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Workshop on Localization Phenomena in Novel Phases of Condensed Matter

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Quantum and Classical Localization Transitions

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QUANTUM AND CLASSICAL LOCALISATION TRANSITIONS

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Work with

Adam Nahum   in progress

M. Ortuño and A. Somoza   Phys. Rev. Lett. 102, 070603 (2009)

Building on

J. Cardy and E. Beamond   Phys. Rev. B 65, 214301 (2002)

I. A. Gruzberg, A. W. W. Ludwig and N. Read   Phys. Rev. Lett. 82, 4254 (1999)

Outline

- **Symmetry classes for random Hamiltonians**

  Discrete symmetries and additions to Wigner-Dyson classification

- **Network models**

  Quantum lattice models for single-particle systems with disorder

- **Quantum - classical mapping**

  For class C network models

- **Applications**

    Spin quantum Hall effect and classical percolation

    Spin quantum Hall effect in bi-layer systems

    3D class C Anderson transition

Symmetry Classes

Dyson random matrix ensembles

Orthogonal

with time-reversal symmetry

Symplectic

with time-reversal symmetry

and Kramers degeneracy

Unitary

without time-reversal symmetry

Additional symmetry classes

Altland and Zirnbauer 1997

Hamiltonian H

2×2 block structure

+ discrete symmetry

Energy levels in pairs $\pm E$

$$X^{-1} H^{*} X = -H$$

(or $X^{-1} H X = -H$)

Given $H\psi = E\psi$

define $\tilde{\psi} = X\psi^{*}$ (or $\tilde{\psi} = X\psi$)

Then $H\tilde{\psi} = -E\tilde{\psi}$

'Class C' $\sigma_y H^{*} \sigma_y = -H$

Disordered Superconductors and Additional Symmetry Classes

Bogoliubov de Gennes Hamiltonian for quasiparticles

Singlet Superconductor

$$\mathcal{H} = \sum_{\alpha\beta} \left[h_{\alpha\beta}(c^\dagger_{\alpha\uparrow} c_{\beta\uparrow} + c^\dagger_{\alpha\downarrow} c_{\beta\downarrow}) + \Delta_{\alpha\beta} c^\dagger_{\alpha\uparrow} c^\dagger_{\beta\downarrow} + \Delta^*_{\alpha\beta} c_{\beta\downarrow} c_{\alpha\uparrow} \right]$$

with spin rotation symmetry $\Delta^{\mathrm{T}} = \Delta$

Put $\mathcal{H}$ into standard form via $\gamma^\dagger_\uparrow = c^\dagger_\uparrow \qquad \gamma^\dagger_\downarrow = c_\downarrow$

Then

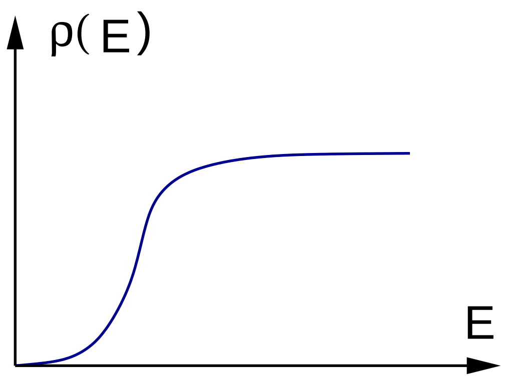
$$\mathcal{H} = \begin{pmatrix} \gamma^\dagger_\uparrow & \gamma_\downarrow \end{pmatrix} \cdot \begin{bmatrix} h & \Delta \\ \Delta^* & -h^{\mathrm{T}} \end{bmatrix} \cdot \begin{pmatrix} \gamma_\uparrow \\ \gamma^\dagger_\downarrow \end{pmatrix}$$

Class C: spin rotation but no time-reversal symmetry

Special features of additional symmetry classes

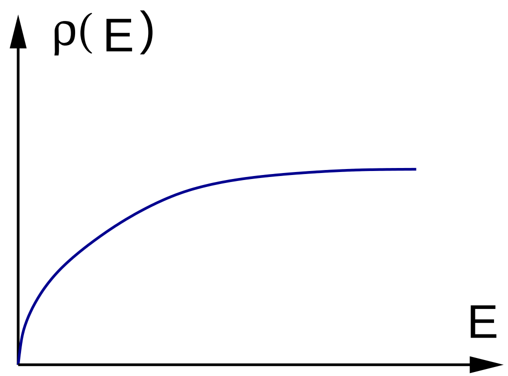
- **Structure in density of states $\rho(E)$ around $E = 0$**
- **Critical behaviour in $\rho(E)$ at Anderson transition**

