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Universal Corrections to Gyromagnetic Ratios of Bound Particles with Arbitrary Spins

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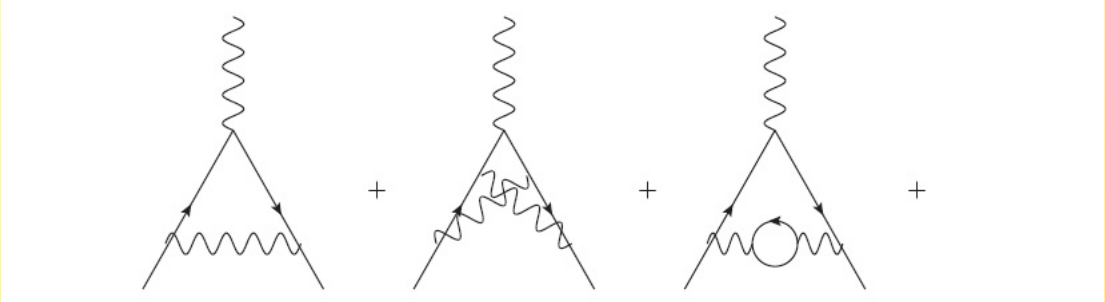
Outline

Reminder on Free g-Factors

- g-Factors of EM and non EM origin
- Free electron g-factor



- QED Series:

$$\frac{g-2}{2} = \frac{\alpha}{2\pi} - 0.328\ldots\left(\frac{\alpha}{\pi}\right)^2 + \ldots$$

- Accuracy of modern theory $\Delta g/g \sim 10^{-13}$, Kinoshita et al (2007)

Experiment and Determination of α

- Experimentally $\Delta g/g \sim 10^{-13}$ (*Hanneke, Fogwell, Gabrielse, 2008*):

$$\frac{g_e}{2} = 1.001\ 159\ 652\ 180\ 73\ (28), \qquad \delta = 2.8 \times 10^{-13}$$

- Measurement of the free electron g-factor – best way to measure α

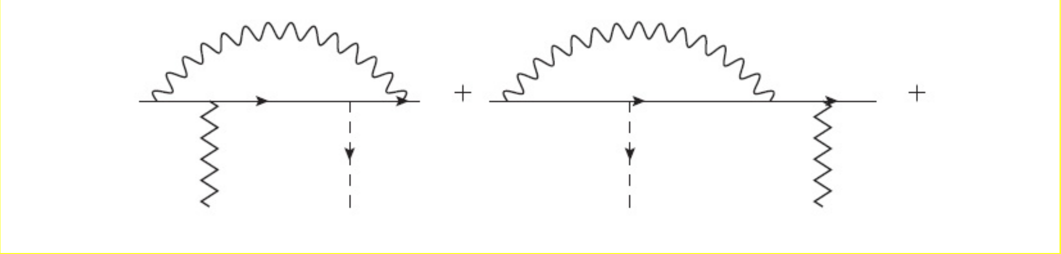
$$\alpha^{-1} = 137.035\ 999\ 084\ (51), \qquad \delta = 3.7 \times 10^{-10}$$

Additional Corrections to g-Factors due to Binding

- Relativistic corrections, matrix element of $e\boldsymbol{\gamma}\cdot\mathbf{A}$ (*Breit, 1928*):

$$g_b = 2\left[1 - \frac{(Z\alpha)^2}{3} - \frac{(Z\alpha)^4}{12} + \dots\right]$$

- Leading radiative-recoil corrections of order $\alpha(Z\alpha)^2$ (*Faustov, 1970; Grotch, 1970; Close, Osborn, 1971*)
- Other corrections series over $\alpha^n(Z\alpha)^k$, recoil corrections m_e/M_N, etc. (*1995-present, review by Mohr et al, 2008*)



- Accuracy of modern theory $\Delta g_b/g_b \sim 10^{-11}$

What is Measured

- Interaction Hamiltonian for hydrogenlike ion in external magnetic field

$$H_{int} = c\boldsymbol{\mu}_e \cdot \boldsymbol{\mu}_N - \boldsymbol{\mu}_e^b \cdot \mathbf{B} - \boldsymbol{\mu}_N^b \cdot \mathbf{B}$$

$$= 2\pi\hbar\Delta\nu_{HFS}\mathbf{s}\cdot\mathbf{I} - g_e^b\frac{e\hbar}{2m_ec}\mathbf{s}_e\cdot\mathbf{B} - g_N^b\frac{e\hbar}{2M_Nc}\mathbf{I}\cdot\mathbf{B}$$

