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# Correlation Numbers in Matrix Model and Liouville Gravity

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# Two dimensional gravity for genus one in Matrix Models, Topological and Liouville approaches.

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## Three approaches to 2D quantum geometry

One is the continuous approach, in which the theory is defined through the functional integral over the Riemannian metric $g_{\mu\nu}(X)$, with appropriate gauge fixing. The choice of the conformal gauge leads to quantum Liouville theory (coupled to matter fields), and for that reason this approach is often called the Liouville Gravity.

The other is the discrete approach, based on the idea of approximating the fluctuating $2D$ geometry by an ensemble of planar graphs, so that the continuous theory is recovered in the scaling limit where the planar graphs of very large size dominate. The discrete approach is usually referred to as the Matrix Models, since technically the ensemble of the graphs is usually generated by the perturbative expansion of the integral over $N \times N$ matrices, with $N$ sent to infinity to guarantee the planarity of the graphs .

There exists the third approach —2d Topological gravity. Witten built axiomatics of this theory by studying intersection theory . It was conjectured and checked (for genus-zero) that correlation numbers in Topological gravity and in Matrix models coincide. It should be mentioned that the coincidence takes place if correlation numbers in OMM are calculated in KdV frame.

Continuous approach $\qquad\qquad$ Discret approach

$\downarrow \qquad\qquad\qquad\qquad \downarrow$

"Liouville Gravity" $\qquad\qquad$ "Matrix Models"

Impressive body of evidence that the two describe the same reality:

- Operators $O_k^{LG}$ and $O_k^{MM}$ have identical scale dimensions
- Some correlation numbers coincide:

$$\langle\, O_1^{LG} ... O_n^{LG} \,\rangle = \langle\, O_1^{MM} ... O_n^{MM} \,\rangle$$

But with "naive" identification many correlation numbers are not in agreement.

Resolution: **Resonance relations**:

$$[O_k] \; = [O_{k_1}] + [O_{k_2}]$$

A conjecture was proposed and checked that there exists a so called “resonance” transformation in MM, such that the correlation functions in the both theories will coincide. Indeed the ”resonance” transformation in MM is the transformation from the ”KdV” background and the choice of the obsevables to a different one (”CFT frame”). The explicit form of the transformation was established for particular case, namely for the Minimal quantum gravity $\mathcal{MG}_{2/2p+1}$ and the One-Matrix Model (OMM) with p critical points. There exist also the tird approach to 2D gravity,namely, Witten topological gravity and conjectured its equivalence to Matrix models. He checked this equivalence for correlators of partition function for genus zero. The concidence takes place when correlation numbers in OMM are calculated in KdV frame.

# 1. Minimal Gravity

## 1.1. Quantum Geometry

$$\sum_{\text{topologies}} \int D[g]\, D[\phi]\, e^{-S[g,\phi]}$$

$g(x)$ - Riemannian metric , $\phi$ - "matter" fields

$$\langle\, \tilde{O}_{k_1} ... \tilde{O}_{k_N} \,\rangle = \int \tilde{O}_{k_1} ... \tilde{O}_{k_N}\, e^{-S[g,\phi]}\, D[g,\phi]$$

$$\tilde{O}_k = \int_{\mathbb{M}} O_k(x)\, d\mu_g(x)$$

$O_k(x)$ - local fields (built from $\phi$ and $g$).
**Generating function**: $\{\lambda\} = \{\lambda_1, ..., \lambda_n\}$

$$Z(\{\lambda\}) = \int D[g,\phi]\, e^{-S_\lambda[g,\phi]}\,,$$

$$S_\lambda[g,\phi] = S_0[g,\phi] + \sum_k \lambda_k\, \tilde{O}_k$$

$$\langle\, \tilde{O}_{k_1} ... \tilde{O}_{k_N} \,\rangle = \left. \frac{\partial^N Z(\{\lambda\})}{\partial\lambda_{k_1} ... \partial\lambda_{k_N}} \right|_{\lambda=0}$$

The parameters $\{\lambda\}$ are the coordinates in the"theory space" $\Sigma$.

## 1.2. Conformal Matter, and Liouville Gravity

$$g^{\mu\nu}\, T_{\mu\nu}^{\text{matter}} = -\frac{c}{12}\, R$$

Conformal Gauge $g_{\mu\nu} = e^{2b\varphi}\, \widehat{g}_{\mu\nu}$: $\quad\Rightarrow\quad$ Decoupling

$$S[g,\phi] \;\rightarrow\; S_{\mathsf{L}}[\varphi] + S_{\text{Ghost}}[B,C] + S_{\text{Matter}}[\phi]$$

with

$$S_L[\phi] = \frac{1}{4\pi} \int \sqrt{\widehat{g}}\, \left[\widehat{g}^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + Q\,\hat{R}\,\varphi + 4\pi\mu\, e^{2b\,\varphi}\right] d^2x\,,$$

$$S_{\text{Ghost}}[B,C] = \frac{1}{2\pi} \int \sqrt{\widehat{g}}\; B_{\mu\nu}\, \nabla^\mu C^\nu\, d^2x\,,$$

$\Big(B_{\mu\nu} = B_{\nu\mu}\,, \quad \widehat{g}^{\mu\nu} B_{\mu\nu} = 0\Big)$,

$$26 - c = 1 + 6\,Q^2 \qquad\qquad Q = b + 1/b\,.$$

($S_{\text{Matter}}[\phi]$ is conformally invariant, with the central charge $c$).

