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Workshop on Localization Phenomena in Novel Phases of Condensed Matter

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Anderson Localization from Classical Trajectories

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Anderson Localization from Classical Trajectories

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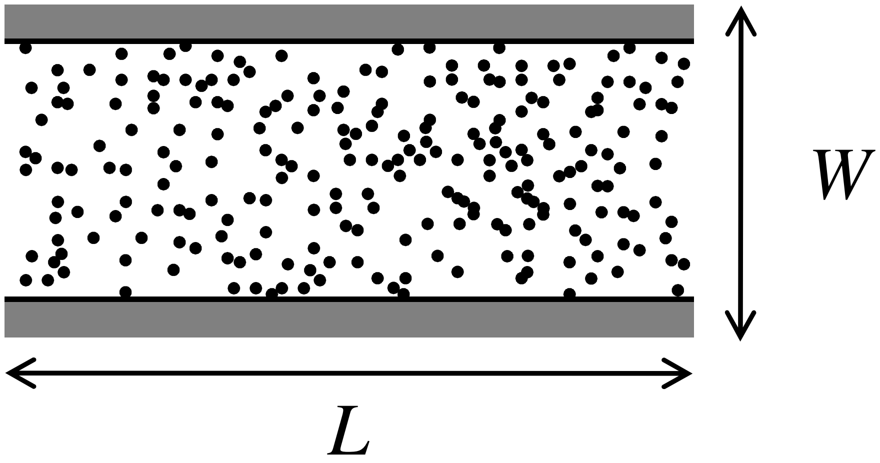
Alexander von Humboldt
Stiftung/Foundation

Anderson localization



disordered wire

Quantum wire: $L \gg W$

$$G = \frac{2e^2}{h} g$$

$$\langle \ln g \rangle = -\frac{2L}{\xi}$$

$$\mathrm{var}\ \ln g = \frac{4L}{\xi}$$

(localized regime: $L \gg \xi$)

- nonlinear sigma model

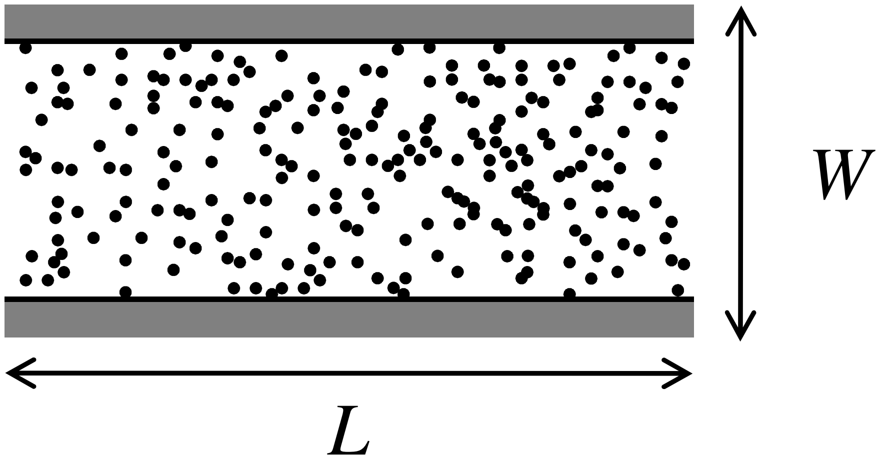
  Efetov and Larkin (1983)

- scaling approach

  Dorokhov (1982)

  Mello, Pereyra, Kumar (1988)

“Precursors” of localization



disordered wire

Quantum wire: $L \gg W$

metallic regime $L \ll \xi$

$$\langle g \rangle = \frac{\xi}{\beta L} + \frac{1}{3}\left(1 - \frac{2}{\beta}\right)$$

weak localization

$$\mathrm{var}\, g = \frac{2}{15\beta}$$

universal conductance fluctuations

β = 1,2,4: orthogonal, unitary, symplectic symmetry

Diagrammatic perturbation theory

Anderson, Abrahams, Ramakrishnan (1979)
Gorkov, Larkin, Khmelnitskii (1979)
Altshuler (1985)
Lee and Stone (1985)

Weak localization

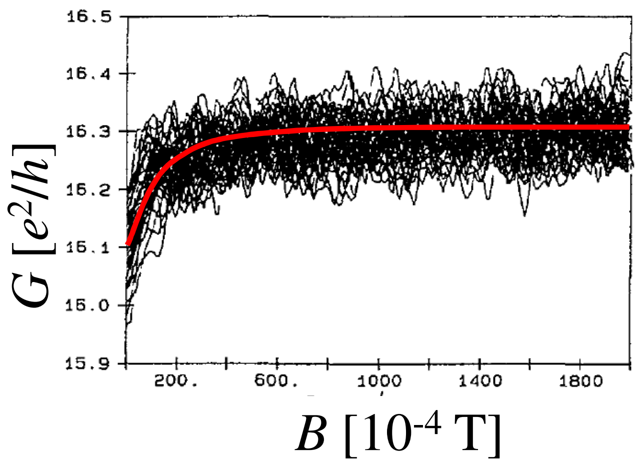
Nonzero (negative) ensemble average

$\langle \delta G \rangle$ at zero magnetic field

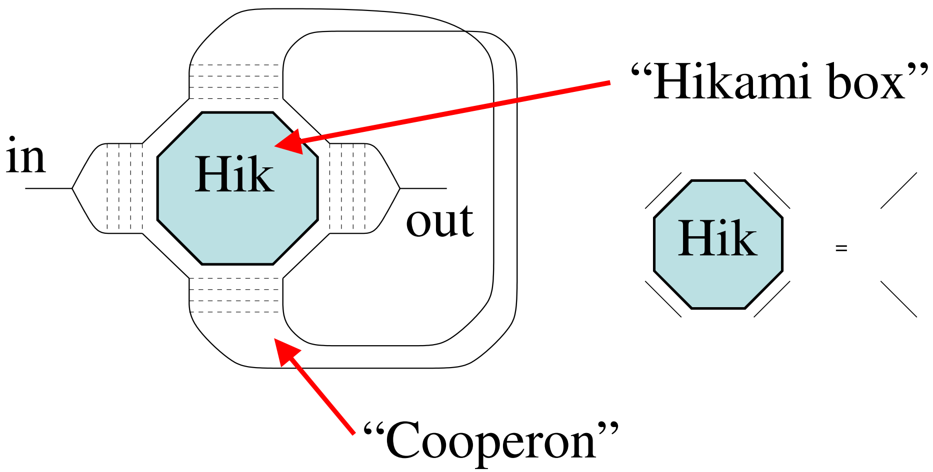
$$G = \left| \sum_{\mu} A_{\mu} \right|^2$$

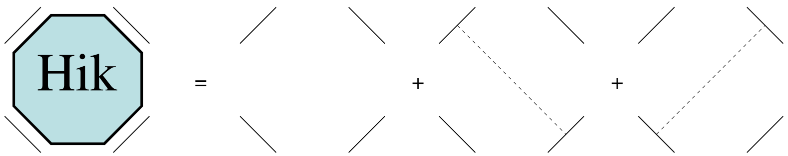
$$= \sum_{\mu} |A_{\mu}|^2 + \boxed{\sum_{\mu \neq \nu} A_{\mu} A_{\nu}^{*}}$$

$\leftarrow \delta G$



Mailly and Sanquer (1991)



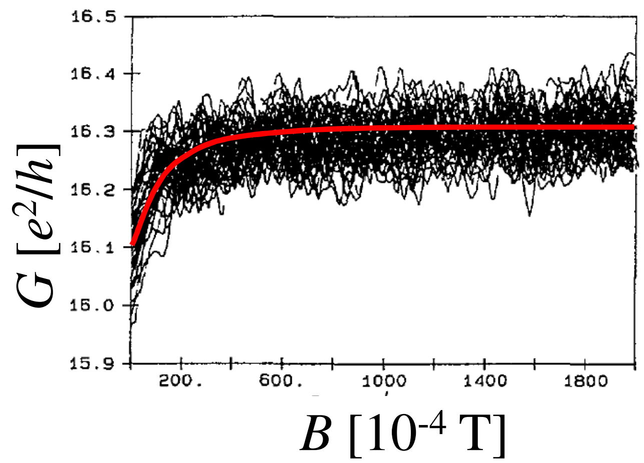


Gorkov, Larkin, Khmelnitskii (1979)