In class C



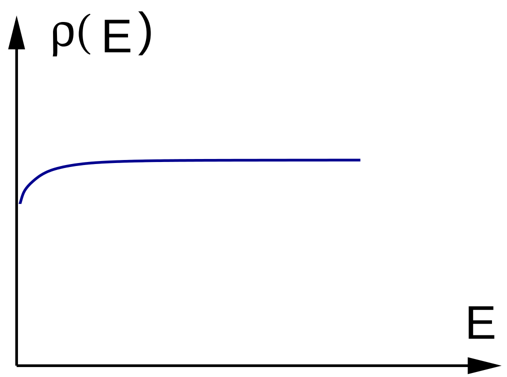
$$\rho(E) \sim E^2$$

RMT and insulator



$$\rho(E) \sim |E|^\alpha$$

Critical point



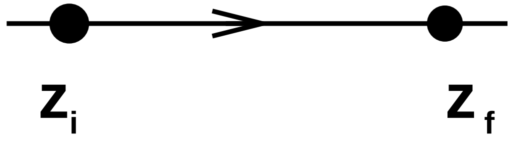
$$\rho(E) - \rho(0) \sim |E|^{d/2-1}$$

Metal

Network Models

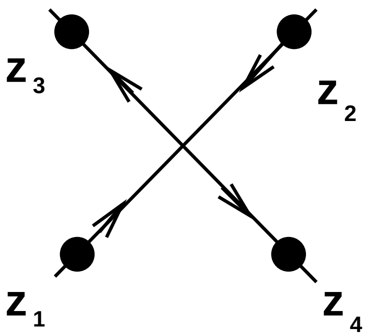
Ingredients

Links:



$$z_f = e^{i\phi} z_i$$

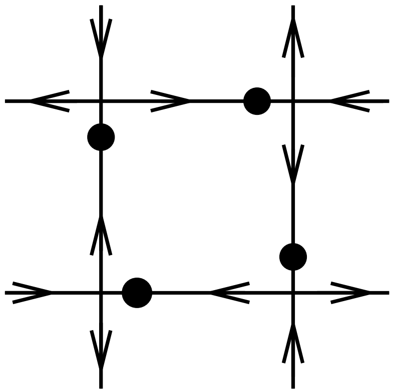
Nodes:



$$\begin{pmatrix} z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

Model

Lattice of links and nodes



Evolution operator

$$W = W_1 W_2$$

W_1: **links** W_2: **nodes**

Disorder introduced via random distribution for link phases ϕ_l

Generalisations of network models

Amplitudes $z_i \longrightarrow$ **n-component vector**

Link phases $e^{i\phi} \longrightarrow n \times n$ **unitary matrices** U

Without further restrictions: U(n) model

not time-reversal invariant, so member of unitary symmetry class

With discrete symmetries:

Class C: $\sigma_y H^* \sigma_y = -H$ **so link phases** $\in$ **Sp(n), with Sp(2)** $\sim$ **SU(2)**

For SU(2) model:

Quantum localisation maps onto classical localisation

SU(2) network model and classical random walks

Feynman path expansion for Green function $G(\zeta) = (1 - \zeta W)^{-1}$

$$[G(\zeta)]_{r_1,r_2} = \sum_{\text{n−step paths}} \zeta^n A_{\text{path}}$$

with weight $A_{\text{path}} \sim \prod_{\text{links}} U_{\text{link}} \left\{ \begin{array}{c} \cos(\alpha) \\ \pm\sin(\alpha) \end{array} \right\}^n$

SU(2) Averages

$$\langle U^n \rangle = \begin{cases} 1 & n = 0 \\ -1/2 & n = \pm 2 \\ 0 & \text{otherwise} \end{cases}$$

– keep only paths that cross each link 0 or 2 times.

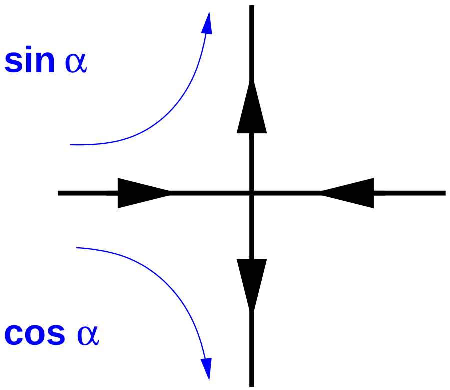
Gruzberg, Ludwig and Read (1999); Beamond, Cardy and Chalker (2002)

Mirlin, Evers and Mildenberger (2003); Cardy (2005)

Quantum to classical mapping

Disorder-average for quantum system $\longrightarrow$ average over classical paths