- Ratio f_s/f_c of spin-flip to cyclotron frequency is

$$\frac{f_s}{f_c} = g_b\frac{e}{2q}\frac{M_i}{m_e} = \frac{g_b}{2(Z-1)}\frac{M_i}{m_e}$$

- Experiments are done in hydrogenlike $^{12}C^{5+}$ $(Z=6)$ and hydrogenlike $^{16}O^{7+}$ $(Z=8)$. Nuclear spin $I=0$

What is Measured

- Experimentally (Häffner, Werth, Verdu, 2003)

$$\frac{f_s(^{12}C^{5+})}{f_c(^{12}C^{5+})} = 4\ 376.210\ 4989\ (23), \qquad \delta = 5.2 \times 10^{-10}$$

$$\frac{f_s(^{16}O^{7+})}{f_c(^{16}O^{7+})} = 4\ 164.376\ 1837\ (32), \qquad \delta = 7.6 \times 10^{-10}$$

- Perspective (*Quint et al, 2008*): error of f_s/f_c about $10^{-12} - 10^{-13}$– comparable to accuracy of free electron g-factor

How to use these precise results?

- f_s/f_c is the best way to measure electron mass in atomic units: $\Delta m_e/m_e = 5 \times 10^{-10}$, $4 - 6$ times more precise than direct comparison of cyclotron frequencies for free electron and ion

Phenomenological Summary

$$\frac{f_s}{f_c} = g_b \frac{e}{2q}\frac{M_i}{m_e} = \frac{g_b}{2(Z-1)}\frac{M_i}{m_e}$$

- We need precise theory of g_b or precise measurement of M_i/m_e to utilize the frequency measurements
- Accuracy of modern theory $\Delta g_b/g_b \sim 10^{-11}$ for $^{12}C^{5+}$ and $^{16}O^{7+}$
- Classical leading relativistic and recoil corrections of order $(Z\alpha)^2$ were calculated for spin one half constituents

Problem of Spin Dependence

- What about corrections of order $(Z\alpha)^2$ for other spins? Spin of $^{12}C^{5+}$ and $^{16}O^{7+}$ nuclei is zero
- No agreement in the literature, results are contradictory!
- Discrepancy between both results for $^{12}C^{5+}$ and $^{16}O^{7+}$ at the level of $(0.2-0.3)\times 10^{-10}$

What is the spin dependence of leading corrections?

NRQED Lagrangian

- Construct most general nonrelativistic Lagrangian compatible with gauge, Galilean invariance and discrete symmetries
- Renormalizability is not important, use all vertices compatible with symmetries
- Predictive power is still there, expansion goes over $v/c \sim p/(mc)$
- Building blocks: $\mathbf{D} = \boldsymbol{\nabla} - ie\mathbf{A} = i(\mathbf{p} - e\mathbf{A})$, $\mathbf{E}$, $\mathbf{B}$, and $\mathbf{S}$.
- For higher spin particles we include polynomials in the components of the spin – higher irreducible intrinsic multipole moments
- Determine coefficients comparing scattering amplitudes of relativistic and nonrelativistic theories
- Use nonrelativistic Hamiltonian for bound state calculations

NRQED Lagrangian

$$\mathcal{L}=\phi^{+}\Bigg\{iD_0+\frac{\mathbf{D}^2}{2m}+\frac{\mathbf{D}^4}{8m^3}+c_F\frac{e\mathbf{s}\cdot\mathbf{B}}{2m}+c_D\frac{e(\mathbf{D}\cdot\mathbf{E}-\mathbf{E}\cdot\mathbf{D})}{8m^2}$$

$$+c_Q\frac{eQ_{ij}(D_iE_j-E_iD_j)}{8m^2}+c_S\frac{ie\mathbf{s}\cdot(\mathbf{D}\times\mathbf{E}-\mathbf{E}\times\mathbf{D})}{8m^2}$$

$$+c_{W1}\frac{e[\mathbf{D}^2(\mathbf{s}\cdot\mathbf{B})+(\mathbf{s}\cdot\mathbf{B})\mathbf{D}^2]}{8m^3}+c_{W2}\frac{-eD^i(\mathbf{s}\cdot\mathbf{B})D^i}{4m^3}$$

$$+c_{p'p}\frac{e[(\mathbf{s}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{D})+(\mathbf{D}\cdot\mathbf{B})(\mathbf{s}\cdot\mathbf{D})]}{8m^3}+\ldots\Bigg\}\phi$$

$D^0=\partial^0+ieA^0$, $\mathbf{D}=\boldsymbol{\nabla}-ie\mathbf{A}=i(\mathbf{p}-e\mathbf{A})$.
$Q_{ij}=s_is_j+s_js_i-(2/3)\mathbf{s}^2\delta_{ij}$

Relativistic Construction

- Next task is to find the coefficients

What about relativistic QED for charged particles with arbitrary spin?