**Correlation numbers** $\langle\, \tilde{O}_{k_1} ... \tilde{O}_{k_N} \,\rangle$ with

$$\tilde{O}_k = \int V_k(x)\, \Phi_k(x)\, d^2x$$

$\Phi_k(x)$ - (spinless) primary fields of the matter CFT, with the conformal dimensions $(\Delta_k, \Delta_k)$ $V_k(x)$ - "gravitational dressings",

$$V_k(x) = e^{2a_k\, \varphi(x)}\,, \qquad a_k(Q - a_k) + \Delta_k = 1$$

**Gravitational dimensions** of $\tilde{O}_k$ control the scale dependence of the corr. functions:

$$\tilde{O}_k \;\sim\; \mu^{\delta_k}\,, \qquad \delta_k = -\frac{a_k}{b}$$

## 1.3. Correlation numbers

$$\langle\, \tilde{O}_{k_1} ... \tilde{O}_{k_n} \,\rangle = |(x_1 - x_2)(x_2 - x_3)(x_3 - x_1)|^2 \times$$

$$\int d^2x_4 ... d^2x_n \underbrace{\langle\, O_{k_1}(x_1) O_{k_2}(x_2) O_{k_3}(x_3) O_{k_4}(x_4) ... O_{k_n}(x_n) \,\rangle}$$

$$\downarrow$$

$$\langle\, V_{k_1}(x_1) ... V_{k_n}(x_n) \,\rangle_{\text{Liouville}} \;\langle\, \Phi_{k_1}(x_1) ... \Phi_{k_n}(x_n) \,\rangle_{\text{Matter}}$$

## 1.4. Matter CFT: "Minimal Models"

$$\mathcal{M}_{p/q} \qquad c = 1 - 6\,\frac{(p-q)^2}{pq}$$

Finite number of primary fields

$$\Phi_{(n,m)} \quad (n = 1, ..., p-1\,, \quad m = 1, ..., q-1\,, \quad n \leq m)\,,$$

with (in principle) computable correlation functions, e.g.

$$\langle\, \Phi_{(n_1,m_1)}(x_1) ... \Phi_{(n_4,m_4)}(x_4)\,\rangle_{MM} =$$

$$\sum_{(n,m)} \mathbb{C}^{(n,m)}_{(n_1,m_1)(n_2,m_2)}\,\mathbb{C}^{(n,m)}_{(n_3,m_3)(n_4,m_4)} |\mathcal{F}_{(n,m)}(\Delta_i|x)|^2$$

Fusion rules:

$$\Phi_{(n_1,m_1)}\Phi_{(n_2,m_2)} = \sum_{n=|n_1-n_2|+1}^{N}\;\sum_{m=|m_1-m_2|+1}^{M} [\Phi_{(n,m)}]\,,$$

with

$$N = \min(n_1 + n_2 - 1, 2p - n_1 - n_2 - 1)\,,$$

$$M = \min(m_1 + m_2 - 1, 2q - m_1 - m_2 - 1)$$

## 1.5. "Yang-Lee series" of the Minimal Models $\mathcal{M}_{2/2p+1}$

- $\mathcal{M}_{2/2p+1}$ has $p$ primary fields

$$\Phi_k \equiv \Phi_{(1,k+1)}\,, \qquad k = 0, 1, ..., p-1 \quad (p, p+1, ..., 2p-1)$$

Fusion rules

$$[\Phi_{k_1}][\Phi_{k_2}] = \sum_{k=|k_1-k_2|\,:\,2}^{k_1+k_2} [\Phi_k]\,, \qquad [\Phi_k] = [\Phi_{2p-k-1}]$$

$$\Phi_k = \Phi_{2p-k-1}$$

- Correlation functions:

$$\langle\, \Phi_k \,\rangle = \delta_{k,0}, \qquad \langle\, \Phi_k \Phi_{k'} \,\rangle \;\sim\; \delta_{k,k'}$$

$$\langle\, \Phi_{k_1} \Phi_{k_2} \Phi_{k_3} \,\rangle = 0$$

$$\text{if} \quad \begin{cases} k_1 + k_2 < k_3, \text{ etc,} & \text{for} \quad k_1 + k_2 + k_3 \text{ even} \\ k_1 + k_2 + k_3 < 2p - 1 & \text{for} \quad k_1 + k_2 + k_3 \text{ odd} \end{cases}$$

$$\langle\, \Phi_{k_1} ... \Phi_{k_n} \,\rangle = 0$$

$$\text{if} \quad \begin{cases} k_1 + ... + k_{n-1} < k_n, & \text{for} \quad k_1 + ... + k_n \text{ even} \\ k_1 + ... + k_n < 2p - 1 & \text{for} \quad k_1 + ... + k_n \text{ odd} \end{cases}$$

• Generating function: $\{\lambda\} = \{\lambda_1, \lambda_2, ..., \lambda_{p-1}\}$

$$Z_{\mathcal{MG}}(\mu, \{\lambda\}) = \left\langle \ \exp\left\{ -\sum_{i=1}^{p-1} \lambda_i \, \tilde{O}_i \right\} \ \right\rangle_{\mathcal{MG}_{2/2p+1}}$$

The cosmological constant $\mu$ may be treated as $\mu = \lambda_0$

$$S[\mathcal{MG}] = ... + \mu \underbrace{\int e^{2b\varphi(x)} \, d^2x}_{} + ...$$

$$\tilde{O}_0 = \int V_0(x) \, \Phi_0(x) \, d^2x \, , \qquad \Phi_0 = I$$

Dimensions:

$$\lambda_k \ \sim \ \mu^{\frac{k+2}{2}} \, , \qquad k = 0, 1, ..., p-1$$

By the definition

$$\langle \, \tilde{O}_{k_1} ... \tilde{O}_{k_n} \, \rangle = \left. \frac{\partial^n Z_{\mathcal{MG}}(\mu, \{\lambda_i\})}{\partial\lambda_{k_1} ... \partial\lambda_{k_n}} \right|_{\{\lambda_i\}=0} , \qquad \{\lambda_i\} = \{\lambda_1, ..., \lambda_n\}$$