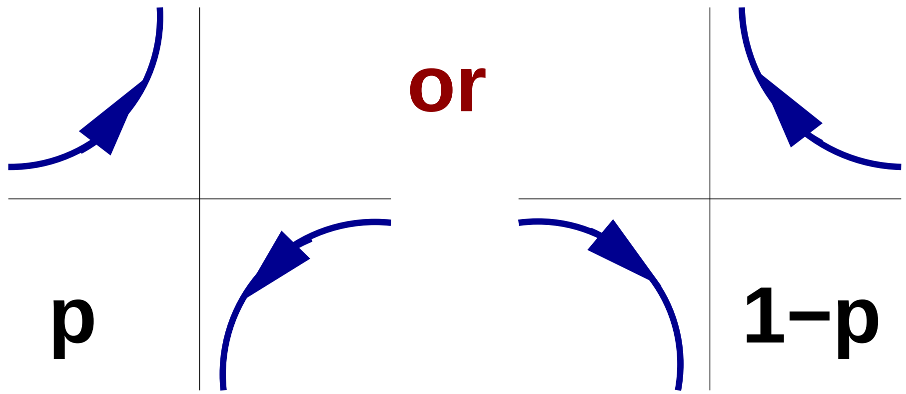
Quantum



Quantum amplitudes

+ random SU(2) phases

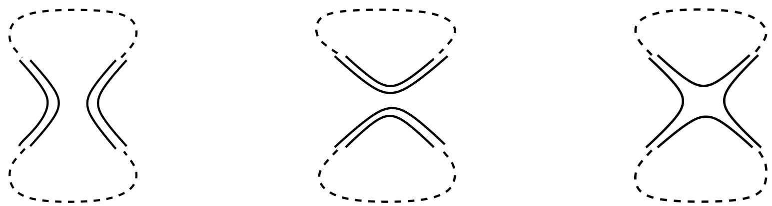
Classical



Classical probabilities

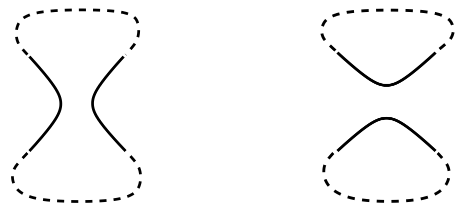
$$p = \sin^2(\alpha) \quad 1 - p = \cos^2(\alpha)$$

Quantum



$\cos^4\alpha$ $\quad \sin^4\alpha$ $\quad -\sin^2\alpha\ \cos^2\alpha$

Classical



p $\quad 1 - p$

Calculating Physical Quantities

Density of states in quantum system

Evolution operator W

eigenvalues $e^{i\varepsilon}$

density $\rho(\varepsilon)$

Classical system

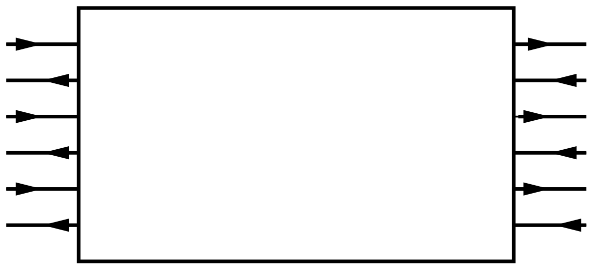
Return probability p_n after n steps

Mapping

$$\rho(\varepsilon) = \frac{1}{2\pi}\left[1 - \sum_n p_n \cos(2n\varepsilon)\right]$$