- For spin one charged particles $W^{\pm}$ sector of renormalizable EW theory ($W^{\pm}$ and photon)
- No Lagrangian theory for higher spins
- One-photon relativistic vertices for arbitrary spin were constructed by Khriplovich et al (1996)
- For spin one the NRQED Hamiltonians obtained from the Lagrangian renormalizable theory and from the relativistic diagram technique are identical

Relativistic Construction

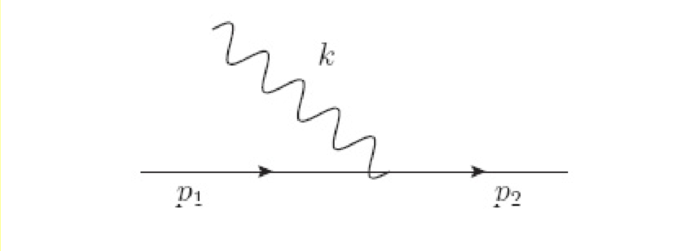
- Charged particles are described by completely symmetric spinors $\xi = \left\{\xi^{\alpha_1\alpha_2\ldots\alpha_p}_{\dot{\beta}_1\dot{\beta}_2\ldots\dot{\beta}_q}\right\}$
- For integer spin $p = q = s$, for half integer $p = s + 1/2$, $q = s - 1/2$.
- Under spatial reflection dotted and undotted indices trade places $\Longrightarrow$ two spinors ξ and η, $\xi^\alpha \to i\eta_{\dot{\alpha}}$, $\eta_{\dot{\alpha}} \to i\xi^\alpha$
- "Standard representation": $\phi = (\xi + \eta)/2$, $\chi = \xi - \eta)/2$, $\chi \to 0$ when $v \to 0$

$$\psi = \sqrt{\frac{mc^2}{E_p}}\begin{pmatrix} (1 + \frac{(\mathbf{\Sigma}\cdot\mathbf{v})^2}{8c^2})\phi \\ \frac{\mathbf{\Sigma}\cdot\mathbf{v}}{2c}\phi \end{pmatrix} \approx \begin{pmatrix} (1 + \frac{(\mathbf{\Sigma}\cdot\mathbf{v})^2}{8c^2} - \frac{\mathbf{p}^2}{4m^2c^2})\phi \\ \frac{\mathbf{\Sigma}\cdot\mathbf{v}}{2c}\phi \end{pmatrix}$$

One-Photon Terms

- Nonrelativistic normalization $E_p/(mc)\bar{\psi}\psi = \phi^+\phi = 1$. Corresponds to Foldy-Wouthuysen tranformation
- EM interaction is $(-ej_\mu A^\mu)$
- One-photon vertex

$$\Gamma_\mu = \frac{(p_1+p_2)_\mu}{2m} F_e(q^2,\tau) - F_m(q^2,\tau)\frac{\Sigma_{\mu\nu}q^\nu}{2m}$$



$q = p_2 - p_2$, $\Sigma_{\mu\nu}$ – the generalization of $\sigma_{u\nu}$, S_μ –covariant spin four-vector, $\tau = (q\cdot S)^2$, and $F_e(0,0) = 1$, $F_m(0,0) = g/2$

One-Photon Terms

- Intrinsic electric and magnetic multipole moments arise in expansion of form factors over $S \cdot q$
- Only $F_e(0,0) = 1$, $F_m(0,0) = g/2$ generate leading corrections to g-factors
- Nonrelativistic expansion of matrix elements $\bar{\psi} J_\mu \psi A^\mu$ generates nonrelativistic vertices with one external field
- Matrix element of J_0 generates

$$V_0 = eA_0 - e(g-1)\frac{\nabla \cdot \mathbf{E}}{8m^2}\frac{\mathbf{\Sigma}^2}{3} - e(g-1)\frac{\mathbf{s} \cdot (\mathbf{E} \times \mathbf{p})}{2m^2} + e(g-1)\lambda\frac{Q_{ij}\nabla_i E_j}{2m^2}$$