## 2. Matrix Models

Continuous limit of the ensemble of planar graphs =Quantum Geometry

**2.1.One-matrix Model** The planar graphs = Feynmann diagrams associated with the perturbative evaluation of the matrix integral

$$Z = \log \int dM \; e^{-N\,\mathrm{tr}\left(\frac{1}{2}M^2 - \sum_{n=3}\frac{\alpha_n}{n!}M^n\right)}$$

$M$- Hermitian $N \times N$ matrix, $N$ being the device for sorting out the topologies

$$Z = N^2 Z_0 + Z_1 + ... + N^{2-2g} Z_g + ...$$

Each term $Z_g$ generates discretized surfaces, of the topology $g$, made of triangles and higher polygons, with the weights determined by $\alpha_i$.

• We concentrate on $g = 0$ (sphere) $\Sigma$ -space of the "potentials" $V(M) = \sum_{n=3}\frac{\alpha_n}{n!}M^n$.

The one-Matrix Model exhibits an infinite set of multi-critical points, labelled by the integer $p = 1, 2, 3, ....$

In the scaling limit the partition function is expressed through the solution of the "string equation"

$$\mathcal{P}(u) = 0\,, \tag{1}$$

where $\mathcal{P}(u)$ is the $p+1$-degree polynomial

$$\mathcal{P}(u) = u^{p+1} + t_0\,u^{p-1} + \sum_{k=1}^{p-1} t_k\,u^{p-k-1}\,, \tag{2}$$

with the parameters $t_k$ describing the relevant deviations from the $p$-critical point . The singular part of the Matrix Model partition function $Z(t_0, t_1, ..., t_{p-1})$ is expressed through $\mathcal{P}(u)$ as follows

$$Z = \frac{1}{2}\int_0^{u_*} \mathcal{P}^2(u)\,du\,, \tag{3}$$

where $u_* = u_*(t_0, t_1, ..., t_{p-1})$ is the suitably chosen root of the polynomial , i.e. $\mathcal{P}(u_*) = 0$.

It is important to remember that $Z$ really gives only the singular part of the Matrix Model partition function.

Take

$$t_0 = \mu \quad -\text{''cosmological constant''}$$

Then

$$[u] = [\mu^{\frac{1}{2}}]\,, \quad [t_k] = [\mu^{\frac{k+2}{2}}]\,, \qquad [Z] = [\mu^{\frac{2p+3}{2}}]\,,$$

exactly the gravitational dimensions of $\mathcal{MG}_{2/2p+1}$,

$$t_k \ \sim \ \lambda_k\,, \quad k = 0, 1, 2, ..., p-1.$$

Convenient to separate $t_0 = \mu$ and $\{t_i\} = \{t_1, t_2, ..., t_{p-1}\}$

Matrix Model correlation numbers:

$$\langle\, \mathcal{O}_{k_1} ... \mathcal{O}_{k_n} \rangle_{MM} \equiv \frac{\partial^n Z_{MM}(\mu, \{t_i\})}{\partial t_{k_1} ... \partial t_{k_n}}\bigg|_{\{t_i\}=0}, \qquad \{t_i\} = \{t_1, ..., t_n\}$$

With the (naive) identification $t_k \sim \lambda_k$ one would expect

$$\langle\, \mathcal{O}_{k_1} ... \mathcal{O}_{k_n} \,\rangle_{MM} = \langle\, \tilde{O}_{k_1} ... \tilde{O}_{k_n} \,\rangle_{\mathcal{MG}} \times \text{[Leg factors]}$$

<u>This expectation fails</u>.

Since

$$\mathcal{P}(u) = u^{p+1} + \mu\, u^{p-1} + \sum_{k=1}^{p-1} t_k\, u^{p-k-1}\,, \qquad Z = \frac{1}{2}\int_0^{u_*} \mathcal{P}^2(u)\, du$$

we have $u_*(\mu, 0, ..., 0) = \sqrt{\mu}$, and

$$\left.\frac{\partial Z}{\partial t_k}\right|_{\{t=0\}} = \int_0^{u_*} \mathcal{P}(u)\, \frac{\partial \mathcal{P}(u)}{\partial t_k}\, du \Big|_{\{t=0\}} = -\frac{2\,\mu^{\frac{2p-k+1}{2}}}{(2p-k-1)(2p-k+1)}$$

$$\left.\frac{\partial^2 Z}{\partial t_k \partial t_{k'}}\right|_{\{t=0\}} = \int_0^{u_*} \frac{\partial \mathcal{P}(u)}{\partial t_k}\, \frac{\partial \mathcal{P}(u)}{\partial t_{k'}}\, du \Big|_{\{t=0\}} = \frac{\mu^{\frac{2p-k-k'-1}{2}}}{2p-k-k'-1}$$

etc

in sharp contrast with

$$\langle\, \tilde{O}_k \,\rangle_{\mathcal{MG}} \;=\; 0\,, \quad k = 1, 2, ..., p-1 \quad \text{(since } \langle\, \Phi_k \,\rangle_{CFT} = 0\text{)}$$

$$\langle\, \tilde{O}_k \tilde{O}_{k'} \,\rangle_{\mathcal{MG}} \;\sim\; \delta_{kk'}\,, \qquad \text{(since } \langle\, \Phi_k \Phi_{k'} \,\rangle_{CFT} \sim \delta_{kk'}\text{)}$$