Conductance

transmission matrix t_{ij}



Landauer formula $G = \sum_{ij} |t_{ij}|^2$

Classical system

transmission probability $p_{i\to j}$

Mapping

$$\langle |t_{ij}|^2 \rangle = p_{i\to j}$$

Applications of Mapping

- In 2d: Spin quantum Hall effect and classical percolation

- Quasi-2D: Spin quantum Hall effect in bi-layer systems

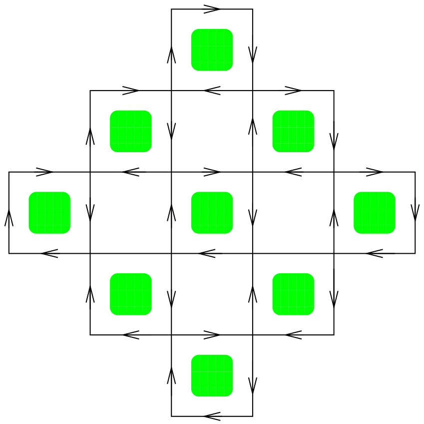
- In 3D: class C Anderson transition

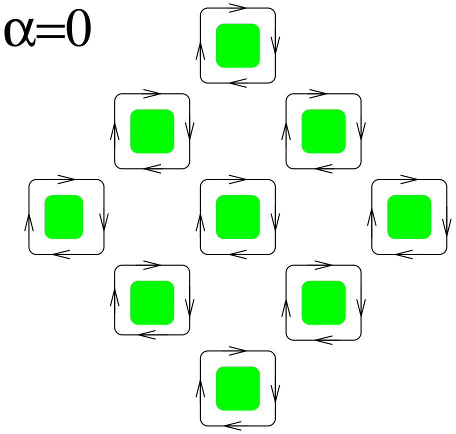
Spin Quantum Hall Effect

Random $\mathrm{SU}(2)$ link phases + uniform scattering angle α at nodes

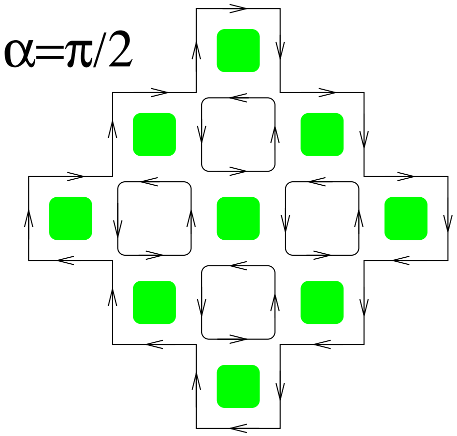
Delocalisation transition as α varied

Model

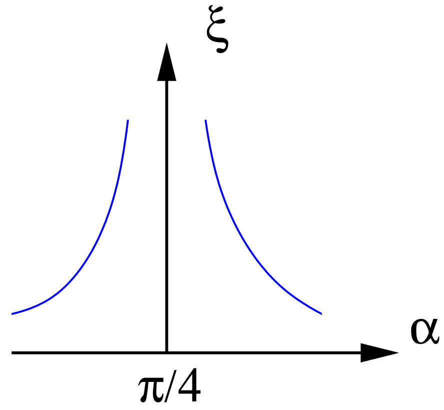




Limiting cases



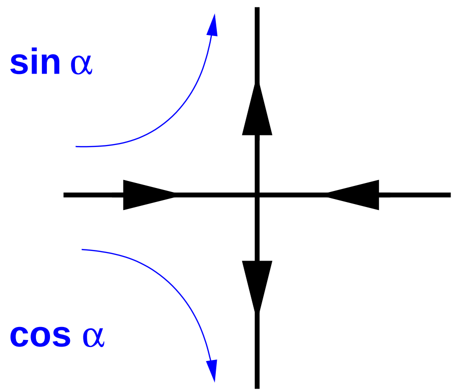
Localisation length



$$\xi \sim |\alpha - \pi/4|^{-\nu}$$

SU(2) network model and percolation

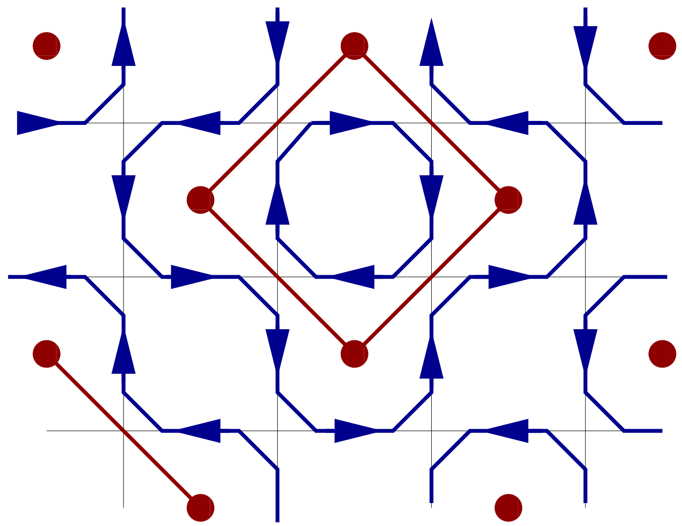
Quantum



Quantum amplitudes

+ random SU(2) phases

Classical



Classical probabilities

$p = \sin^2(\alpha), 1 - p = \cos^2(\alpha)$

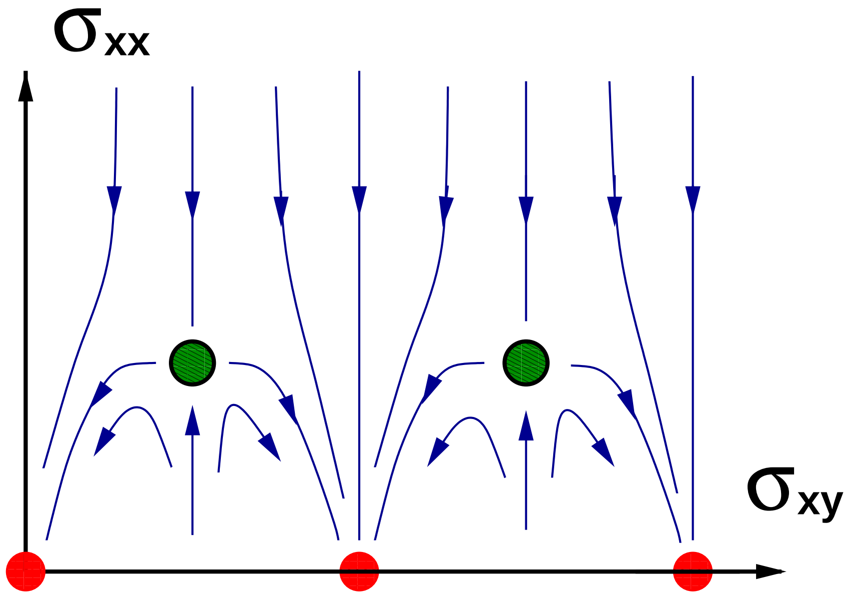
Consequences: $\xi_{Quantum} \sim |\alpha - \pi/4|^{-4/3}$

