is universal, i.e. its characteristic properties – such as the **critical exponent** - do not depend on the microscopic details of the system.
- We have numerically checked this universality for the quasi-periodically kicked rotor, by varying the effective Planck's constant, irrational ratio between the three frequencies, and the depth ε of the modulation.
- We always observe (G. Lemarié et al, Europhys. Lett. **87**, 37007 (2009), arXiv:0904.2324)

$$\nu = 1.58 \pm 0.02$$

- It is identical to the exponent numerically measured on the Anderson model (Slevin and Ohtsuki).

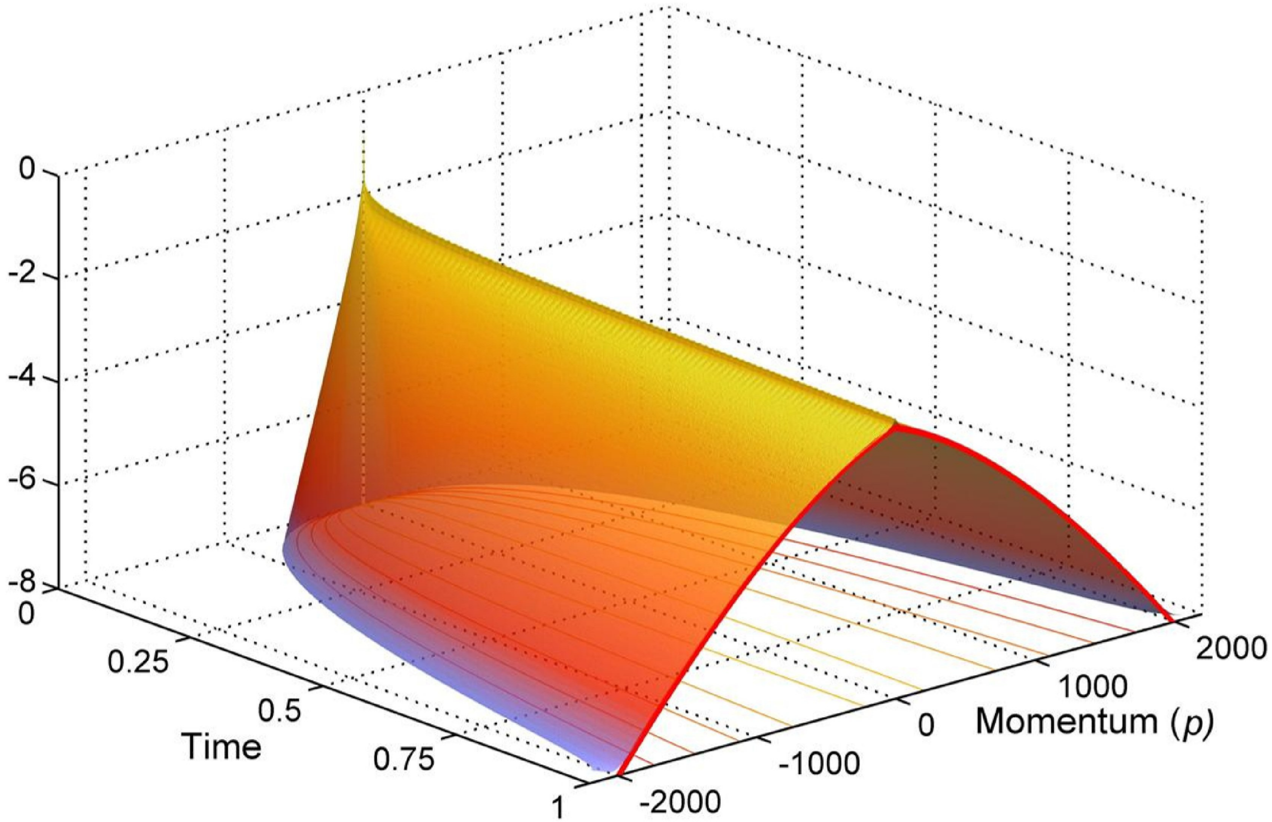
- Anderson localization
- Anderson localization with cold atoms
- The periodically kicked rotor: dynamical localization and Anderson localization
- Experimental realization of the periodically kicked rotor
- Anderson localization and Anderson transition for the “three-color” kicked rotor
- Finite-time scaling
- Critical regime

Momentum distribution at the critical point

The initial state is almost perfectly localized in momentum space => the momentum distribution at time t is nothing but a direct measure of the average intensity Green function $\Pi(0, p; t)$

- Numerical experiment at the critical point:

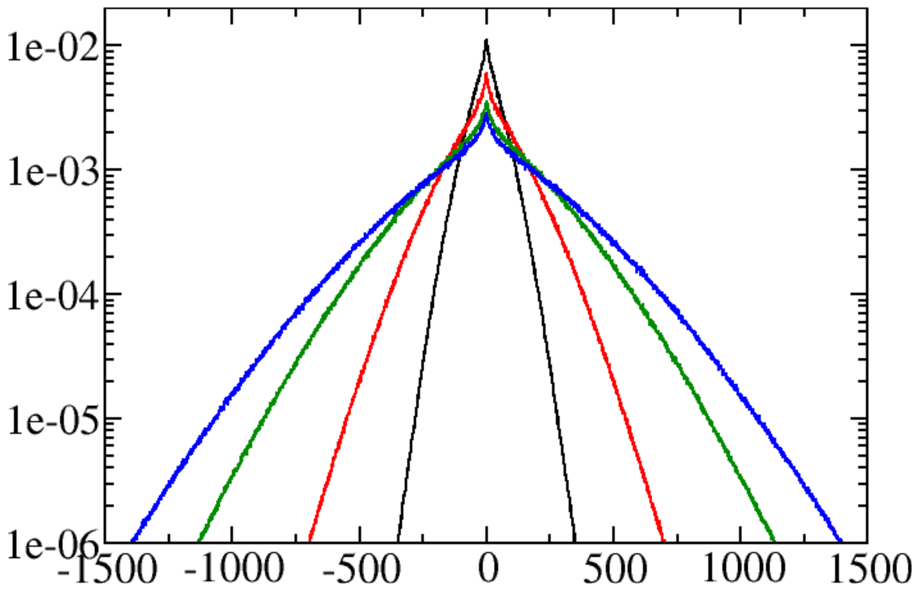
Momentum Green Function (log scale)



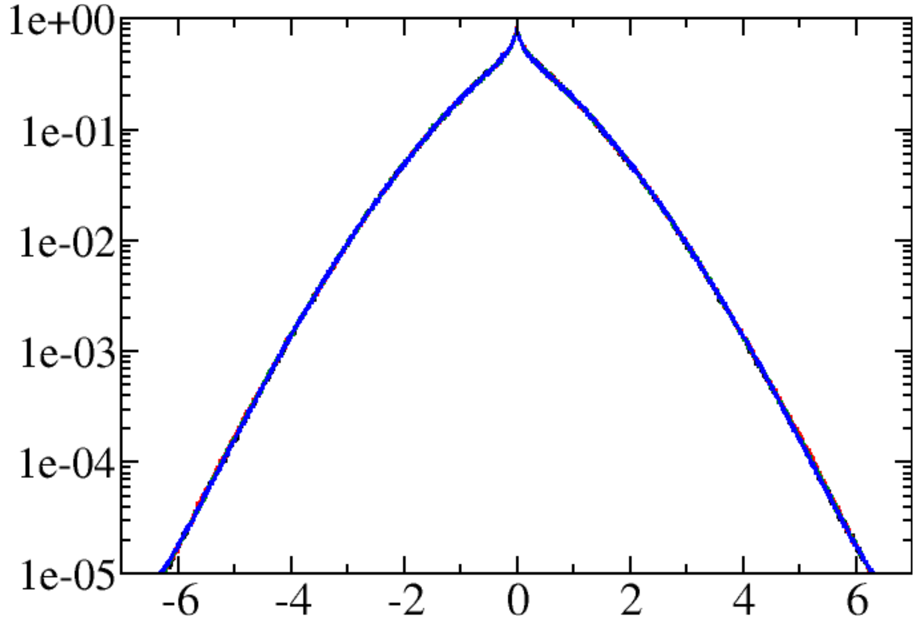
(millions of kicks)

- Time invariant shape (neither Gaussian, nor exponential)