- $\mathbf{\Sigma}^2 = 4s$, $\lambda = 1/(2s-1)$ for integer spin;
  $\mathbf{\Sigma}^2 = 4s+1$, $\lambda = 1/(2s)$ for half integer spin

One-Photon Terms

- $Q_{ij} = s_i s_i + s_j s_i - \frac{2}{3}\delta_i j \mathbf{s}^2$ is the quadrupole moment
- Coefficients before the Darwin and quadrupole terms do depend on spin!

Matrix element of **J** generates

$$V_s = -\frac{e(\mathbf{A}\cdot\mathbf{p}+\mathbf{p}\cdot\mathbf{A})}{2m} - g\frac{e}{2m}(\mathbf{s}\cdot\mathbf{B})(1-\frac{\mathbf{p}^2}{2m^2})$$
$$-(g-2)\frac{e}{2m}(\mathbf{s}\cdot\mathbf{B})\frac{\mathbf{p}^2}{2m^2} + (g-2)\frac{e}{2mc}\frac{(\mathbf{p}\cdot\mathbf{B})(\mathbf{s}\cdot\mathbf{p})}{2m^2}$$

One-photon Hamiltonian

$$H = \frac{\mathbf{p}^2}{2m} + V_0 + V_s$$

One-Photon Terms

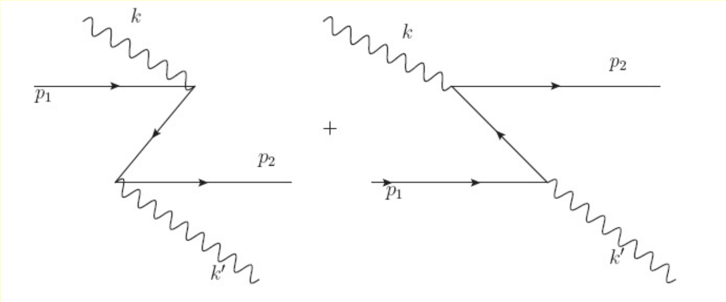
One-photon Hamiltonian

$$H = \frac{\mathbf{p}^2}{2m} - \frac{e(\mathbf{A}\cdot\mathbf{p} + \mathbf{p}\cdot\mathbf{A})}{2mc} + eA_0 - (g-1)e\frac{\nabla\cdot\mathbf{E}}{8m^2}\frac{\boldsymbol{\Sigma}^2}{3}$$
$$-(g-1)e\frac{\mathbf{s}\cdot(\mathbf{E}\times\mathbf{p})}{2m^2} + e(g-1)\lambda\frac{Q_{ij}\nabla_i E_j}{2m^2} - g\frac{e}{2m}(\mathbf{s}\cdot\mathbf{B})(1-\frac{\mathbf{p}^2}{2m^2})$$
$$-(g-2)\frac{e}{2m}(\mathbf{s}\cdot\mathbf{B})\frac{\mathbf{p}^2}{2m^2} + (g-2)\frac{e}{2m}\frac{(\mathbf{p}\cdot\mathbf{B})(\mathbf{s}\cdot\mathbf{p})}{2m^2}$$

- Coefficients before the Darwin and quadrupole terms do depend on spin!
- Magnetic terms exactly like in nonrelativistic reduction in spin one half case after Foldy-Wouthuysen transformation
- This Hamiltonian is not gauge invariant, two-photon terms are missing

Two-Photon Terms

- Regular recipe is to restore two-photon terms from Z-diagrams



- Easy calculation for spin one half gives

$$H_{2\gamma} = \frac{e^2}{2m}\mathbf{A}^2 + (g-1)\frac{e^2}{2m^2}\mathbf{s}\cdot(\mathbf{E}\times\mathbf{A})$$

- Another idea: restore two-photon terms from gauge invariance
- Recall: NRQED Hamiltonian is constructed from $\mathbf{D} = \nabla - ie\mathbf{A} = i(\mathbf{p} - e\mathbf{A})$, $\mathbf{E}$, $\mathbf{B}$, and $\mathbf{S}$.
- Any gauge noninvariant term with two fields has a partner with one field