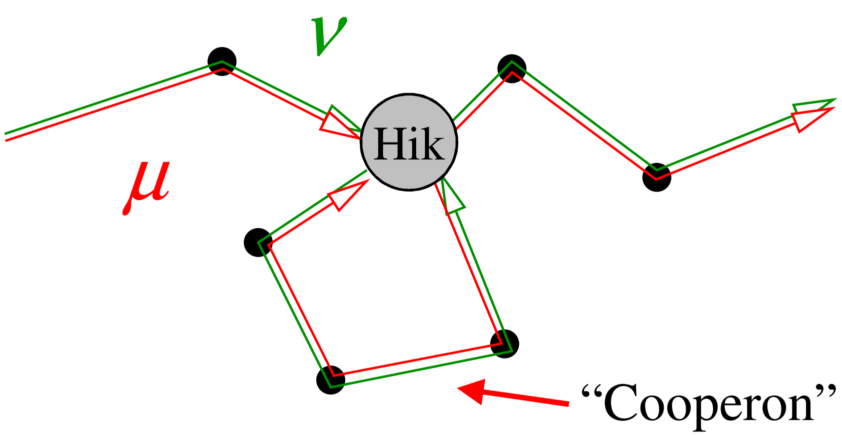
Weak localization

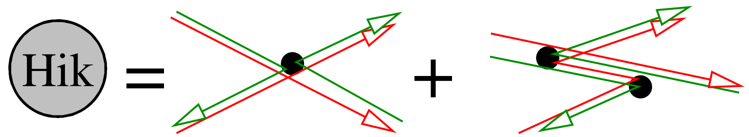
Nonzero (negative) ensemble average
$\langle \delta G \rangle$ at zero magnetic field

$$G = \left| \sum_{\mu} A_{\mu} \right|^2$$
$$= \sum_{\mu} |A_{\mu}|^2 + \underbrace{\sum_{\mu \neq \nu} A_{\mu} A_{\nu}^*}_{\delta G}$$



Mailly and Sanquer (1991)





+ permutations

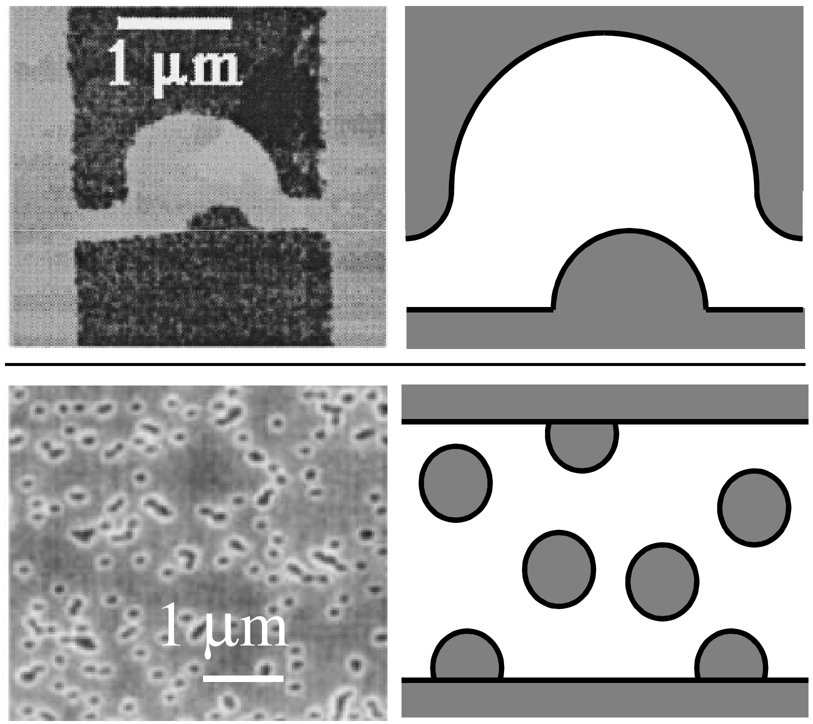
'Hikami box'

Ballistic conductors

- no impurities
- smooth boundaries
- chaotic classical dynamics

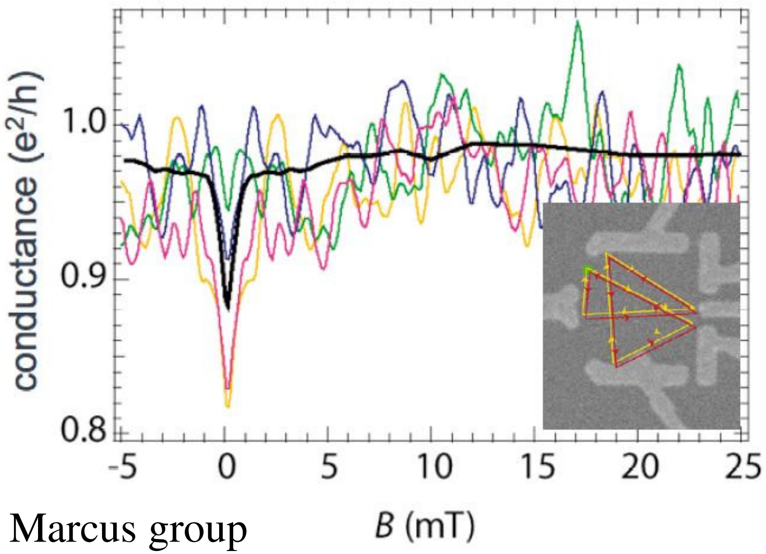
<u>Examples</u>:
Chaotic cavity
(quantum dot)

Lorentz gas
(antidot lattice)

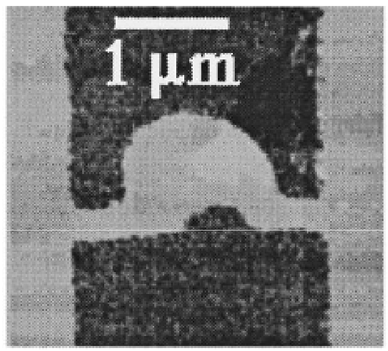


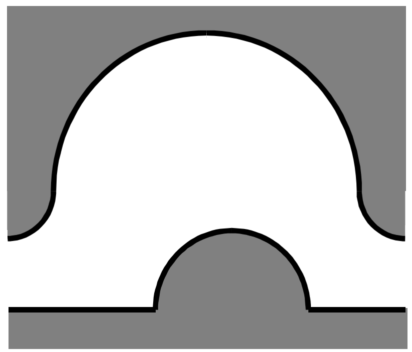
dot: Keller *et al*.; antidot lattice: Yevtushenko *et al*.