and $\rho(\varepsilon) \sim |\varepsilon|^{1/7}$ **at** $\alpha = \pi/4$

Gruzberg, Ludwig, Read

Spin quantum Hall effect in bi-layer systems

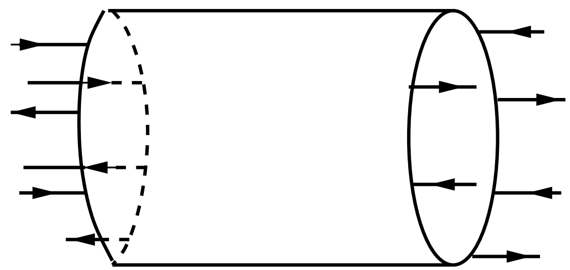
Expected scaling flow



Khmelnitskii

Pruisken

Conductance vs. geometry

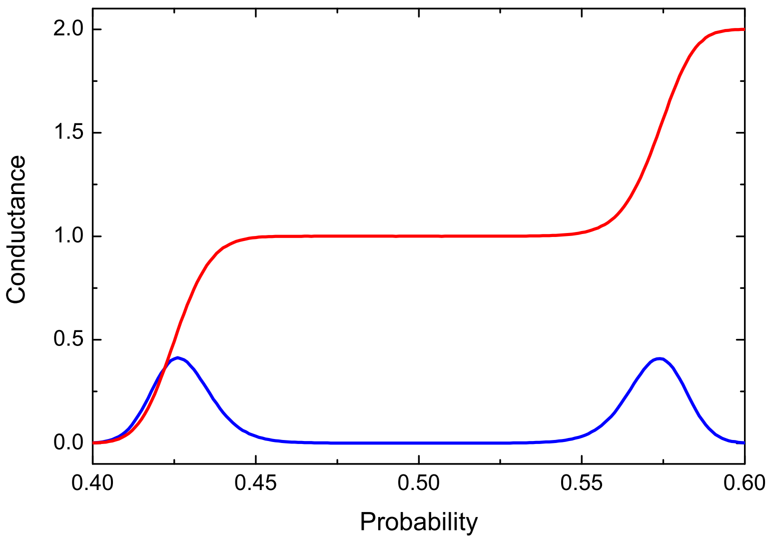


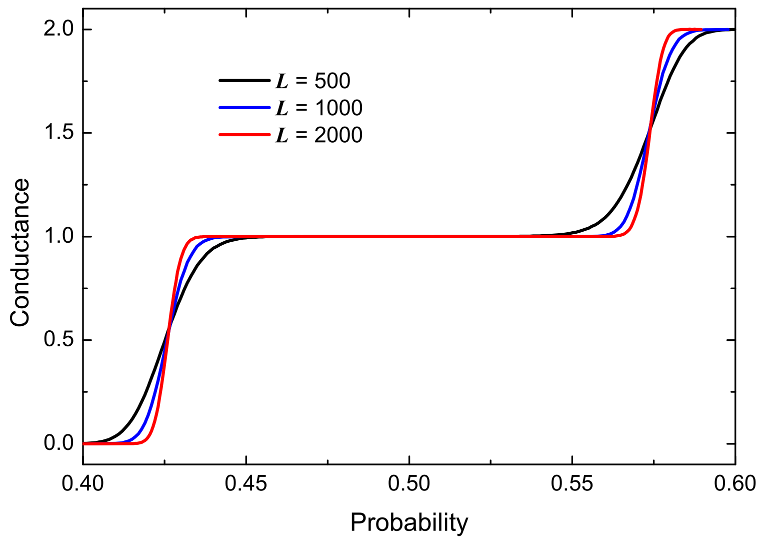
$$G \propto \sigma_{xx}$$

$$G \propto \sigma_{xy}$$

in quantum Hall plateaus

Bi-layer simulations



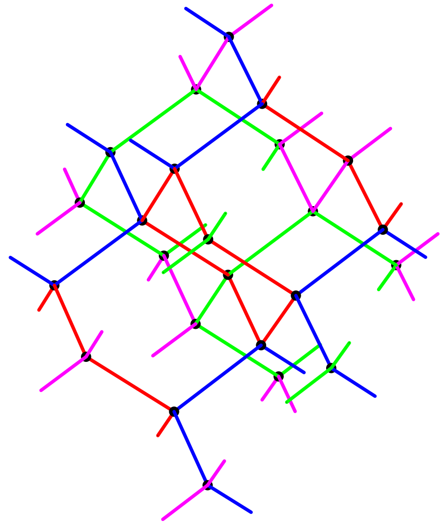


Conductance σ_{xx} and σ_{xy} vs p

Hall conductance vs p and size

3D Class C Anderson Transition

SU(2) network model on diamond lattice



Link directions and nodes chosen so classical walks are:

short closed loops at $p = 0$ infinite trajectories at $p = 1$

Expect transition at $p = p_{\mathrm{c}}$

Very large simulations possible for classical walks

Insulator $p < p_{\mathrm{c}}$

walks have size $\sim \xi(p)$

$$\xi(p) \sim |p - p_{\mathrm{c}}|^{-\nu}$$

Density of states

$$\rho(\varepsilon) \sim \varepsilon^2$$

Critical point $p = p_{\mathrm{c}}$

fractal walks, dimension d_{f}

return probability

$$p_n \sim n^{-3/d_{\mathrm{f}}}$$

Density of states

$$\rho(\varepsilon) \sim |\varepsilon|^{(3/d_{\mathrm{f}})-1}$$

Metal $p > p_{\mathrm{c}}$

free random walks

at distances $\gg \xi(p)$

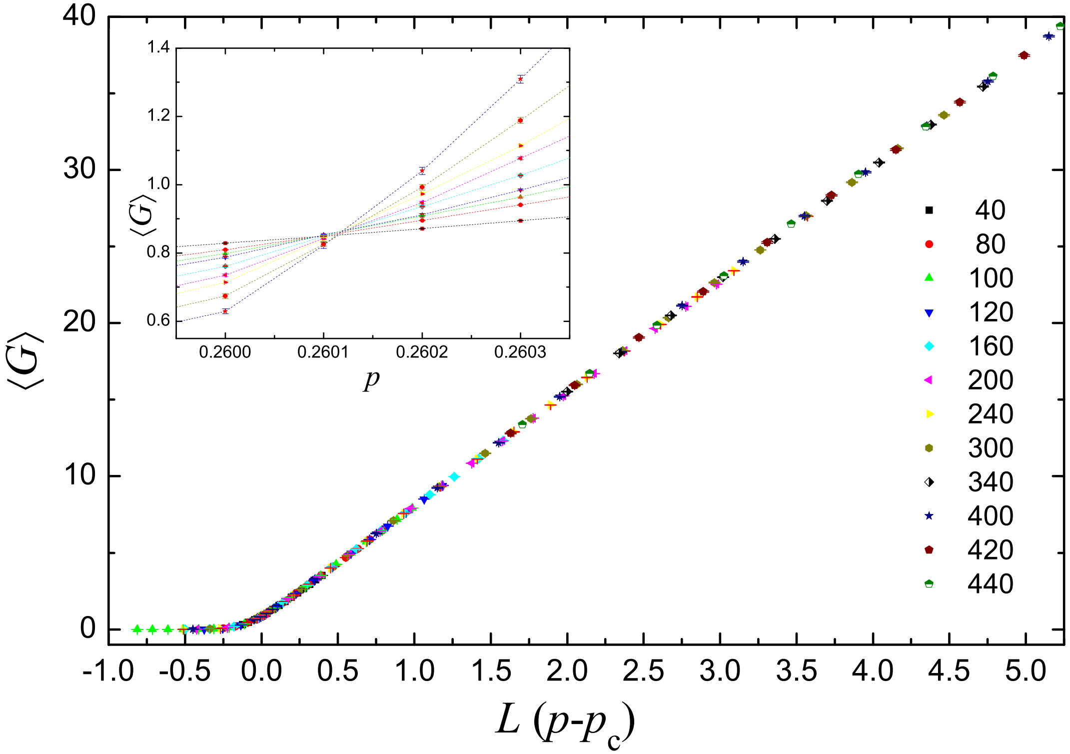
Fractal dimension two

Ohm's law scaling

$$G \propto \frac{\mathrm{Area}}{\mathrm{Length}}$$

3D Class C Simulations: Conductance

Scaling of conductance with sample size and correlation length



$$G(L,p) = f(L/\xi(p))$$

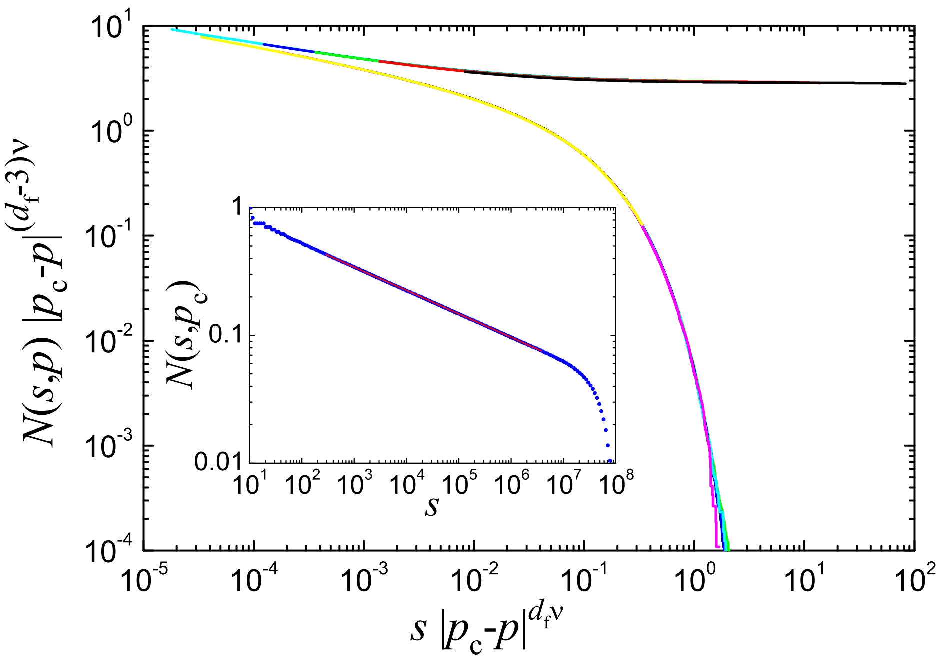
with $\xi(p) \sim |p - p_c|^{-\nu}$ **and** $\nu = 0.9985 \pm 0.0015$

See also: Kagalovsky, Horovitz and Avishai, PRL (2004)

