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A possible origin of the Jaffe-Witten mass gap in QCD

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A POSSIBLE ORIGIN OF THE JAFFE-WITTEN MASS GAP IN QCD

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Lagrangian of QCD

$$L_{QCD} = L_{YM} + L_{qg}$$

$$L_{YM} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + L_{g.f.} + L_{gh.}$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu, \quad a = N_c^2 - 1, \quad N_c = 3$$

$$L_{qg} = i\bar{q}^j_\alpha D_{\alpha\beta} q^j_\beta + \bar{q}^j_\alpha m^j_0 q^j_\beta, \quad \alpha, \beta = 1, 2, 3, \quad j = 1, 2, 3, ...N_f$$

$$D_{\alpha\beta} q^j_\beta = (\delta_{\alpha\beta}\partial_\mu - ig(1/2)\lambda^a_{\alpha\beta} A^a_\mu)\gamma_\mu q^j_\beta \qquad (cov.\ der.)$$

The Jaff-Witten (JW) theorem:

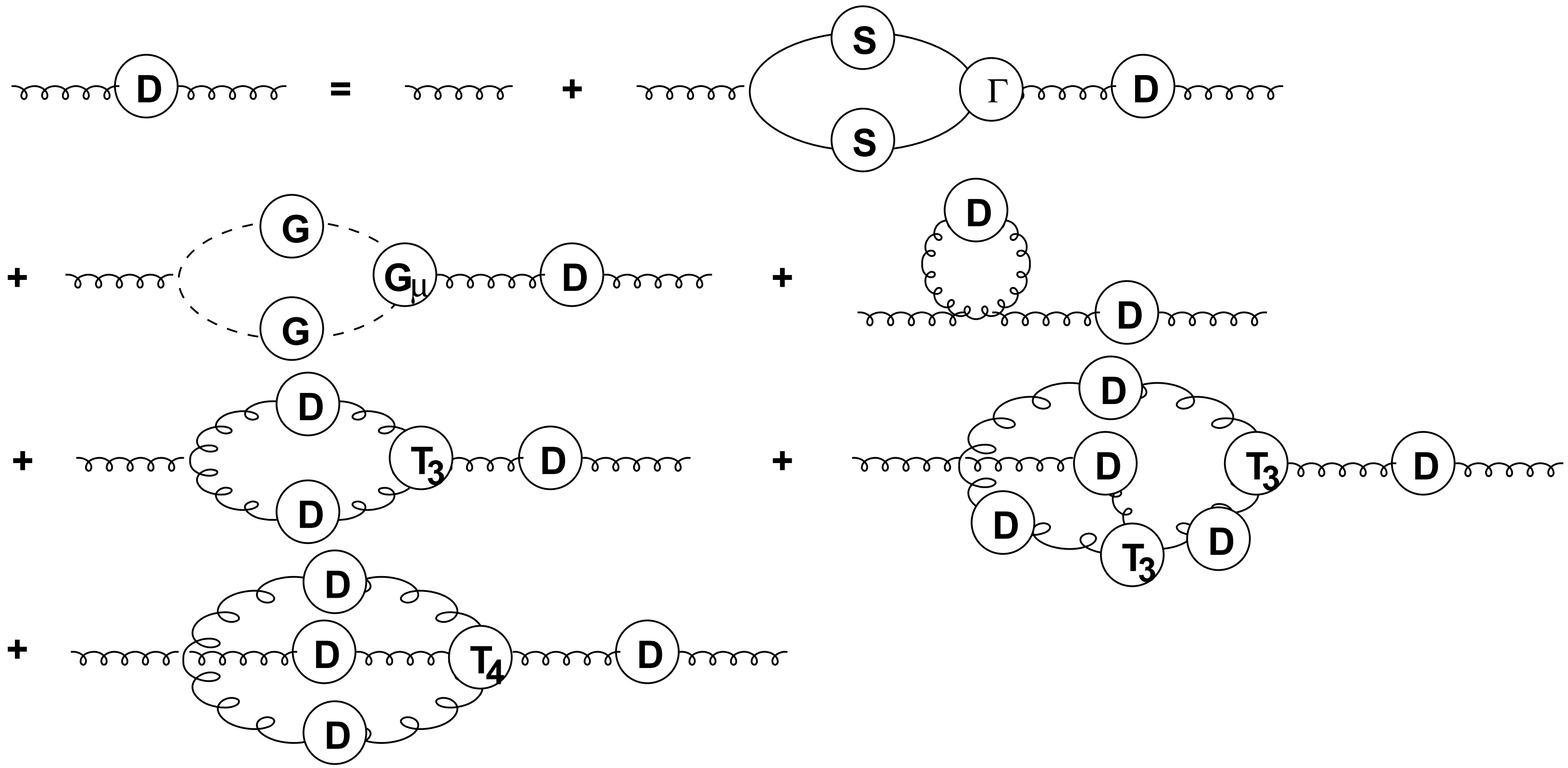
Yang-Mills Existence And Mass Gap: Prove that for any compact simple gauge group G, quantum Yang-Mills theory on R^4 exists and has a mass gap $\Delta > 0$.