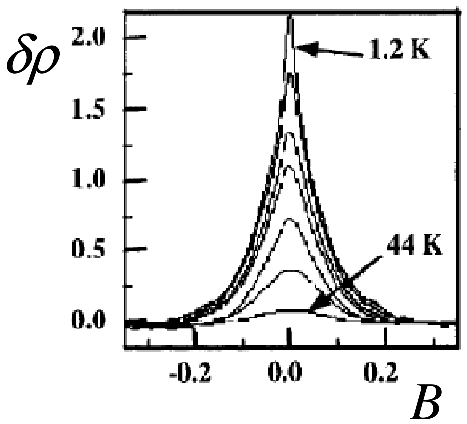
Ballistic conductors



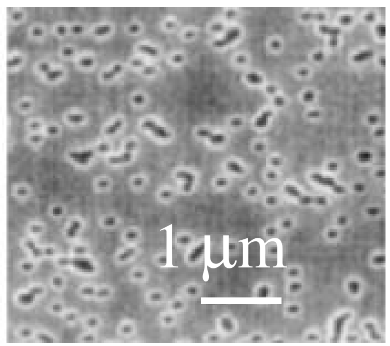
Marcus group

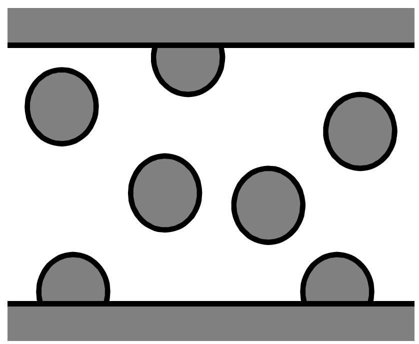






Weiss *et al.*

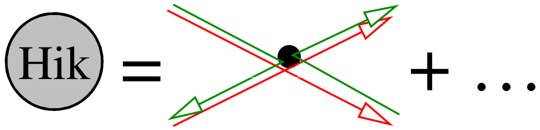


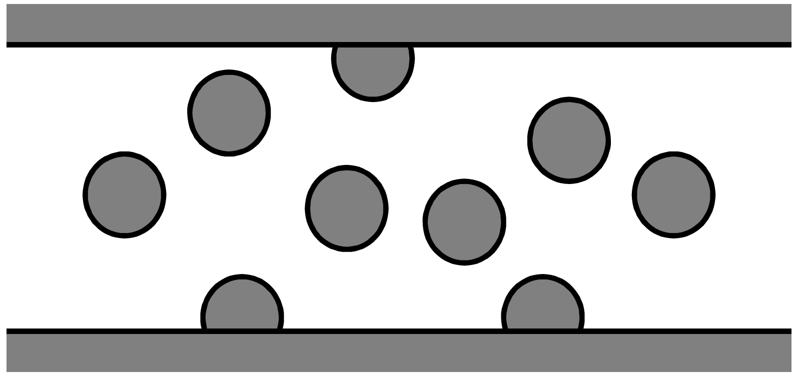


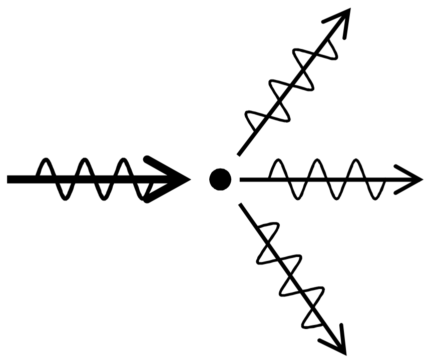
dot: Keller *et al.*; antidot lattice: Yevtushenko *et al.*

Ballistic conductors

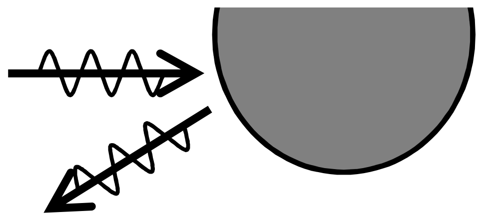
- Theory based on diffractive scattering off point-like impurities not possible;
- No impurity average.





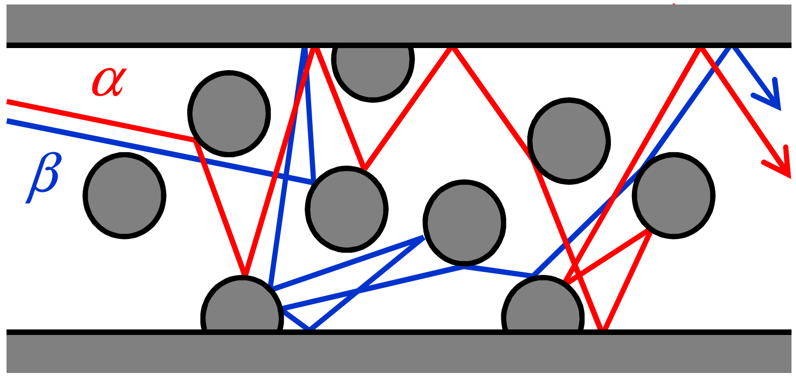


“disordered”



“ballistic”

Ballistic conductors



- Theory based on diffractive scattering off point-like impurities not possible;
- No impurity average.

Instead: Semiclassics

$$g = \sum_{\alpha,\beta} A_\alpha A_\beta e^{i(\mathcal{S}_\alpha - \mathcal{S}_\beta)/\hbar},$$

- α and β have equal angles upon entrance/exit
- $S_{\alpha,\beta}$: classical action
- $A_{\alpha,\beta}$: stability amplitudes

Needed: Careful summation over classical trajectories α, β.

Jalabert, Baranger, Stone (1990)
Argaman (1995)
Aleiner, Larkin (1996)
Richter, Sieber (2002)
Heusler, Müller, Braun, Haake (2006)

Weak localization
ballistic conductors

$$g \sim \sum_{\alpha,\beta} A_\alpha A_\beta e^{i(\mathcal{S}_\alpha - \mathcal{S}_\beta)/\hbar},$$

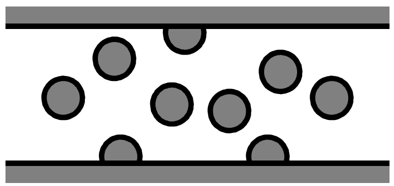
Weak localization: Trajectory pairs with small-angle self encounter

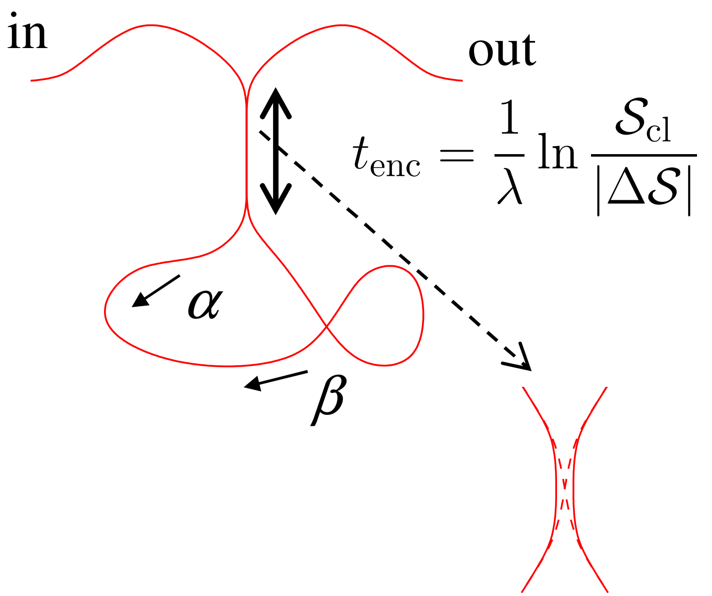
Aleiner, Larkin (1996)
Sieber, Richter (2001)

Encounter duration $t_{\mathrm{enc}} = \tau_{\mathrm{E}} = \frac{1}{\lambda} \ln \frac{\mathcal{S}_{\mathrm{cl}}}{\hbar}$

If $\tau_{\mathrm{E}} << $ dwell time: Semiclassics recovers weak localization correction of disordered metal

$$\langle \delta g \rangle = \frac{1}{3}\left(1 - \frac{2}{\beta}\right)$$





"ballistic Hikami box"

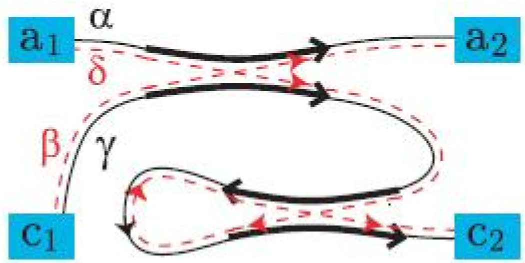
Aleiner, Larkin (1996)
Richter, Sieber (2002)
Heusler, Müller, Braun, Haake (2006)
Brouwer (2007)

Beyond weak localization

ballistic conductors

$$g \sim \sum_{\alpha,\beta} A_\alpha A_\beta e^{i(\mathcal{S}_\alpha - \mathcal{S}_\beta)/\hbar},$$

One or more small-angle self encounters



Braun *et al.* (2006)

- shot noise
- conductance fluctuations
- quantum pump
- full counting statistics
- time delay
- …

Agam, Aleiner, Larkin (2000)
Whitney and Jacquod (2006)
Brouwer and Rahav (2006)
Berkolaiko *et al.* (2007)
Kuipers and Sieber (2007)
Brouwer (2007)

If τ_E << dwell time: Recover quantum corrections of disordered metals

But all of these are perturbative effects!