NRQED Hamiltonian

- Can there be gauge invariant terms with two fields?
- They are of too high order in $Z\alpha$
- We use gauge invariance to restore two-field terms from one-field terms $\mathbf{p} \to \mathbf{p} - e\mathbf{A}$

NRQED Hamiltonian

$$H = \phi^+ \Big\{ \frac{(\mathbf{p} - e\mathbf{A})^2}{2m} + eA_0 - (g-1)e\frac{\nabla \cdot \mathbf{E}}{8m^2}\frac{\mathbf{\Sigma}^2}{3}$$

$$-(g-1)e\frac{\mathbf{s}\cdot(\mathbf{E}\times(\mathbf{p}-e\mathbf{A}))}{2m^2} + e(g-1)\lambda\frac{Q_{ij}\nabla_i E_j}{2m^2} - g\frac{e}{2m}(\mathbf{s}\cdot\mathbf{B})(1-\frac{\mathbf{p}^2}{2m^2})$$

$$-(g-2)\frac{e}{2m}(\mathbf{s}\cdot\mathbf{B})\frac{\mathbf{p}^2}{2m^2} + (g-2)\frac{e}{2m}\frac{(\mathbf{p}\cdot\mathbf{B})(\mathbf{s}\cdot\mathbf{p})}{2m^2}\Big\}\phi$$

NRQED Hamiltonian

- What about loop diagrams?
- Value of g includes all loop corrections to free g-factor
- Other loop corrections are of too high order in $Z\alpha$
- For spin one case this Hamiltonian follows from Electroweak Theory
- NRQED Hamiltonian is sufficient for calculation of nonrecoil corrections
- Nonrecoil correction of order $(Z\alpha)^2$ are universal!

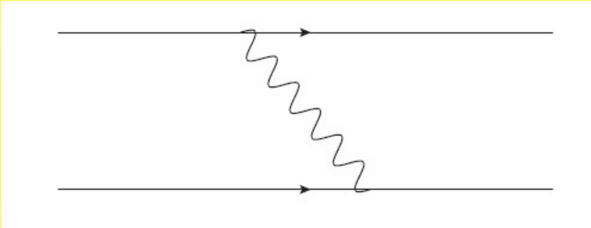
Two-Particle Hamiltonian

From Field Theory to Quantum Mechanics

- Effective two-particle QM Hamiltonian

$$H = H_1 + H_2 + H_{int}$$

- Free Hamiltonians H_i – one-particle sector of NRQED
- Interaction Hamiltonian from one-photon exchange



One-photon exchange interaction for arbitrary spins ($\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$)

$$V_{int} = e_1 e_2 \Bigg[\frac{1}{4\pi r} - (g_1 - 1)\frac{1}{8m_1^2}\frac{\mathbf{\Sigma}_1^2}{3}\delta(\mathbf{r}) - (g_1 - 1)\frac{3\lambda_1}{\pi}\frac{r^i r^j Q_{ij}^{(1)}}{8m_1^2 r^5}$$

$$-(g_2-1)\frac{1}{8m_2^2}\frac{\mathbf{\Sigma}_2^2}{3}\delta(\mathbf{r}) - (g_2-1)\frac{3\lambda_2}{\pi}\frac{r^i r^j Q_{ij}^{(2)}}{8m_2^2 r^5} - \frac{\mathbf{r}(\mathbf{r}\cdot\mathbf{p}_1)\cdot\mathbf{p}_2}{8\pi m_1 m_2 r^3} - \frac{\mathbf{p}_1\cdot\mathbf{p}_2}{8\pi m_1 m_2 r}$$

$$-(g_1 - 1)\frac{2\mathbf{s}_1\cdot(\mathbf{r}\times\mathbf{p}_1)}{16\pi m_1^2 r^3} + g_1\frac{2\mathbf{s}_1\cdot(\mathbf{r}\times\mathbf{p}_2)}{16\pi m_1 m_2 r^3}$$

$$+(g_2 - 1)\frac{2\mathbf{s}_2\cdot(\mathbf{r}\times\mathbf{p}_2)}{16\pi m_2^2 r^3} - g_2\frac{2\mathbf{s}_2\cdot(\mathbf{r}\times\mathbf{p}_1)}{16\pi m_1 m_2 r^3}$$

$$+\frac{g_1 g_2}{16\pi m_1 m_2}\left(\frac{\mathbf{s}_1\cdot\mathbf{s}_2}{r^3} - \frac{3(\mathbf{s}_1\cdot\mathbf{r})(\mathbf{s}_2\cdot\mathbf{r})}{r^5} - \frac{8\pi}{3}\mathbf{s}_1\cdot\mathbf{s}_2\delta(\mathbf{r})\right)\Bigg]$$