## 2.3. Resonance transformations

$$[t_k] = [\mu^{\frac{k+2}{2}}], \qquad [\lambda_k] = [\mu^{\frac{k+2}{2}}]$$

It is possible to have, e.g.

$$[t_k] = [\lambda_{k_1}][\lambda_{k_2}] \qquad (k = k_1 + k_2 + 2 \geq 2)$$

$(k = 0, 1, 2, ..., p-1)$. I.e.

$$t_k \;=\; \lambda_k \;+\; \sum_{\substack{k_1,k_2=0 \\ k_1+k_2=k+2}}^{p-1} c_k^{k_1 k_2}\, \lambda_{k_1}\lambda_{k_2} \;+\; \text{higher order terms}$$

Thus

$$t_0 = \lambda_0 = \mu\,,$$

$$t_1 = \lambda_1\;, \qquad\qquad ([t_1] = [\mu^{3/2}])$$

$$t_2 = \lambda_2 + A_2\,\mu^2\;, \qquad ([t_2] = [\mu^2])$$

$$t_3 = \lambda_3 + B_3\,\mu\,\lambda_1\;, \qquad ([t_3] = [\mu][t_1])$$

$$t_4 = \lambda_4 + A_4\,\mu^3 + B_4\,\mu\,\lambda_2 + C_4\,\lambda_1^2$$

etc

generally

$$t_k = \lambda_k + \underbrace{A_k \, \mu^{\frac{k+2}{2}}} + \sum_{n=0}^{n \leq k/2} \underbrace{B_k^{k-2n} \, \mu^n \, \lambda_{k-2n}} +$$

$$\frac{1}{2} \sum_{n=0} \sum_{k_1+k_2=k-2-2n} \underbrace{C_k^{k_1,k_2} \, \mu^n \, \lambda_{k_1} \lambda_{k_2}} + \ldots$$

$$\uparrow$$

$$Z_{MM}(\{t\}) \;\to\; \tilde{Z}_{MM}(\{\lambda\}) \equiv Z_{MM}(\{t(\lambda)\})$$

The right thing to expect is

$$\frac{\partial^N \tilde{Z}_{MM}(\{\lambda\})}{\partial \lambda_{k_1} \ldots \partial \lambda_{k_N}} = \langle \, \tilde{O}_{k_1} \ldots \tilde{O}_{k_n} \, \rangle_{\mathcal{MG}}$$

under special choice of the "Liouville coordinates" $\{\lambda_1, ..., \lambda_n\}$.

Thus,Problem: Finding the "Liouville coordinates" $\{\lambda\}$, such that

- One-point numbers:

$$\langle\, \tilde{O}_k \,\rangle_{MM} = \left. \frac{\partial \tilde{Z}(\mu, \{\lambda\})}{\partial \lambda_k} \right|_{\{\lambda\}=0} = 0 \qquad \text{for} \quad k = 1, 2, ..., p-1$$

- Two-point numbers:

$$\langle\, \tilde{O}_k \tilde{O}_{k'} \,\rangle_{MM} = \left. \frac{\partial^2 \tilde{Z}(\mu, \{\lambda\})}{\partial \lambda_k \partial \lambda_{k'}} \right|_{\{\lambda\}=0} \sim \ \delta_{kk'}$$

- Three-point numbers:

$$\langle\, \tilde{O}_{k_1} \tilde{O}_{k_2} \tilde{O}_{k_3} \,\rangle_{MM} = \left. \frac{\partial^3 \tilde{Z}(\mu, \{\lambda\})}{\partial \lambda_{k_1} \partial \lambda_{k_2} \partial \lambda_{k_3}} \right|_{\{\lambda\}=0} = 0$$

obey the fusion rules.

• Multi-point numbers obey fusion rules, e.g. For even $k_1+...+k_n$

$$\langle \tilde{O}_{k_1}\tilde{O}_{k_2}...\tilde{O}_{k_n}\rangle_{MM} = 0 \qquad \text{if} \quad k_n > k_1+k_2+...+k_{n-1}$$

For odd $k_1+...+k_n$

$$\langle \tilde{O}_{k_1}\tilde{O}_{k_2}...\tilde{O}_{k_n}\rangle_{MM} = 0 \qquad \text{if} \quad k_1+k_2+...+k_n < 2p-1$$

Building the Liouville coordinates order by order in $\{\lambda\}$:

∘ The resonance transforms do not affect odd parity correlation functions.

∘ Starting from $n = 4$ there are not enough parameters to exterminate the "wrong" correlation numbers:

$$[\lambda_k] = [\mu^{\frac{k+2}{2}}] \;\rightarrow\; [\lambda_{k_1+k_2}] = [\lambda_{k_1}][\lambda_{k_2}][\mu^2]$$

## 3. Finding the Liouville coordinates

When one plugs $t_k(\lambda)$ , the polynomial

$$\mathcal{P}(u) = u^{p+1} + t_0\, u^{p-1} + \sum_{k=1}^{p-1} t_k\, u^{p-k-1}\,, \tag{4}$$

takes the form

$$\mathcal{P}(u) = \mathcal{P}_0(u) + \sum_{k=1}^{p-1} \lambda_k\, \mathcal{P}_k(u) + \dots + \sum_{k_i=1}^{p-1} \frac{\lambda_{k_1}\dots\lambda_{k_n}}{n!}\, \mathcal{P}_{k_1\dots k_n}(u) + \dots$$

where $\mathcal{P}_0(u)$ and $\mathcal{P}_{k_1\dots k_n}(u)$ are the polynomials of $u$ whose coefficients involve non-negative powers of $\mu$.

$$\begin{aligned}
\mathcal{P}_0(u) &= \quad u^{p+1} \quad + C_0'\, \mu\, u^{p-1} \quad + C_0''\, \mu^2\, u^{p-3} \quad + \dots \\
\mathcal{P}_k(u) &= C_k\, u^{p-k-1} + C_k'\, \mu\, u^{p-k-3} + C_k''\, \mu^2\, u^{p-k-5} + \dots \\
&\dots
\end{aligned}$$

$C_k', C_k'', \dots$ are dimensionless constants related to the higher-order coefficients in $t_k(\lambda)$ , and in general $\mathcal{P}_{k_1\dots k_n}(u)$ are polynomials of the degree

$$p + 1 - 2n - \sum k_i\,,$$

of similar structure. Of course, only polynomials of non-negative degree appear, so that the sum in $\mathcal{P}(u)$ is finite.