Momentum distributions at criticality



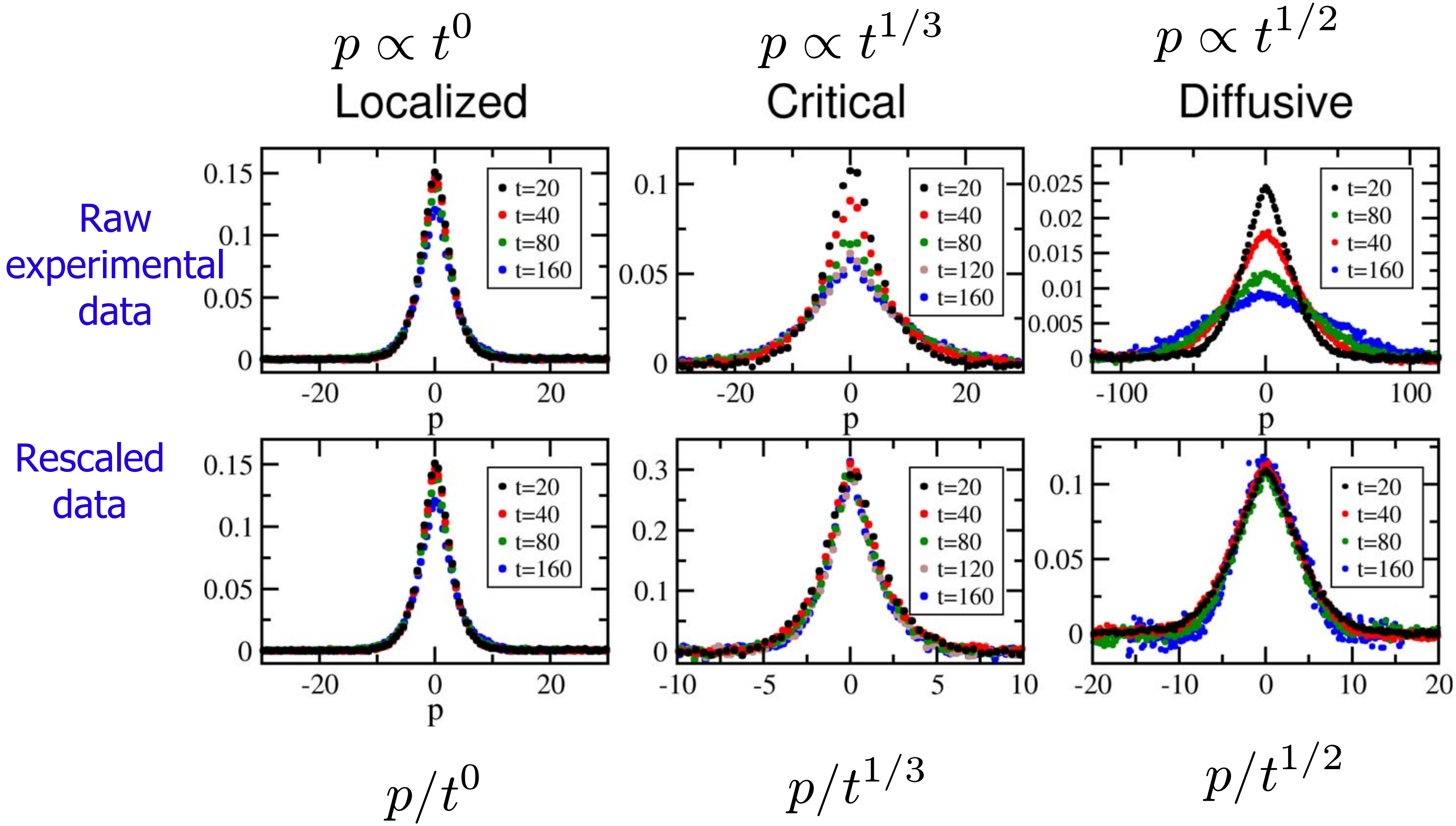
Distributions at various times



Distributions at various times rescaled by the critical $t^{1/3}$ law

Experimental measurements in the critical regime

- Characterized by a specific scaling: $p \propto t^{1/3}$



Lemarié et al, to be published (2010)

Critical wavefunction

- At the critical point, the diffusion constant scales as:

$$D(\omega) = (-i\omega)^{1/3}$$

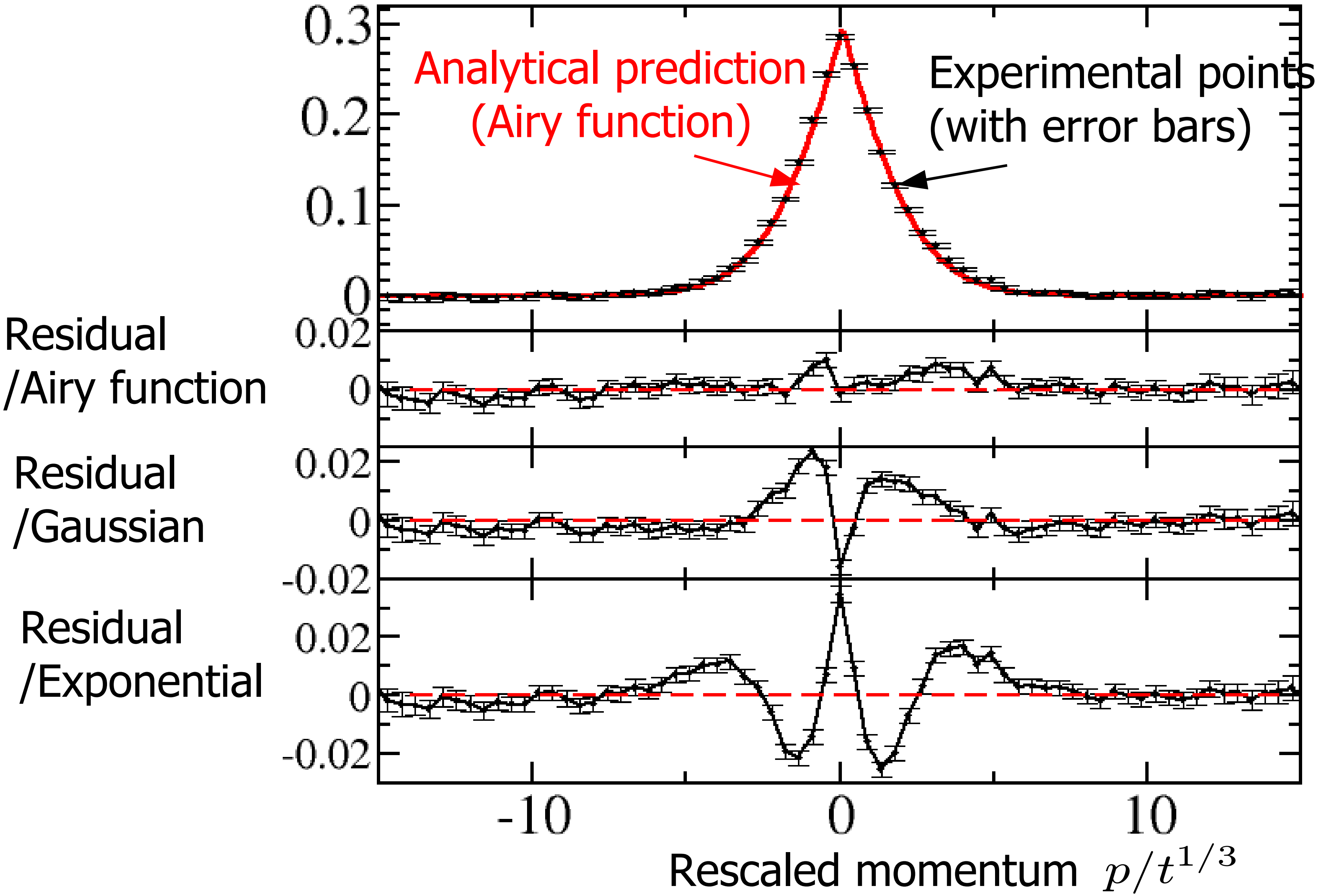
- Using self-consistent theory of localization *(à la Vollhardt-Wölfle)* and the mapping of the quasiperiodic kicked rotor on a 3D Anderson model, we obtain the analytical prediction at the critical point of the Anderson metal-insulator transition:

$$|\psi(p,t)|^2 = \frac{3}{2}\left(3\rho^{3/2}t\right)^{-1/3} \mathrm{Ai}\left[\left(3\rho^{3/2}t\right)^{-1/3}|p|\right]$$

- Behaves asymptotically as $\exp(-\alpha|x|^{3/2})$

- In excellent agreement with numerics and experimental results

Experimentally measured critical Green function



Summary

- Cold atoms are good systems for studying quantum interference, quantum chaos and the quantum dynamics of complex systems.
- Experimental observation of a transition from the localized to the diffusive regime, in a one-body system at zero-temperature, due to quantum interference, driven by the amount of disorder in the system => Anderson transition
- Finite time scaling makes it possible to extract the critical parameters and the critical exponent. First experimental measurement of the critical exponent with atomic matter waves.
- No characteristic scale at the critical point. Self-consistent theory of localization accurately predicts the average intensity Green function.

Perspectives

- Fluctuations of the wave function at the critical point (multifractality). Can be measured in principle, but:
  - We never observe a single Floquet state;
  - The mapping of the 3d Anderson model onto the 1d quasiperiodic kicked rotor involves an averaging over two transverse dimensions => fluctuations are strongly reduced;
  - Short duration of the experiment;
  - Experimental observation of multifractality is unlikely.
- What about two dimensions? Experiment in progress.
- Adding additional temporal quasi-periods => Anderson model in dimension 4, 5...
  - Numerics in *d*=4 (May 2010): $\nu = 1.15 \pm 0.03$
    Good agreement with Anderson model
  - Most probably, the upper critical dimension is infinite.
- Controlled addition of decoherence.
- Interaction between atoms (dynamical localization of a Bose-Einstein condensate) => possible modification of the critical exponent.

Rescaled dynamics at various times (numerics)