(i). It must have a "mass gap". Every excitation of the vacuum has energy at least Δ (to explain why the nuclear force is strong but short-range).

(ii). It must have "quark confinement" (why the physical particles are $SU(3)$-invariant).

(iii). It must have "chiral symmetry breaking" (to account for the "current algebra" theory of soft pions).

We need Mass Gap responsible for the NP dynamics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Gluon SD equation

$$D_{\mu\nu}(q) = D^0_{\mu\nu}(q) + D^0_{\mu\rho}(q) i\Pi_{\rho\sigma}(q; D) D_{\sigma\nu}(q)$$

$$D^0_{\mu\nu}(q) = i \left\{ T_{\mu\nu}(q) + \xi L_{\mu\nu}(q) \right\} \frac{1}{q^2}$$

$$T_{\mu\nu}(q) = \delta_{\mu\nu} - (q_\mu q_\nu / q^2) = \delta_{\mu\nu} - L_{\mu\nu}(q)$$

$$\Pi_{\rho\sigma}(q; D) = \Pi^q_{\rho\sigma}(q) + \Pi^{gh}_{\rho\sigma}(q) + \Pi^t_{\rho\sigma}(D) + \Pi^{(1)}_{\rho\sigma}(q; D^2)$$

$$+ \Pi^{(2)}_{\rho\sigma}(q; D^4) + \Pi^{(2')}_{\rho\sigma}(q; D^3).$$

Subtractions

$$\Pi^s_{\rho\sigma}(q;D) = \Pi_{\rho\sigma}(q;D) - \Pi_{\rho\sigma}(0;D) = \Pi_{\rho\sigma}(q;D) - \delta_{\rho\sigma}\Delta^2(D)$$

$$\Delta^2(D) \equiv \Delta^2(\lambda,\alpha;D)$$

$$\Delta^2(D) = \Pi_t(D) + \Pi_q(0) + \Pi_g(0;D) = \Delta^2_t(D) + \Delta^2_q + \Delta^2_g(D)$$

$$\Delta^2_g(D) \equiv \Pi_g(0;D) = \sum_a \Pi_a(0;D) = \sum_a \Delta^2_a(D), \quad a = gh, (1), (2), (2')$$

$$\Pi^s_{\rho\sigma}(q;D) = \Pi_{\rho\sigma}(q;D) - \Pi_{\rho\sigma}(0;D) = \Pi_{\rho\sigma}(q;D) - \delta_{\rho\sigma}\Delta^2(D)$$

$$\Pi_{\rho\sigma}(q;D) = T_{\rho\sigma}(q)q^2\Pi(q^2;D) + q_\rho q_\sigma \tilde{\Pi}(q^2;D)$$

$$\Pi^s_{\rho\sigma}(q;D) = T_{\rho\sigma}(q)q^2\Pi^s(q^2;D) + q_\rho q_\sigma \tilde{\Pi}^s(q^2;D)$$

$$\Pi(q^2;D) = \Pi^s(q^2;D) + \frac{\Delta^2(D)}{q^2}.$$

$$\tilde{\Pi}(q^2;D) = \tilde{\Pi}^s(q^2;D) + \frac{\Delta^2(D)}{q^2}.$$

ST identity

$$q_\mu q_\nu D_{\mu\nu}(q) = i\xi$$

$$D_{\mu\nu}(q) = i\left\{T_{\mu\nu}(q)d(q^2) + \xi L_{\mu\nu}(q)\right\}\frac{1}{q^2}$$

$$D_{\mu\nu}(q) = D^0_{\mu\nu}(q) + D^0_{\mu\rho}(q)i\Pi_{\rho\sigma}(q;D)D_{\sigma\nu}(q) =$$

$$D^0_{\mu\nu}(q) + D^0_{\mu\rho}(q)iT_{\rho\sigma}(q)[q^2\Pi^s(q^2;D) + \Delta^2(D)]D_{\sigma\nu}(q)$$

$$+D^0_{\mu\rho}(q)iL_{\rho\sigma}(q)q^2\tilde{\Pi}(q^2;D)D_{\sigma\nu}(q)$$

$$q_\mu q_\nu D_{\mu\nu}(q) = i\xi - i\xi^2 \tilde{\Pi}(q^2; D)$$

$$\tilde{\Pi}(q^2; D) = \tilde{\Pi}^s(q^2; D) + \frac{\Delta^2(D)}{q^2} = 0$$

$$\tilde{\Pi}^s(q^2; D) = -\frac{\Delta^2(D)}{q^2}$$

$$\Delta^2(D) = 0$$

$$\tilde{\Pi}^s(q^2; D) = 0$$

$$q_\mu q_\nu D^{PT}_{\mu\nu}(q) = i\xi, \quad D_{\mu\nu}(q) = D^{PT}_{\mu\nu}(q) \to \Delta^2(D) = 0$$