When the fusion rules are violated, the correlation numbers then vanish as well. This requirement for the $n$-point numbers imposes strong conditions on the form of the polynomials $Q_{k_1\dots k_{n-1}}(x)$, which fix them uniquely.

Technically, this is done by constructing the polynomial $Q(u)$, order by order in $\lambda_k$. We have executed this program up to the fifth order. For higher $n$ direct calculations become rather involved. But a quick glance at a few first orders immediately suggests the general form,

$$Q_{k_1\dots k_n}(u) = \left(\frac{d}{du}\right)^{n-1} L_{p-\sum k-n}(u)\,,$$

where again $\sum k = k_1 + \dots + k_n$.

## The conjecture

The partition function of the one-MM is expressed through $Q(u)$

$$\mathcal{Z} = \frac{1}{2}\int_0^{u_*} Q^2(u)\,du\,,$$

$u_*$ is the solution of the "string equation"

$$Q(u_*) = 0$$

$$Q(u) = \sum_{n=0} \sum_{k_1,\dots k_n=1}^{p-1} \frac{\lambda_{k_1}\dots\lambda_{k_n}}{n!} L^{(n-1)}_{p-\sum k-n}(u)$$

Here we denote

$$L_k^{(n)}(u) = \left(\frac{d}{du}\right)^n L_k(u)$$

The partition fuction $\mathcal{Z}$ coincides with the generating functions of the correlation numbers in $\mathcal{MG}_{2/2p+1}$

$$\mathcal{Z} = \left\langle\ \exp\left\{-\sum_{i=1}^{p-1}\lambda_i\,\tilde{O}_i\right\}\ \right\rangle_{\mathcal{MG}_{2/2p+1}}$$

## The method of orthogonal polynomials

The partition function in One-Matrix Model

$$Z(v_k, N) = \log \int dM e^{-V(v_k, M)}$$

where $M$ is hermitian matrix $N \times N$ and potential

$$V(v_k, M) = N \sum_{k=1}^{p+1} v_k M^{2k}$$

Expansion to Feynman diagrams in respect to the coupling constants $v_k$ can be interpreted as genus expansion

$$Z = \sum_{h=0}^{\infty} N^{2-2h} Z_h,$$

$h -$ genus of surfaces

Now we want to compute the integral over $M$. The first step is dioganlization the matrix $M$ in the integral giving

$$\begin{aligned} & Z(v_k, N) = \log \int \prod_{i=1}^{N} d\lambda_i \Delta^2(\lambda) e^{-\sum_i V(v_k, \lambda_i)} \\ & \{\lambda_i\} - \text{eigenvalues of } M \\ & \\ & \Delta(\lambda) = \prod_{i<j} (\lambda_i - \lambda_j) - \text{Vandermonde determinant} \end{aligned} \tag{5}$$

Introducing the set of orthogonal polynomials $P_n(\lambda) = \lambda^n + ...$,

$$\int_{-\infty}^{\infty} d\lambda e^{-V(\lambda)} P_n(\lambda) P_m(\lambda) = s_n \delta_{nm}.$$

one obtains for the partition function

$$Z = N \sum_{k=1}^{N-1} (1 - k/N) \log(s_k/s_{k-1}).$$

Using the relation

$$\lambda P_k(\lambda) = P_{k+1}(\lambda) + R_k P_{k-1}(\lambda)$$

One gets

$$\int e^{-V} P_k \lambda P_{k-1} d\lambda = R_k s_{k-1} = s_k \tag{6}$$

$$R_k = s_k/s_{k-1}$$

Therefore

$$Z = N \sum_{k=1}^{N-1} (1 - k/N) \log R_k$$

We obtain the relation for $R_k$ using

$$ks_{k-1} = \int e^{-V} P_k' P_{k-1} = \int e^{-V} V' P_k P_{k-1}.$$

$$V'(\lambda) = \sum_{k=1}^{p+1} 2k v_k \lambda^{2k-1}$$

and applying $2n-1$ times the previous relation for $\lambda P_k(\lambda)$

$$\lambda^{2n-1} P_k = \lambda^{2n-2}(P_{k+1} + R_k P_{k-1}) =$$
$$= \lambda^{2n-3}(P_{k+2} + (R_{k+1})P_k + (R_k)P_k + (R_k R_{k-1})P_{k-2}) = ...$$

Thus we arrive to the following formula for $R_k$

$$\boxed{\frac{k}{N} = \tilde{W}(R_k, R_{k\pm1}, ..., R_{k\pm p})}$$

where

$$\tilde{W}(R_k, R_{k\pm1}, ..., R_{k\pm p}) =$$

$$= \sum_{n=1}^{p+1} 2n v_n \sum_{\{\sigma_{2n-1}\}} R_{k+m_1} \cdot ... \cdot R_{k+m_n}$$

$\{\sigma_{2n-1}\}$ denotes all "walks" which consist of $2n-1$ steps, starting in $k$ and finishing in $k-1$.