Two-Particle Hamiltonian

- Electric field is due to Coulomb potential
- This is interaction potential without external field
- Restore vector potential by the minimal substitution $\mathbf{p}_i \to \mathbf{p}_i - e\mathbf{A}_i$, $\mathbf{A}_i = \mathbf{B} \times \mathbf{r}_i/2$
- Total QM two-particle Hamiltonian (only relevant terms)

$$H = H_1 + H_2 + H_{int}$$

$$H_1 = \frac{(\mathbf{p}_1 - e_1\mathbf{A}_1)^2}{2m_1} - g_1\frac{e_1}{2m_1}(\mathbf{s}_1\cdot\mathbf{B})(1 - \frac{\mathbf{p}_1^2}{2m_1^2})$$

$$-(g_1 - 2)\frac{e_1}{2m_1}(\mathbf{s}_1\cdot\mathbf{B})\frac{\mathbf{p}_1^2}{2m_1^2} + (g_1 - 2)\frac{e_1}{2m_1}\frac{(\mathbf{p}_1\cdot\mathbf{B})(\mathbf{s}_1\cdot\mathbf{p}_1)}{2m_1^2}$$

Terms with electric field are included in H_{int}

Two-Particle Hamiltonian

- $H_1 \to H_2$ when $1 \to 2$
- Interaction Hamiltonian (only the Coulomb term and spin-orbit terms with magnetic field)

$$H_{int} = \frac{e_1 e_2}{4\pi r}$$

$$+e_1 e_2 \Big[-(g_1 - 1)\frac{2\mathbf{s}_1 \cdot (\mathbf{r} \times (\mathbf{p}_1 - e_1 \mathbf{A}_1))}{16\pi m_1^2 r^3} + g_1 \frac{2\mathbf{s}_1 \cdot (\mathbf{r} \times (\mathbf{p}_2 - e_2 \mathbf{A}_2))}{16\pi m_1 m_2 r^3}$$

$$+(g_2 - 1)\frac{2\mathbf{s}_2 \cdot (\mathbf{r} \times (\mathbf{p}_2 - e_2 \mathbf{A}_2))}{16\pi m_2^2 r^3} - g_2 \frac{2\mathbf{s}_2 \cdot (\mathbf{r} \times (\mathbf{p}_1 - e_1 \mathbf{A}_1))}{16\pi m_1 m_2 r^3} \Big]$$

Problem with Center of Mass

- Goal: separate internal properties of the bound system
- Center of mass motion does not separate in magnetic field
- Analogy with degenerate PT in QM: unperturbed wave functions should diagonalize perturbation
- Idea: composite particle in weak external field should respond to field like charged elementary particle
- Charged particle in magnetic field rotates on Landau orbit, its momentum is not conserved $[H, \mathbf{p}] \neq 0$
- Position of the center of Landau orbit is conserved (we use symmetric gauge, $\mathbf{A} = \mathbf{B} \times \mathbf{r}/2$), hence *pseudomomentum* is conserved

$$[H, \mathbf{p} + e\mathbf{A}] = 0$$

Problem with Center of Mass

- CM coordinates: $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, $\mathbf{R} = \mu_1\mathbf{r}_1 + \mu_2\mathbf{r}_2$, $\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2$, $\mathbf{p} = (\mathbf{p}_1 - \mathbf{p}_2)/2 + (\mu_2 - \mu_1)\mathbf{P}/2$, $\mu_i = m_i/(m_1 + m_2)$
- Unperturbed Hamiltonian for bound system

$$H_0 = \frac{(\mathbf{p}_1 - e_1\mathbf{A}_1)^2}{2m_1} + \frac{(\mathbf{p}_2 - e_2\mathbf{A}_2)^2}{2m_2} + V_C(|\mathbf{r}_1 - \mathbf{r}_2|))$$

- Neither total momentum, nor pseudomomentum are conserved: $[H_0, \mathbf{P}] \neq 0$, $[H_0, \mathbf{P} + (e_1 + e_2)\mathbf{A}(\mathbf{R})] \neq 0$
- Sum of pseudomomenta of constituents is conserved

$$[H_0, \mathbf{P} + (e_1 + e_2)\mathbf{A}(\mathbf{R}) + (e_1\mu_2 - e_2\mu_1)\mathbf{A}(\mathbf{r})] = 0$$