Non-perturbative effects

Level correlations:
Form factor $K(t)$ for $|t| > \tau_H$

Heusler, Müller, Altland, Braun, Haake (2007)

"inspired by field theoretical formulation of RMT correlation functions"

Heusler et al. (2007)

Today: Anderson localization

… inspired by theory of Anderson localization in disordered metals

• one-dimensional nonlinear sigma model

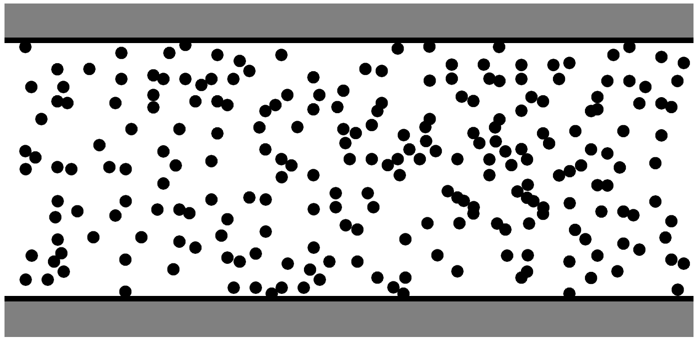
Efetov and Larkin (1983)

• scaling approach

Dorokhov (1982)
Mello, Pereyra, Kumar (1988)

Anderson localization

disordered metals



quantum wire of length L, width W

Impurity scattering: mean free path l.
Take limit $W \to \infty$ while keeping ratio $k_F W\, l/L$ fixed.

- nonlinear sigma model

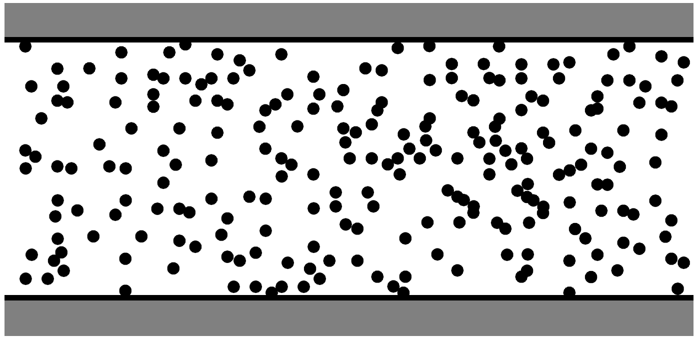
Efetov and Larkin (1983)

- scaling approach

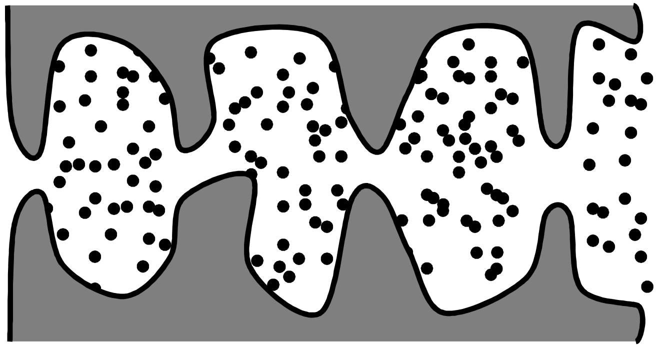
Dorokhov (1982)
Mello, Pereyra, Kumar (1988)

Anderson localization

disordered metals



quantum wire



array of *n* "quantum dots"

Dots are connected via ballistic contacts with conductance $g_c \gg 1$.
Take limit $g_c \rightarrow \infty$ while keeping ratio g_c/n fixed.

Disordered quantum dots: random matrix theory

Mirlin, Müller-Groeling, Zirnbauer (1994)
Brouwer, Frahm (1996)

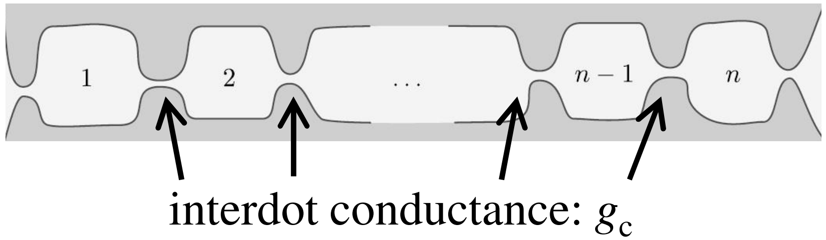
Anderson localization

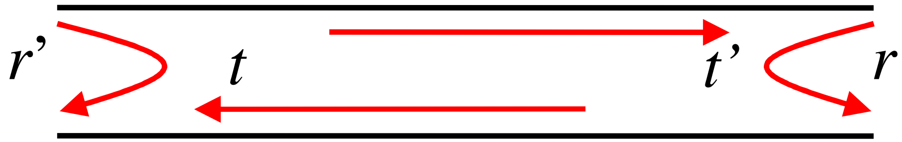
disordered metals

$$S(n) = \begin{pmatrix} r'(n) & t(n) \\ t'(n) & r(n) \end{pmatrix}$$

$$\mathcal{T}(n) = t(n)t^{\dagger}(n)$$

$$T_m(n) = \mathrm{tr}\,\mathcal{T}(n)^m$$





$g(n) = T_1(n)$: conductance of array of n dots

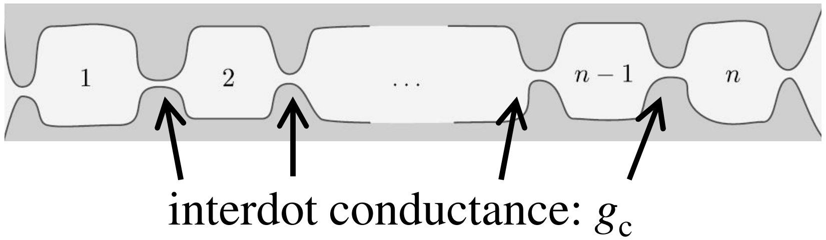
Anderson localization

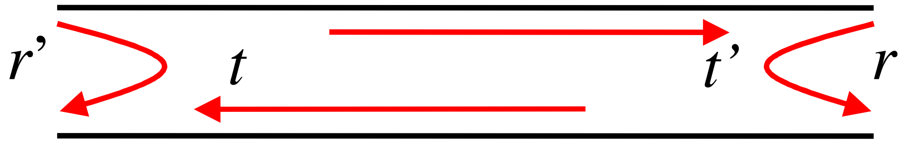
disordered metals

$$S(n) = \begin{pmatrix} r'(n) & t(n) \\ t'(n) & r(n) \end{pmatrix}$$

$$\mathcal{T}(n) = t(n)t^\dagger(n)$$

$$T_m(n) = \mathrm{tr}\,\mathcal{T}(n)^m$$





random matrix theory: recursion relation for moments of the T_i:

$$\begin{aligned} \delta\langle T_1\rangle &= \langle T_1(n)\rangle - \langle T_1(n-1)\rangle \\ &= -\frac{1}{g_c}\langle T_1(n-1)^2\rangle + \mathcal{O}(g_c^{-2}) \end{aligned}$$

(no time-reversal symmetry, β=2)

Replace difference equation by differential equation:

$$\frac{\partial}{\partial L}\langle T_1\rangle = -\frac{2}{\xi}\langle T_1^2\rangle$$

$L/\xi = n/2g_c$ $\quad$ ξ: "localization length"