Classical walks: return probability

Integrated return probability

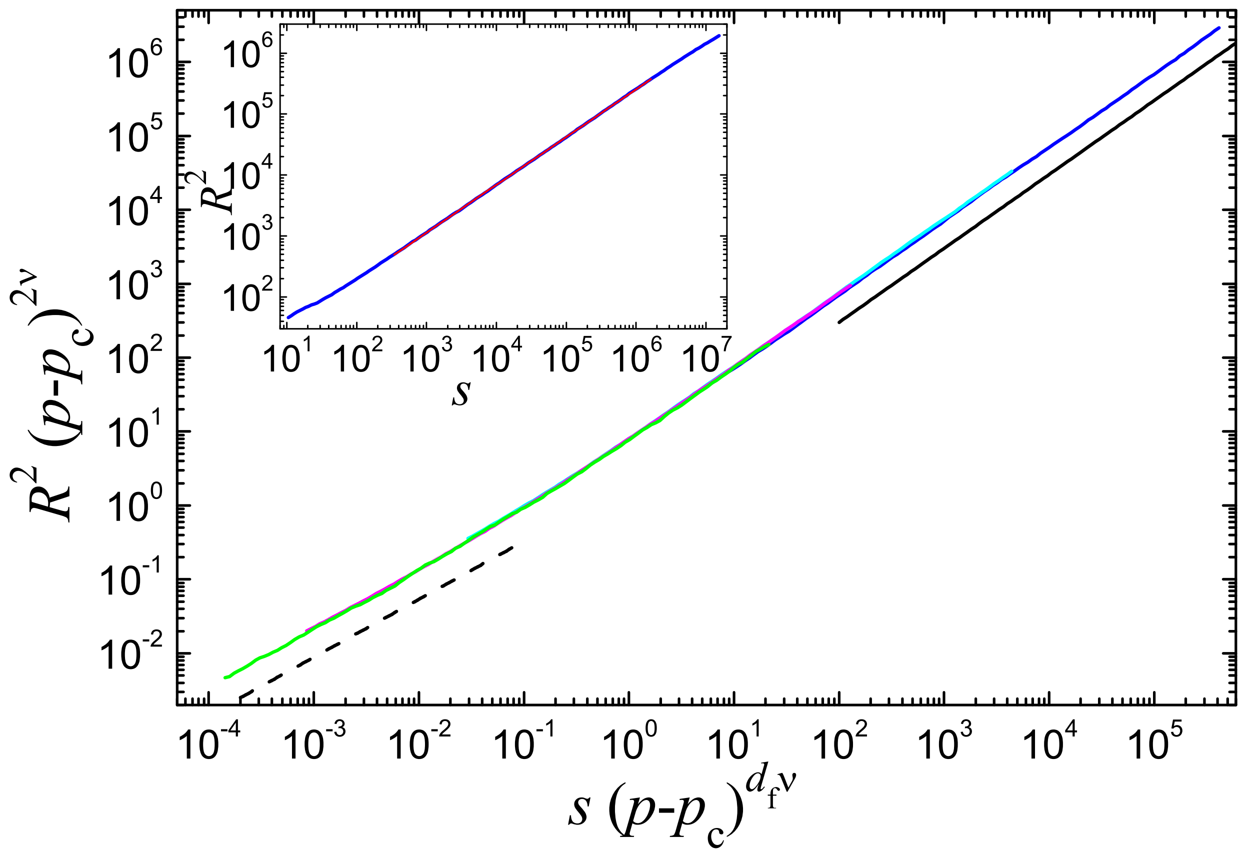
$$N(s,p) = \sum_{t \geq s} P(s,p) = \xi(p)^{d_f - 3} h_{\pm}(s/\xi(p)^{d_f})$$



Scaling collapse: $d_f = 2.53 \pm 0.01$

Classical walks: end-to-end distance vs. length

Critical at short distances, free walks at long distances



small s**:** $\langle R^2 \rangle \sim s^{2/d_f}$ **large** s**:** $\langle R^2 \rangle \sim s$

What is universality class for these 3D walks?

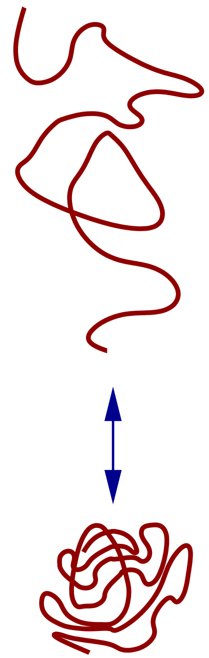
Compare with collapse transition for polymers