- Remedy: Unitary transformation $U = e^{-i(e_1\mu_2 - e_2\mu_1)\mathbf{A}(\mathbf{r})\cdot\mathbf{R}}$

Unitary Transformation

$$H_0' = U^{-1} H_0 U = \frac{[\mathbf{P} - Q\mathbf{A}(\mathbf{R}) - q\mathbf{A}(\mathbf{r})]^2}{2(m_1 + m_2)} + \frac{[\mathbf{p} - (e_1\mu_2^2 + e_2\mu_1^2)\mathbf{A}(\mathbf{r})]^2}{2m_r}$$

$$Q = e_1 + e_2, \; q = 2(e_1\mu_2 - e_2\mu_1)$$

$$U^{-1}\Big(\mathbf{P} + (e_1+e_2)\mathbf{A}(\mathbf{R}) + (e_1\mu_2 - e_2\mu_1)\mathbf{A}(\mathbf{r})\Big) U = \mathbf{P} + (e_1+e_2)\mathbf{A}(\mathbf{R})$$

- After transformation pseudomomentum is conserved

$$[H_0', \mathbf{P} + (e_1 + e_2)\mathbf{A}(\mathbf{R})] = 0$$

- In all calculations one should use the unitary transformed Hamiltonian

Spin Part of the Hamiltonian

Spin-dependent terms contain factors

$$\mathbf{p}_1 - e_1\mathbf{A}(\mathbf{r}_1) \to \mu_1\Big[\mathbf{P} - (e_1+e_2)\mathbf{A}(\mathbf{R})\Big] + \Big[\mathbf{p} - [e_1 - (e_1+e_2)\mu_1^2]\mathbf{A}(\mathbf{r})\Big]$$

$$\mathbf{p}_2 - e_2\mathbf{A}(\mathbf{r}_2) \to \mu_2\Big[\mathbf{P} - (e_1+e_2)\mathbf{A}(\mathbf{R})\Big] - \Big[\mathbf{p} - [e_2 - (e_1+e_2)\mu_2^2]\mathbf{A}(\mathbf{r})\Big]$$

After unitary transformation interaction terms with vector potential change form

Transformed spin-dependent Hamiltonian for the first particle (only terms magnetic field)

Bound g-Factors

$$H'^{(1)}_{spin} = -g_1\frac{e_1}{2m_1}(\mathbf{s}_1\mathbf{B})\left(1-\frac{\mathbf{p}^2}{2m_1^2}\right)-(g_1-2)\frac{e_1}{2m_1}(\mathbf{s}_1\mathbf{B})\frac{\mathbf{p}^2}{2m_1^2}$$

$$+(g_1-2)\frac{e_1}{2m_1}\frac{(\mathbf{pB})(\mathbf{s}_1\mathbf{p})}{2m_1^2}$$

$$-e_1e_2(g_1-1)\frac{2\mathbf{s}_1\cdot\left(\mathbf{r}\times\left[\mathbf{p}-[e_1-(e_1+e_2)\mu_1^2]\mathbf{A}(\mathbf{r})\right]\right)}{16\pi m_1^2r^3}$$

$$-e_1e_2g_1\frac{2\mathbf{s}_1\cdot\left(\mathbf{r}\times\left[\mathbf{p}-[e_2-(e_1+e_2)\mu_2^2]\mathbf{A}(\mathbf{r})\right]\right)}{16\pi m_1m_2r^3}$$

Similar Hamiltonian for the second particle

Bound g-Factors

Calculation of matrix elements is trivial

$$g_1^{bound} = g_1\Bigg[\left(1 - \frac{\mu_2^2 e_1^2 e_2^2}{2(4\pi)^2 n^2}\right) + \frac{\mu_2 e_1 e_2^2[e_1 - (e_1+e_2)\mu_1^2]}{6(4\pi)^2 n^2} + \frac{\mu_1 e_1 e_2^2[e_2 - (e_1+e_2)\mu_2^2]}{3(4\pi)^2 n^2}\Bigg] + (g_1 - 2)\left[\frac{\mu_2^2 e_1^2 e_2^2}{3(4\pi)^2 n^2} + \frac{\mu_2 e_1 e_2^2[e_1 - (e_1+e_2)\mu_1^2]}{6(4\pi)^2 n^2}\right]$$

g_2 is obtained by the substitution $1 \leftrightarrow 2$

Corrections are universal for particles of any spin; depend only on the g-factors, not on the magnitude of their spin. Why?