## Evaluation of $Z_0$ and $Z_1$

Propose existence of smooth function $R(\xi, N)$ of variable $\xi \in [0,1]$, and $R(\frac{k}{N}, N) = R_k$, and Taylor expansion for $R(\xi + m/N, N)$

$$R(\xi + m/N, N) = R(\xi, N) + \frac{m}{N} R_\xi(\xi, N) + \frac{m^2}{2N^2} R_{\xi\xi}(\xi, N) + O\Big(\frac{1}{N^3}\Big),$$

Thus

$$\tilde{W}(R(\xi, N)) = W(R(\xi, N)) + \frac{1}{N} W_1(R(\xi, N)) + \frac{1}{N^2} W_2(R(\xi, N)) + O\Big(\frac{1}{N^3}\Big),$$

After calculation

$$W(R(\xi, N)) = \sum_{n=1}^{p+1} \frac{(2n)!}{n!(n-1)!} v_n R^n(\xi, N),$$

$$W_1(R(\xi, N)) = 0,$$

$$W_2(R(\xi, N)) = \frac{RR_{\xi\xi}}{6} W''(R(\xi, N)) + \frac{RR_\xi^2}{12} W'''(R(\xi, N))$$

As a result we have

$$Z = N \sum_{k=1}^{N-1} (1 - k/N) \log R(\xi, N)$$

where $R(\xi, N)$ is solution of equation

$$\xi = W(R(\xi, N)) + \frac{RR_{\xi\xi}}{6N^2} W''(R(\xi, N)) + \frac{RR_\xi^2}{12N^2} W'''(R(\xi, N)) + O\Big(\frac{1}{N^4}\Big),$$

Assuming also the expansion

$$R(\xi, N) = R(\xi) + \frac{1}{N} R_1(\xi) + \frac{1}{N^2} R_2(\xi) + ...,$$

thus

$$\begin{aligned}
&\xi = W(R(\xi)),\\
&R_1(\xi) = 0,\\
&R_2(\xi) = -\frac{R(\xi)}{12W'(R)} \left(2R_{\xi\xi} W''(R(\xi)) + R_\xi^2 W'''(R(\xi))\right).
\end{aligned}$$

Passing from sum to integral in partition function, we use Euler-Maclorein formula up to $N^0$ terms

$$Z = N^2 \int_0^1 d\xi (1-\xi) \log R(\xi, N) - \frac{N}{2}(F(1) - F(1/N)) + \\ + \frac{1}{12}(F'(1) - F'(1/N)) + O(1/N),$$

where $F(\xi) = (1-\xi)\log R(\xi, N)$.

Then for partition function in genus-zero and genus-one we obtain

$$Z_0 = \int_0^1 d\xi (1-\xi) \log R,$$

$$Z_1 = -\frac{1}{12}\int_0^1 d\xi (1-\xi) \frac{2R_{\xi\xi}W''(R) + R_\xi^2 W'''(R)}{W'(R)},$$

where $R = R(\xi)$.

## The vicinity of $p$-critical point

The $p$-critical point are defined by the system of equations

$$W(R_c) = 1, \quad W'(R_c) = 0, \quad ... \quad W^{(p)}(R_c) = 0.$$

This system of equations, which determine coefficients $v_k^c$, $k = 1, ..., p$, and define the $R_c$.

Consider small deviations $\delta v_k = v_k - v_k^c$, and new coordinates $t_k$ in vicinity of the critical point

$$W(R_c) = 1 + t_{p+1}, \quad W'(R_c) = t_p, \quad ... \\ W^{(p-1)}(R_c) = t_0, \quad W^{(p)}(R_c) = 0.$$

Denoting $u = R - R_c$ one can obtain

$$\xi = W(u) = u^{p+1} + t_0 u^{p-1} + \sum_{k=1}^{p-1} t_k u^{p-k-1} + 1.$$

Making a substitution $\xi = 1 - y$, one can get

$$\mathcal{P}(u) + y = 0,$$

and the string polynomial $\mathcal{P}(u)$ defined as

$$\mathcal{P}(u) = u^{p+1} + t_0 u^{p-1} + \sum_{k=1}^{p-1} t_k u^{p-k-1}$$

and $u(y)$ is its solution.
Therefore for the partiton functions we obtain

$$Z_0 = \frac{1}{R_c} \int_0^1 dy\, y\, u(y),$$

$$Z_1 = -\frac{1}{12} \int_0^1 dy\, y \left( \frac{2\mathcal{P}''(u) u_{yy} + \mathcal{P}'''(u) u_y^2}{\mathcal{P}'(u)} \right)$$

These expressions can be efficiently simplified and we arrive to the final answer

$$Z_0 = \frac{1}{2}\int_0^{u^*} \mathcal{P}^2(u)du$$

$$Z_1 = -\frac{\log \mathcal{P}'(u^*)}{12}$$

where

$$u^* = u^*(t_0, t_1, ..., t_{p-1})$$

is the "'maximal" root of the polynomial $\mathcal{P}(u)$.

These formalae are indeed the explict expressions of the generating finctions for the correlation numbers in genus zero and genus one.

The same expressions can be obtained from **the double scaling limit** and **Douglas string equation**. The double scaling limit arises when $N$ goes to $\infty$, while $\mu$ and $t_k$ lead to 0 proportionally $(N^{-2}\varepsilon^2)^{\frac{2}{2p+3}}$ and $(N^{-2}\varepsilon^2)^{\frac{k+2}{2p+3}}$ correspondingly,
and $\varepsilon$ is some finite parameter. Making suitable replacement of variables, using the rescaling and performing the substitution $Z/N^2 \to Z$ for simplicity, we arrive to the expression for the partition function in the double scaling limit $Z[\mu, t_k, \varepsilon]$

$$Z[\mu, t_k, \varepsilon] = \sum_{h=0}^{\infty} \varepsilon^{2h} Z_h[\mu, t_k],$$

where $\varepsilon$ is the parameter, which is responsible for genus expansion.