Self avoidance + local attraction

Swollen phase: self avoiding walks $\langle R^2 \rangle \propto s^{1.18}$

Theta-point: $\langle R^2 \rangle \sim s$

Collapsed phase: $\langle R^2 \rangle \sim s^{2/3}$



In contrast, Ohm's law requires $\langle R^2(s) \rangle \propto s$

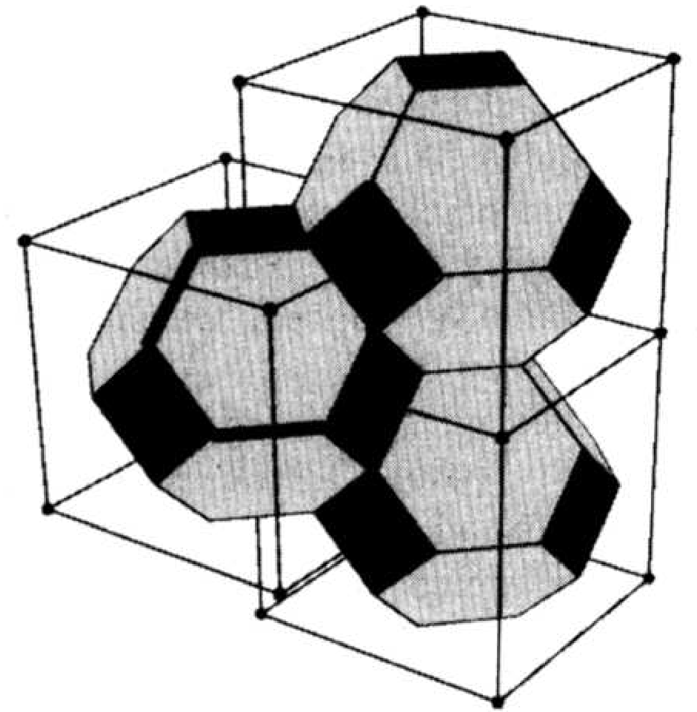
for class C walks in metallic phase

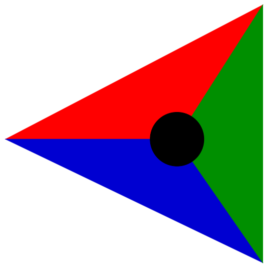
Tricolour percolation and tricolour walks

- Pick lattice in which each edge of Wigner-Seitz cells is shared by three sites.

- Colour sites red, blue or green with probabilities p, q and $1-p-q$.

- Tricolour walks formed from edges where three colours meet.





Tricolour percolation and tricolour walks

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- Tricolour walks formed from edges where three colours meet.

On body-centred cubic lattice:

[Bradley *et al.*, PRL (1992)]

- Some walks extended near $p = q = 1/3$

- All walks localised for p, q both small

- Exponent values match ones for class C walks

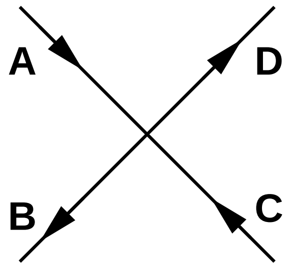
Continuum Theory

Generalise classical walks to n flavours

$$\mathcal{Z} = \sum_{\text{configs}} p^{n_{\text{left}}}(1-p)^{n_{\text{right}}} n^{n_{\text{loops}}}$$

Introduce n component complex

unit vector $\vec{z}_l$ on each link l



Calculate $\mathcal{Z} = \mathcal{N} \prod_l \int \mathrm{d}\vec{z}_l \; \mathrm{e}^{-\mathcal{S}}$

with $\mathrm{e}^{-\mathcal{S}} = \prod_{\text{nodes}} \left[p(\vec{z}_A^{\dagger} \cdot \vec{z}_B)(\vec{z}_C^{\dagger} \cdot \vec{z}_D) + (1-p)(\vec{z}_A^{\dagger} \cdot \vec{z}_D)(\vec{z}_C^{\dagger} \cdot \vec{z}_B) \right]$

Expand $\prod_{\text{nodes}}[\ldots]$ Loops contribute factors

$$\sum_{\alpha,\beta,\ldots\gamma} \int \mathrm{d}\vec{z}_1 \ldots \int \mathrm{d}\vec{z}_L \; z_1^{*\alpha} z_2^{\alpha} z_2^{*\beta} \ldots z_L^{*\gamma} z_1^{\gamma}$$

Hence: (i) factor of n per loop (ii) invariance under $\vec{z}_l \longrightarrow \mathrm{e}^{\mathrm{i}\varphi_l}$

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Generalise classical walks to n flavours

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Continuum limit CP(n-1) model

$$S = \int \mathrm{d}^d \mathbf{r} \, |(\nabla - \mathrm{i}A)\vec{z}|^2 \quad \textbf{with} \quad A = \tfrac{\mathrm{i}}{2}(z^{*\alpha}\nabla z^\alpha - z^\alpha \nabla z^{*\alpha})$$

with $|\vec{z}|^2 = 1$ **and invariance under** $\vec{z} \longrightarrow \mathrm{e}^{\mathrm{i}\varphi(\mathrm{r})}\vec{z}$

Critical dimensions: $d_l = 2$ **and** $d_u = 4$. **First order for** $n > n_c$

see also: Candu, Jacobsen, Read and Saleur (2009)

Summary

Quantum-classical mapping

for class C localisation problems

Class C Anderson transition

Critical behaviour in density of states

Correspondence between quantum and classical localisation

Critical behavour known exactly for 2D classical problem

Classical problem:
efficient starting point for simulations in quasi-2D and 3D

3D transition: same universality class as tricolour walks

Continuum description

CP(n-1) model with $n \rightarrow 1$