Bargmann-Michel-Telegdi Equation

- Terms in NRQED Hamiltonian with derivatives of electric field are irrelevant for corrections to g-factors
- In semiclassical approximation trajectory of charged particle with spin in external magnetic field does not depend on spin. Spin is a QM correction of order $\hbar$.
- The BMT equation for spin motion (Bargmann-Michel-Telegdi, 1959) is valid when we neglect field gradients and preserve only linear in field terms
- Only relativistic invariance and nonrelativistic limit are needed for derivation!

Bargmann-Michel-Telegdi Equation

- In three-dimensional form

$$\frac{d\mathbf{s}}{dt} = \frac{e}{2mc}\mathbf{s} \times \left\{ \left(g_s - 2 + \frac{2}{\gamma} \right) \mathbf{B} - \frac{(g_s - 2)\gamma}{1+\gamma} \frac{\mathbf{v} \cdot \mathbf{B}\mathbf{v}}{c^2} + \left(g_s - \frac{2\gamma}{1+\gamma} \right) \frac{[\mathbf{E} \times \mathbf{v}]}{c} \right\}$$

- The coefficients are universal for all spins!
- BMT is a Heisenberg equation for spin

$$i\hbar \frac{d\mathbf{s}}{dt} = [\mathbf{s}, H]$$

Bargmann-Michel-Telegdi Equation

$$H = -\frac{e\hbar}{2mc}\mathbf{s}\cdot\left\{\left(g_s - 2 + \frac{2}{\gamma}\right)\mathbf{B} - \frac{(g_s-2)\gamma}{1+\gamma}\frac{\mathbf{v}\cdot\mathbf{B}\mathbf{v}}{c^2} + \left(g_s - \frac{2\gamma}{1+\gamma}\right)\frac{[\mathbf{E}\times\mathbf{v}]}{c}\right\}$$

Nonrelativistic limit and minimal substitution $\mathbf{v}\to(\mathbf{p}-e\mathbf{A})/m$

$$H \approx -\frac{e\hbar}{2mc}\left\{\left(g_s - \frac{(\mathbf{p}-e\mathbf{A})^2}{m^2c^2}\right)\mathbf{s}\cdot\mathbf{B} - (g_s-2)\frac{[(\mathbf{p}-e\mathbf{A})\cdot\mathbf{B}][\mathbf{s}\cdot(\mathbf{p}-e\mathbf{A})]}{2m^2c^2} + (g_s-1)\frac{\mathbf{s}\cdot[\mathbf{E}\times(\mathbf{p}-e\mathbf{A})]}{mc}\right\}$$

Use this Hamiltonian to calculate corrections to bound state g-factor (Eides, Grotch, 1997)

Universal Nonrecoil g-Factor

Linear in external magnetic field terms

$$H \approx -\frac{e\hbar}{2mc}\left\{\left(g_s - \frac{\mathbf{p}^2}{m^2c^2}\right)\mathbf{s}\cdot\mathbf{B} - (g_s-2)\frac{(\mathbf{p}\cdot\mathbf{B})(\mathbf{s}\cdot\mathbf{p})}{2m^2c^2} - e\,(g_s-1)\,\frac{\mathbf{s}\cdot[\mathbf{E}\times\mathbf{A}]}{mc}\right\}$$

- Nonrecoil bound state g-factor

$$g_{bound} = g_s\left(1 - \frac{(Z\alpha)^2}{3n^2}\right) + (g_s-2)\frac{(Z\alpha)^2}{2n^2}$$

- Bound state g-factor naturally does not depend on magnitude of spin
- The source of leading recoil relativistic corrections is the one-photon exchange

Summary

- NRQED Hamiltonian for charged particles of arbitrary spin with all terms of order $(Z\alpha)^2$ is constructed
- For spin one case this Hamiltonian follows from the renormalizable Lagrangian QED of charged vector bosons
- An explicit expression for all (nonrecoil and recoil) leading binding ($\sim (Z\alpha)^2$) corrections to free g-factors is obtained
- Leading relativistic ($\sim (Z\alpha)^2$) corrections to bound state g-factors are universal and do not depend on the magnitude of particle spin