## String equation

We can compute the partition functions $Z_h$ using the String equation which is the equation for function $u(x, \varepsilon, \mu, t_k)$ , connected with the partition function $Z[\mu, t_k, \varepsilon]$ as

$$u(x, \varepsilon) = \frac{d^2 Z}{dx^2}$$

It looks as

$$[\hat{P}, \hat{Q}] = 1$$

where

$$\hat{Q} = \varepsilon^2 d^2 + u(x), \quad d \equiv \frac{d}{dx}$$

$$\hat{P} = -\sum_{k=1}^{p+1} t_{p-1-k} \hat{Q}_+^{k-1/2}$$

are two differential operators

and $\hat{Q}_+^{k-1/2}$ stands for the non-negative part of the pseudo-differential operator $\hat{Q}^{k-1/2}$

We look for $u(x)$ in the form

$$u(x,\varepsilon) = \sum_{h=0}^{\infty} \varepsilon^{2h} u_h(x)$$

where, obviously, $u_h$

$$u_h(x) = \frac{d^2 Z_h}{dx^2}.$$

It is known, that

$$[\hat{Q}_+^{k-1/2}, \hat{Q}] = \frac{dS_k}{dx},$$

where the coefficients $S_k(u)$ obey the recursion relation

$$\frac{dS_{k+1}}{dx} = u\frac{dS_k}{dx} + \frac{1}{2}u_x S_k + \frac{\varepsilon^2}{4}\frac{d^3 S_k}{dx^3},$$

with the boundary conditions $S_0 = \frac{1}{2}$ and $S_k(k \neq 0)$ vanish at $u = 0$.

Use the equations above one can obtain

$$[\hat{P},\hat{Q}]=1 \quad \Longrightarrow \quad \sum_{k=1}^{p+1} t_{p-1-k}S_k(u)=-x$$

The solution of the recursion relations, including the first three terms is

$$S_k(u)=\frac{C_{2k}^k}{2^{2k+1}}\left(u^k+\frac{\varepsilon^2 k(k-1)}{6}u^{k-2}u_{xx}+\frac{\varepsilon^2 k(k-1)(k-2)}{12}u^{k-3}u_x^2\right).$$

where $\quad C_{2k}^k=\dfrac{(2k)!}{k!k!}$

Thus after rescaling the parameter $t_k \to \dfrac{2^{2k+1}}{C_{2k}^k}t_k$, we can obtain that

$$\mathcal{P}(u)+\varepsilon^2\left(\frac{1}{6}\mathcal{P}''(u)u_{xx}+\frac{1}{12}\mathcal{P}'''(u)u_x^2\right)=O(\varepsilon^4),$$

where $\mathcal{P}(u)$ is the string polynomial and $x=t_{p-1}$, $t_{-2}=1$, $t_{-1}=0$.

Using the expansion for $u(x, \varepsilon)$, we get to the zeroth order in the $\varepsilon$, that $u_0(x)$ obeys

$$\mathcal{P}(u_0) = 0,$$

therefore

$$u_0 = u^*(t_1, ..., t_{p-2}, x),$$

where $u^*$ is the real maximal root of polynomial $\mathcal{P}(u)$. To the second order in the $\varepsilon$ gives for the $u_1(t_1, ..., t_{p-2}, x)$ the following expression

$$u_1 = -\frac{\mathcal{P}'''(u^*)(u^*_x)^2 + 2\mathcal{P}''(u^*)u^*_{xx}}{12\mathcal{P}'(u^*)}.$$

Knowing $u_0$ and $u_1$ we can find corresponding the partition functions $Z_0$ and $Z_1$, using the fact that if $Z$ and $u^*$ are connected by relation

$$\frac{\partial^2 Z}{\partial x^2} = f(u^*),$$

then

$$Z = -\int_0^{u^*} \mathcal{P}(u)\mathcal{P}'(u)f(u)du.$$

This formula can be checked by straightforward calculation.

Integrating by parts and omitting the regular terms, we arrived to the expressions obtained above

$$Z_0 = \frac{1}{2}\int_0^{u^*} \mathcal{P}^2(u)du$$

and

$$Z_1 = -\frac{\log \mathcal{P}'(u^*)}{12}.$$

## Evaluation of correlation numbers in genus-one in KdV frame

The singular part of the partition function on torus $Z_1(t_0, t_1, ...t_{p-1})$ is

$$Z_1 = -\frac{\log \mathcal{P}'(u^*)}{12},$$

where $\mathcal{P}(u)$ is the polynomial of degree $p+1$ ($p$ is natural number)

$$\mathcal{P}(u) = u^{p+1} + t_0 u^{p-1} + \sum_{k=1}^{p-1} t_k u^{p-k-1},$$

Formula for correlation numbers is

$$\langle O_{k_1} ... O_{k_n} \rangle_1 = \frac{\partial^n Z_1}{\partial t_{k_1} ... \partial t_{k_n}}\bigg|_{t_1=...=t_{p-1}=0}$$

$$\langle O_k \rangle_1 = \frac{p+k}{24} u_c^{-k-2},$$

$$\langle O_{k_1} O_{k_2} \rangle_1 = \frac{(p+2+k_1+k_2)(k_1+k_2)+2p-k_1k_2}{48} u_c^{-k_1-k_2-4},$$