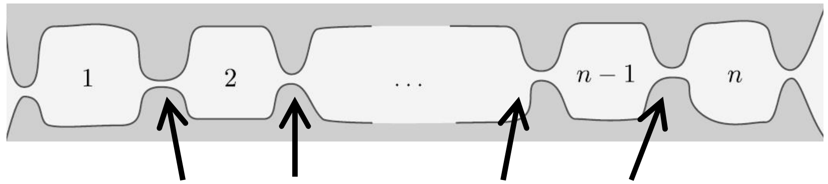
Anderson localization

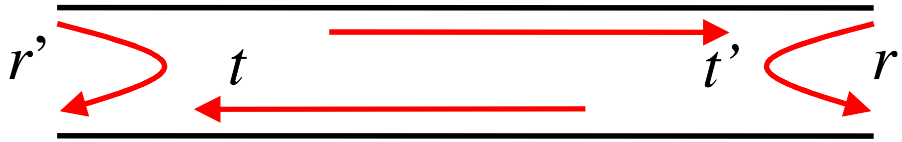
disordered metals

$$S(n) = \begin{pmatrix} r'(n) & t(n) \\ t'(n) & r(n) \end{pmatrix}$$

$$\mathcal{T}(n) = t(n)t^\dagger(n)$$

$$T_m(n) = \mathrm{tr}\,\mathcal{T}(n)^m$$





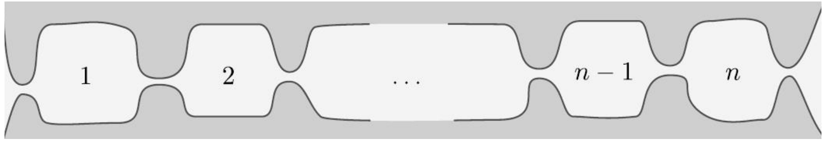
General recursion relations:

$$\begin{aligned}
\delta\left\langle \prod_{m=1}^{n} T_{i_m} \right\rangle &= -\frac{1}{g_c}\left(\sum_{k=1}^{n} i_k\right)\left\langle T_1 \prod_{m=1}^{n} T_{i_m}\right\rangle \\
&+ \frac{1}{g_c}\sum_{k=1}^{n}\sum_{j=1}^{i_k-1} i_k \left\langle (T_j(T_{i_k-j} - T_{i_k-j+1})) \prod_{\substack{m=1\\ m\neq k}}^{n} T_{i_m}\right\rangle \\
&+ \frac{2}{g_c}\sum_{k=1}^{n}\sum_{l=1}^{k-1} i_k i_l \left\langle (T_{i_k+i_l} - T_{i_k+i_l+1}) \prod_{\substack{m=1\\ m\neq k,l}}^{n} T_{i_m}\right\rangle + \mathcal{O}(g_c^{-2})
\end{aligned}$$

Anderson localization

disordered metals

$$\delta\left\langle\prod_{m=1}^{n}T_{i_m}\right\rangle = -\frac{1}{g_c}\left(\sum_{k=1}^{n}i_k\right)\left\langle T_1\prod_{m=1}^{n}T_{i_m}\right\rangle + \frac{1}{g_c}\sum_{k=1}^{n}\sum_{j=1}^{i_k-1}i_k\left\langle (T_j(T_{i_k-j}-T_{i_k-j+1}))\prod_{\substack{m=1\\m\neq k}}^{n}T_{i_m}\right\rangle + \frac{2}{g_c}\sum_{k=1}^{n}\sum_{l=1}^{k-1}i_k i_l\left\langle (T_{i_k+i_l}-T_{i_k+i_l+1})\prod_{\substack{m=1\\m\neq k,l}}^{n}T_{i_m}\right\rangle + \mathcal{O}(g_c^{-2})$$



Transform recursion relations into differential equation for generating function:

$$F_2(\theta_1,\theta_3) = \left\langle \det\left(\frac{2+(\cos(\theta_3)-1)\mathcal{T}}{2+(\cosh(\theta_1)-1)\mathcal{T}}\right)\right\rangle$$

$$\frac{\partial}{\partial L}F_2 = \frac{2}{\xi}\sum_{j=1,3}\frac{1}{J(\theta_1,\theta_3)}\frac{\partial}{\partial\theta_j}J(\theta_1,\theta_3)\frac{\partial}{\partial\theta_j}F_2,$$

$$J(\theta_1,\theta_3) = \frac{\sin(\theta_3)\sinh(\theta_1)}{(\cosh(\theta_1)-\cos(\theta_3))^2}.$$

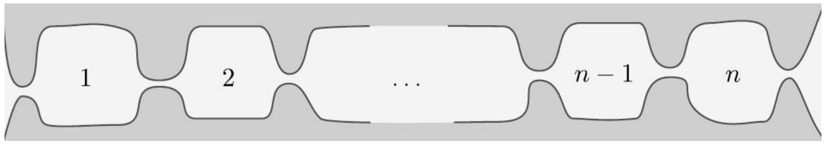
Description equivalent to existing theory of localization in quantum wires

Efetov and Larkin (1983)
Dorokhov (1982)
Mello, Pereyra, Kumar (1988)

Anderson localization

disordered metals

$$\delta \left\langle \prod_{m=1}^{n} T_{i_m} \right\rangle = -\frac{1}{g_c} \left(\sum_{k=1}^{n} i_k \right) \left\langle T_1 \prod_{m=1}^{n} T_{i_m} \right\rangle + \frac{1}{g_c} \sum_{k=1}^{n} \sum_{j=1}^{i_k-1} i_k \left\langle (T_j (T_{i_k-j} - T_{i_k-j+1})) \prod_{\substack{m=1 \\ m\neq k}}^{n} T_{i_m} \right\rangle + \frac{2}{g_c} \sum_{k=1}^{n} \sum_{l=1}^{k-1} i_k i_l \left\langle (T_{i_k+i_l} - T_{i_k+i_l+1}) \prod_{\substack{m=1 \\ m\neq k,l}}^{n} T_{i_m} \right\rangle + \mathcal{O}(g_c^{-2})$$

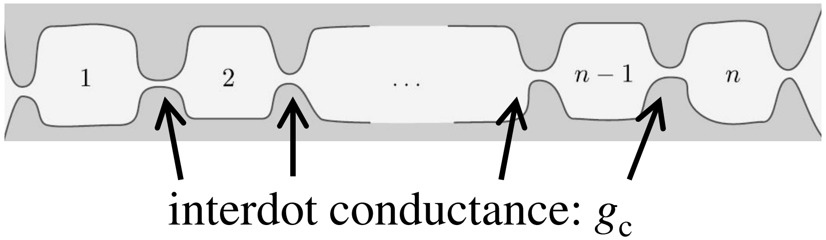


Can one derive the same set of recursion relations from semiclassics?

Anderson localization

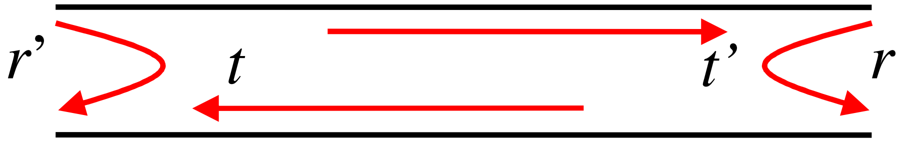
ballistic conductors

$$S(n) = \begin{pmatrix} r'(n) & t(n) \\ t'(n) & r(n) \end{pmatrix}$$



$$\mathcal{T}(n) = t(n)t^\dagger(n)$$

$$T_m(n) = \mathrm{tr}\, \mathcal{T}(n)^m$$



Can we show that
$$\delta\langle T_1\rangle = \langle T_1(n)\rangle - \langle T_1(n-1)\rangle = -\frac{1}{g_c}\langle T_1(n-1)^2\rangle + \mathcal{O}(g_c^{-2})$$