## Comparison with Topological Gravity

E.Witten recursion relation

$$\langle\sigma_{k_1}\sigma_{k_2}...\sigma_{k_s}\rangle_0 = k_1 \sum_{S=X\cup Y}\langle\sigma_{k_1-1}\prod_{i\in X}\sigma_{k_i}\sigma_0\rangle_0\langle\sigma_0\prod_{j\in Y}\sigma_{k_j}\sigma_{k_{s-1}}\sigma_{k_s}\rangle_0,$$

$$\begin{aligned}\langle\sigma_{k_1}\sigma_{k_2}...\sigma_{k_s}\rangle_1 = \frac{1}{12}k_1\langle\sigma_{k_1-1}\sigma_{k_2}...\sigma_{k_s}\sigma_0\sigma_0\rangle_0 + \\ + k_1\sum_{S=X\cup Y}\langle\sigma_{k_1-1}\prod_{i\in X}\sigma_{k_i}\sigma_0\rangle_0\langle\sigma_0\prod_{j\in Y}\sigma_{k_j}\rangle_1,\end{aligned}$$

It follows from basis recursion relation

$$\langle\sigma_{k_1}\sigma_{k_2}\sigma_{k_3}\rangle_0 = k_1\langle\sigma_{k_1-1}\sigma_0\rangle_0\langle\sigma_0\sigma_{k_2}\sigma_{k_3}\rangle_0,$$
$$\langle\sigma_k\rangle_1 = \frac{1}{12}k\langle\sigma_{k-1}\sigma_0\sigma_0\rangle_0 + k\langle\sigma_{k-1}\sigma_0\rangle_0\langle\sigma_0\rangle_1$$

and

$$\frac{\partial}{\partial a_k}\langle N\rangle = \langle\sigma_k N\rangle,$$

In One-Matrix Model

$$\sigma_k \leftrightarrow O_{p-k-1}, \qquad a_k \leftrightarrow t_{p-k-1}$$

We need to check

$$\langle O_{p-k_1-1} O_{p-k_2-1} O_{p-k_3-1}\rangle_0 = k_1 \langle O_{p-k_1} O_{p-1}\rangle_0 \langle O_{p-1} O_{p-k_2-1} O_{p-k_3-1}\rangle_0,$$
$$\langle O_{p-k-1}\rangle_1 = \frac{1}{12} k \langle O_{p-k} O_{p-1} O_{p-1}\rangle_0 + k\langle O_{p-k} O_{p-1}\rangle_0 \langle O_{p-1}\rangle_1$$

At arbitrary $\{t_k\}$ one can get

$$\langle O_{k_1} O_{k_2}\rangle_0 = \frac{\partial^2 Z_0}{\partial t_{k_1} \partial t_{k_2}} = \frac{(u^*)^{2p-k_1-k_2-1}}{2p-k_1-k_2-1},$$
$$\langle O_{k_1} O_{k_2} O_{k_3}\rangle_0 = \frac{\partial^3 Z_0}{\partial t_{k_1} \partial t_{k_2} \partial t_{k_3}} = -\frac{(u^*)^{3p-k_1-k_2-k_3-3}}{\mathcal{P}'(u^*)}.$$
$$\langle O_k\rangle_1 = \frac{\partial Z_1}{\partial t_k} = -\frac{p-k-1}{12\mathcal{P}'(u^*)}(u^*)^{p-k-2} + \frac{\mathcal{P}''(u^*)}{12(\mathcal{P}'(u^*))^2}(u^*)^{p-k-1}$$

Use this expressions, we see that recursion relation are fulfilled.

# Evaluation of correlation numbers in CFT frame

**KdV frame** $\longrightarrow$ **CFT frame**

$\updownarrow$ $\updownarrow$

$\{t_k\} \longrightarrow$ "resonanse" transformation $t_k = t_k(\{\lambda_k\}) \longrightarrow \{\lambda_k\}$

As a result

$$\mathcal{P}(u,\{t_k\}) = u^{p+1} + t_0 u^{p-1} + \sum_{k=1}^{p-1} t_k u^{p-k-1}$$

$$\downarrow$$

$$Q(x,\{\lambda_k\}) = \sum_{n=0}^{\infty} \sum_{k_1 \ldots k_n=1}^{p-1} \frac{\lambda_{k_1} \ldots \lambda_{k_n}}{n!} \frac{d^{n-1}}{dx^{n-1}} L_{p-\sum k_i - n}(x),$$

where

$$x = u/u_c, \quad u_c = u^*(\{\lambda_k\} = 0)$$

$L_n(x)$ – Legendre polynomials.

Formula for correlation numbers is

$$\langle O_{k_1} ... O_{k_n} \rangle_1 = \frac{\partial^n Z_1}{\partial \lambda_{k_1} ... \partial \lambda_{k_n}} \Big|_{\lambda_1 = ... = \lambda_{p-1} = 0}$$

First two correlation numbers in CFT frame

$$\langle O_k \rangle_1 = \frac{(2p-k)(k+1)}{24},$$

$$\langle O_{k_1} O_{k_2} \rangle_1 = -\frac{(1+k_1)(1+k_2)\Big((k_1+k_2-2p+2)(k_1+k_2) - k_1 k_2 - 4p\Big)}{24}.$$

## Conclusion

• We have derived the torus partition function $Z_1$ in $p$-critical One-Matrix Model. Using the explicit expression for the partition function in genus-one we compute the correlation numbers in KdV, as well as in CFT frames.

• The results in CFT frame should be compared against the correlation numbers in the Minimal Liouville gravity, which have not been computed yet. We expect the coincidence in genus-one similarly one observed in genus-zero.

• The results in KdV frame have been compared with Witten's results for the correlation numbers of the 2d topological gravity and found to coincide.