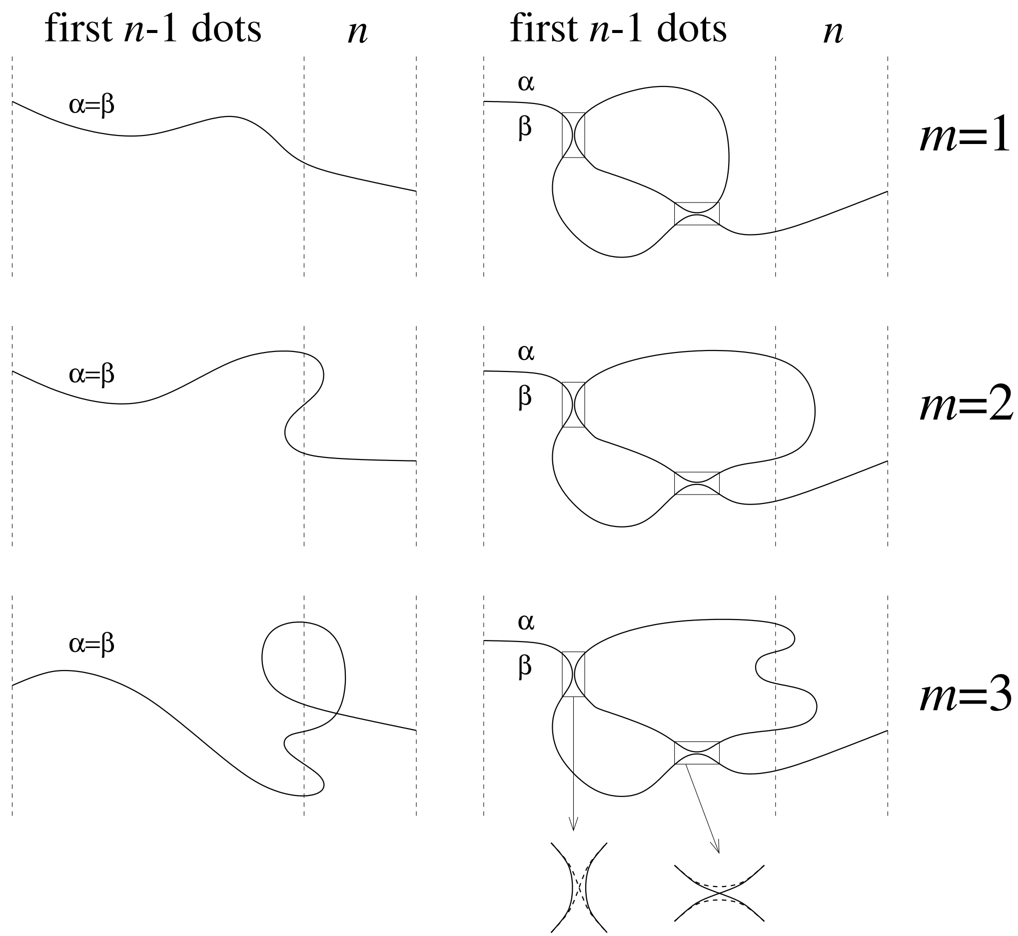
from the semiclassical expression for T_1?

$$T_1 = \sum_{\alpha,\beta} A_\alpha A_\beta e^{i(\mathcal{S}_\alpha - \mathcal{S}_\beta)/\hbar}$$

Anderson localization
ballistic conductors

$$T_1 = \sum_{\alpha,\beta} A_\alpha A_\beta e^{i(\mathcal{S}_\alpha - \mathcal{S}_\beta)/\hbar}$$

- α and β each have m segments in n^{th} dot, $\alpha_1,\ldots,\alpha_m$; $\beta_1,\ldots,\beta_m$.



Anderson localization
ballistic conductors

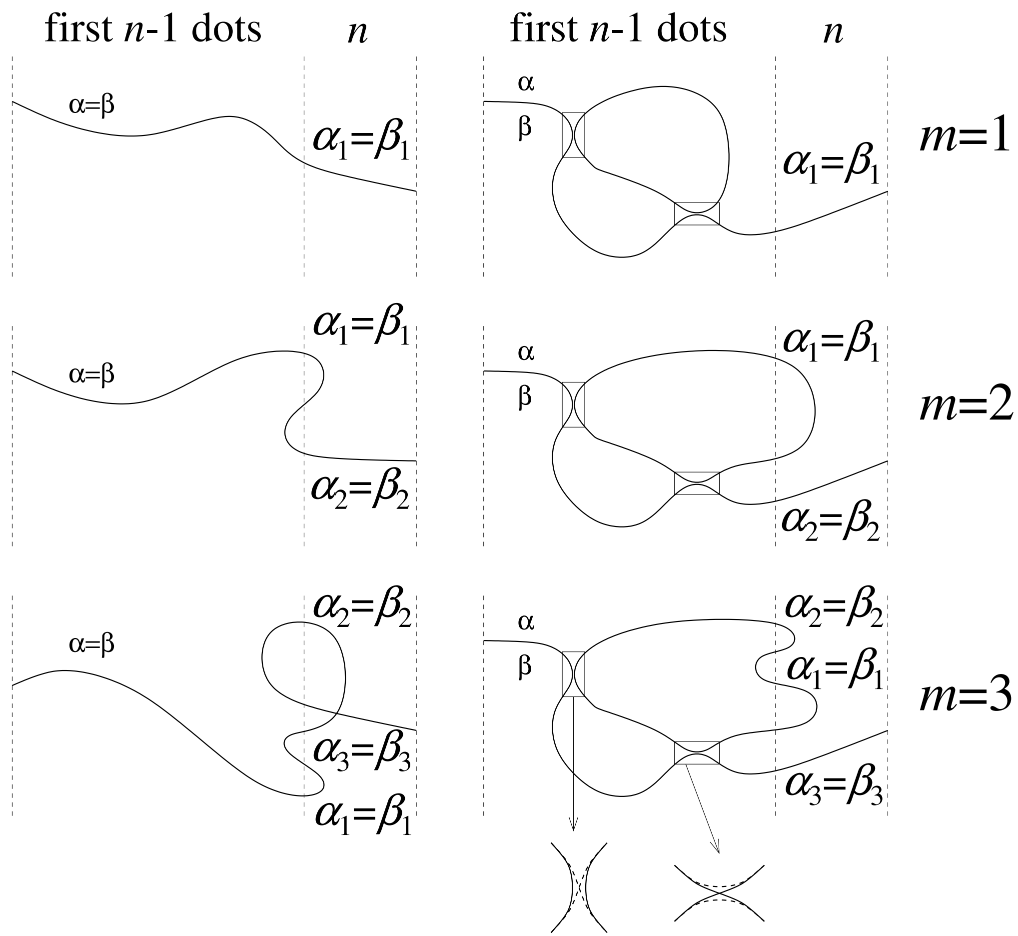
$$T_1 = \sum_{\alpha,\beta} A_\alpha A_\beta e^{i(\mathcal{S}_\alpha - \mathcal{S}_\beta)/\hbar}$$

- α and β each have m segments in n^{th} dot, $\alpha_1,\ldots,\alpha_m$; $\beta_1,\ldots,\beta_m$.

To leading order in g_c:

- diagonal approximation in n^{th} dot
- pair α_i with β_i, $i=1,\ldots,m$

No restriction on number of small-angle self encounters in first n-1 dots



Anderson localization

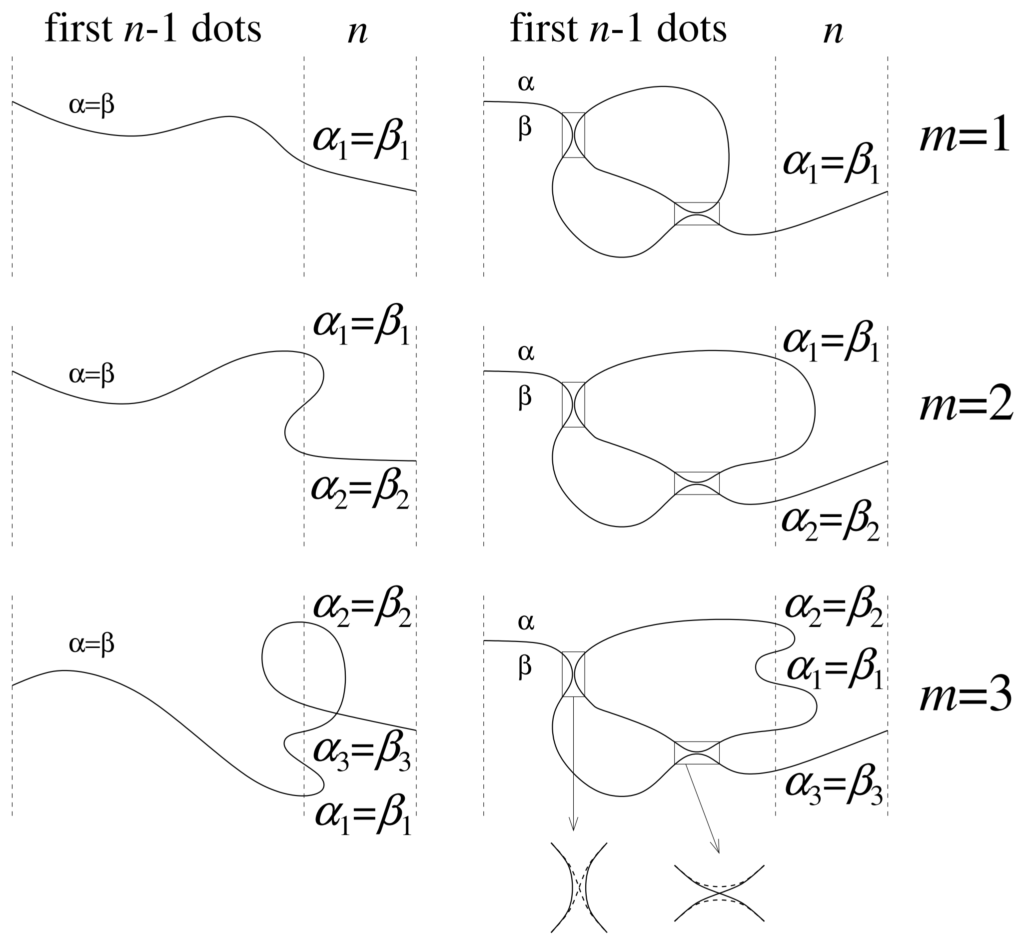
ballistic conductors

$$\frac{1}{2}\langle T_1(n-1)\rangle$$

$$\frac{1}{4g_{\rm c}}\langle T_1(n-1)R_1(n-1)\rangle$$

reflection:
$R_1 = g_{\rm c} - T_1$

$$\frac{1}{8g_{\rm c}^2}\langle T_1(n-1)R_1(n-1)^2\rangle$$



Anderson localization

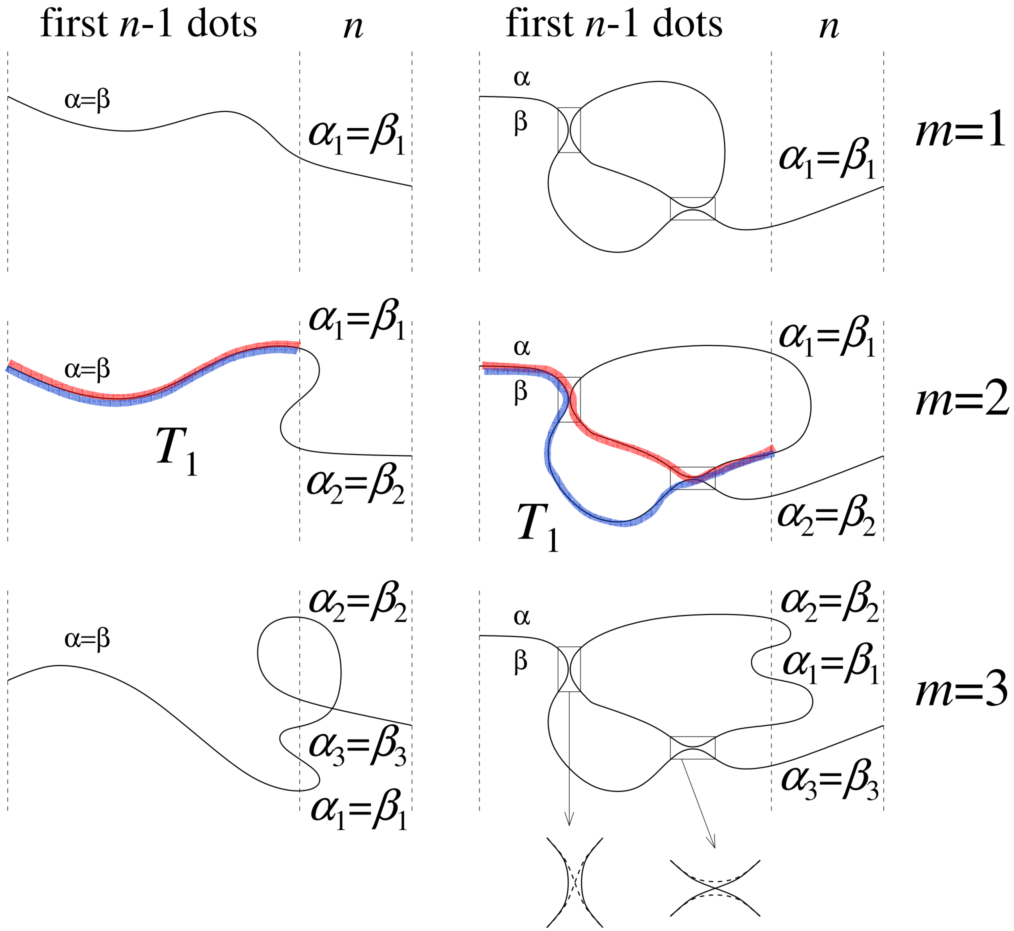
ballistic conductors

$\frac{1}{2}\langle T_1(n-1)\rangle$

$\frac{1}{4g_c}\langle T_1(n-1)R_1(n-1)\rangle$

reflection:
$R_1 = g_c - T_1$

$\frac{1}{8g_c^2}\langle T_1(n-1)R_1(n-1)^2\rangle$



Anderson localization

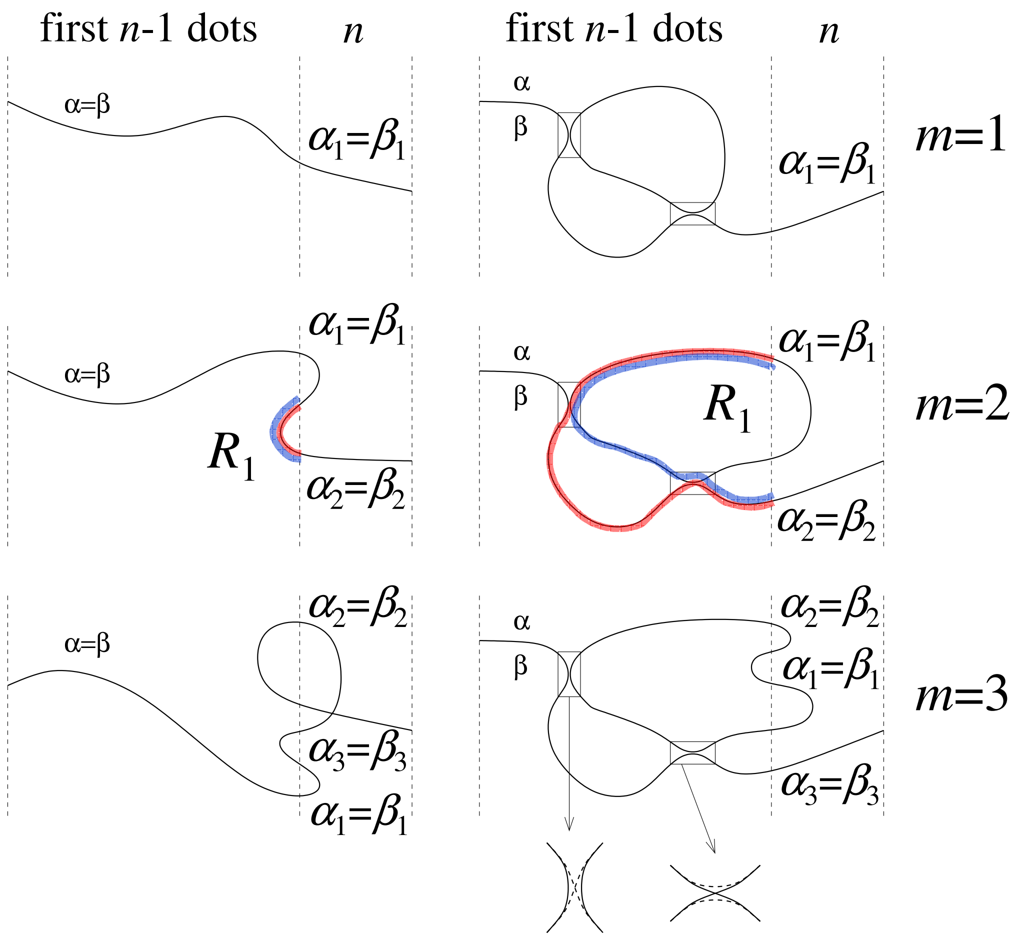
ballistic conductors

$$\frac{1}{2}\langle T_1(n-1)\rangle$$

$$\frac{1}{4g_c}\langle T_1(n-1)R_1(n-1)\rangle$$

reflection:
$R_1 = g_c - T_1$

$$\frac{1}{8g_c^2}\langle T_1(n-1)R_1(n-1)^2\rangle$$



Anderson localization

ballistic conductors

$$\frac{1}{2}\langle T_1(n-1)\rangle$$

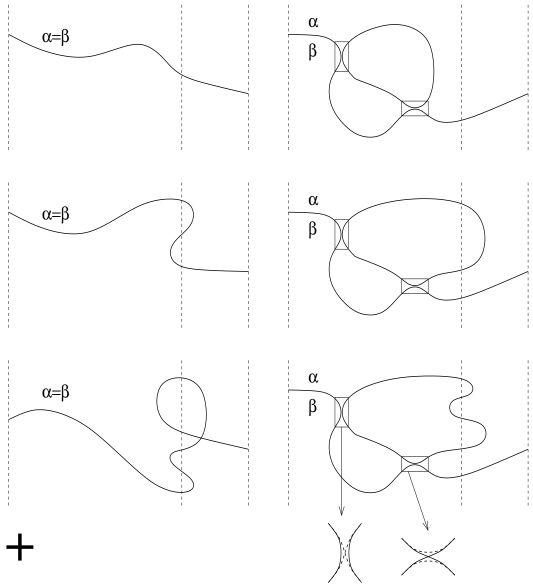
$$\frac{1}{4g_c}\langle T_1(n-1)R_1(n-1)\rangle$$

$$\frac{1}{8g_c^2}\langle T_1(n-1)R_1(n-1)^2\rangle$$

$$\cdots$$

+

reflection:
$R_1 = g_c - T_1$



m=1

m=2

m=3

$$\begin{aligned}\langle T_1(n)\rangle &= \sum_{m=1}^{\infty} \frac{1}{2^m g_c^{m-1}} \left\langle T_1(n-1)R_1(n-1)^{m-1}\right\rangle \\ &= \left\langle \frac{g_c T_1(n-1)}{2g_c - R_1(n-1)} \right\rangle \\ &= \langle T_1(n-1)\rangle - \frac{1}{g_c}\langle T_1(n-1)^2\rangle + \mathcal{O}(g_c^{-2})\end{aligned}$$

Anderson localization

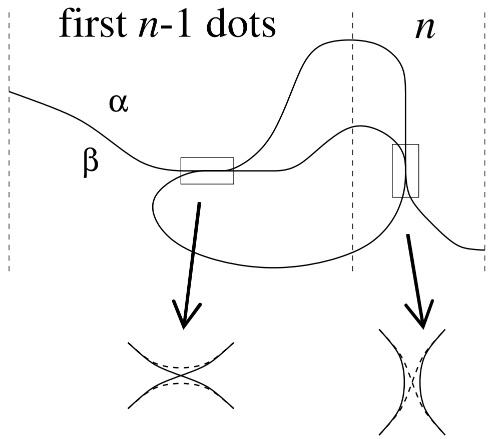
ballistic conductors

Beyond diagonal approximation in n^{th} dot:

- contribution of order g_c^{-2}

$$-\frac{1}{8g_c^2}\langle \mathrm{tr}\, S_{12}S_{22}^\dagger S_{22}S_{12}^\dagger\rangle$$

$$= \frac{1}{8g_c^2}\langle T_2(n-1) - T_1(n-1)\rangle$$



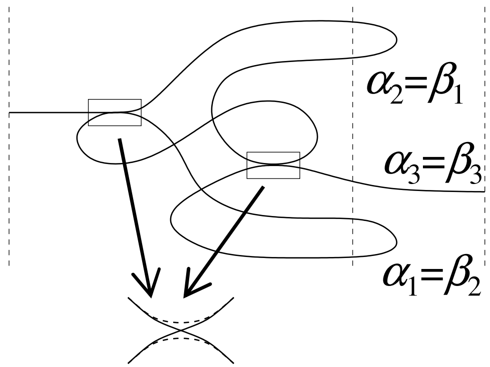
m=2

Pairing α_i with β_j, $i \neq j$:

- contribution of order g_c^{-2}

$$\frac{1}{8g_c^2}\langle \mathrm{tr}\, S_{12}(S_{22}^\dagger S_{22})^2 S_{12}^\dagger\rangle =$$

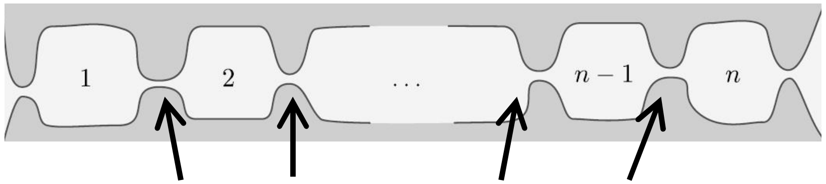
$$= \frac{1}{8g_c^2}\langle T_1(n-1) - 2T_2(n-1) + T_3(n-1)\rangle$$



m=3

Anderson localization

ballistic conductors



<u>Summarizing…</u>

$$\delta\langle T_1\rangle = \langle T_1(n)\rangle - \langle T_1(n-1)\rangle = -\frac{1}{g_c}\langle T_1(n-1)^2\rangle + \mathcal{O}(g_c^{-2})$$

Extension to higher moments and β=1 (time-reversal symmetry):

- Need to consider up to one encounter in n^{th} dot;
- Need to go (slightly) beyond pairing α_i with β_i, i=1,…,m.

… …

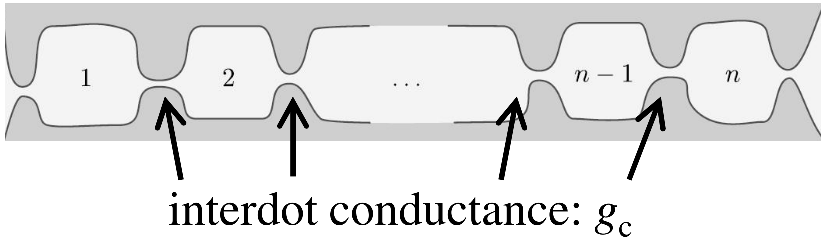
full set of recursion relations

$$\delta\left\langle \prod_{m=1}^{n} T_{i_m}\right\rangle = -\frac{1}{g_c}\left(\sum_{k=1}^{n} i_k\right)\left\langle T_1 \prod_{m=1}^{n} T_{i_m}\right\rangle + \frac{1}{g_c}\sum_{k=1}^{n}\sum_{j=1}^{i_k-1} i_k \left\langle (T_j(T_{i_k-j} - T_{i_k-j+1})) \prod_{\substack{m=1\\ m\neq k}}^{n} T_{i_m}\right\rangle + \frac{2}{g_c}\sum_{k=1}^{n}\sum_{l=1}^{k-1} i_k i_l \left\langle (T_{i_k+i_l} - T_{i_k+i_l+1}) \prod_{\substack{m=1\\ m\neq k,l}}^{n} T_{i_m}\right\rangle + \mathcal{O}(g_c^{-2})$$

Anderson localization

ballistic conductors

Summarizing…



Theory of Anderson localization in array of ballistic chaotic cavities, formulated in terms of classical trajectories only.