$$D^{PT}_{\mu\nu}(q) = i\left\{T_{\mu\nu}(q) d^{PT}(q^2) + \xi L_{\mu\nu}(q)\right\}\frac{1}{q^2}$$

$$D^{PT}_{\mu\nu}(q) = D^0_{\mu\nu}(q) + D^0_{\mu\rho}(q) i T_{\rho\sigma}(q) q^2 \Pi^s(q^2; D^{PT}) D^{PT}_{\sigma\nu}(q)$$

$$d^{PT}(q^2) = \frac{1}{1 + \Pi^s(q^2; D^{PT})}$$

$$q_\rho \Pi_{\rho\sigma}(q; D^{PT}) = q_\rho \Pi^s_{\rho\sigma}(q; D^{PT}) = 0$$

Transversality of the full gluon self-energy

$$q_\rho \Pi_{\rho\sigma}(q; D) = q_\rho \Pi^q_{\rho\sigma}(q) + q_\rho \Pi^g_{\rho\sigma}(q; D)$$

$$\Pi^g_{\rho\sigma}(q; D) = \Pi^t_{\rho\sigma}(D) + \Pi^{gh}_{\rho\sigma}(q) + \Pi^{(1)}_{\rho\sigma}(q; D^2) + \Pi^{(2)}_{\rho\sigma}(q; D^4) + \Pi^{(2')}_{\rho\sigma}(q; D^3).$$

A. The quark contribution

The color currents conservation condition implies

$$q_\rho \Pi^q_{\rho\sigma}(q) = q_\sigma \Pi^q_{\rho\sigma}(q) = 0, \Longrightarrow \Delta^2_q = 0.$$

$$\Pi^{s(q)}_{\rho\sigma}(q) = \Pi^{q}_{\rho\sigma}(q) - \Pi^{q}_{\rho\sigma}(0) = \Pi^{q}_{\rho\sigma}(q) - \delta_{\rho\sigma}\Delta^2_q$$

$$\Pi^{q}_{\rho\sigma}(q) = T_{\rho\sigma}(q) q^2 \Pi_1(q^2) + q_\rho q_\sigma \Pi_2(q^2),$$

$$\Pi^{s(q)}_{\rho\sigma}(q) = T_{\rho\sigma}(q) q^2 \Pi^s_1(q^2) + q_\rho q_\sigma \Pi^s_2(q^2).$$

$$\Pi_2(q^2) = \Pi^s_2(q^2) + \frac{\Delta^2_q}{q^2},$$

$$\Pi_1(q^2) = \Pi^s_1(q^2) + \frac{\Delta^2_q}{q^2}$$

$$\Pi_2(q^2) = \Pi_2^s(q^2) + \frac{\Delta_q^2}{q^2} = 0$$

$$\Pi_2^s(q^2) = -\frac{\Delta_q^2}{q^2}$$

$$\Delta_q^2 = 0, \quad \Pi_2(q^2) = \Pi_2^s(q^2) = 0$$

$$\Pi_{\rho\sigma}^q(q) = \Pi_{\rho\sigma}^{s(q)}(q) = T_{\rho\sigma}(q) q^2 \Pi_1^s(q^2).$$

In complete analogy with QED

B. The pure gluon contribution

$$q_\rho \Pi^g_{\rho\sigma}(q; D) =$$

$$q_\rho \Big[\Pi^t_{\rho\sigma}(D) + \Pi^{gh}_{\rho\sigma}(q) + \Pi^{(1)}_{\rho\sigma}(q; D^2) + \Pi^{(2)}_{\rho\sigma}(q; D^4) + \Pi^{(2')}_{\rho\sigma}(q; D^3)\Big] \neq 0,$$

and hence $q_\rho \Pi_{\rho\sigma}(q; D) \neq 0$, unless the constant skeleton tadpole term $\Pi^t_{\rho\sigma}(D)$ is discarded from the very beginning

$$q_\rho \Big[\Pi^{gh}_{\rho\sigma}(q) + \Pi^{(1)}_{\rho\sigma}(q; D^2) + \Pi^{(2)}_{\rho\sigma}(q; D^4) + \Pi^{(2')}_{\rho\sigma}(q; D^3)\Big] = 0$$

However, in general case

$$\Delta^2(D) = \Delta^2_t(D) + \Delta^2_g(D)$$

On the other hand

$$D_{\mu\nu}(q) = i\left\{T_{\mu\nu}(q)d(q^2) + \xi L_{\mu\nu}(q)\right\}\frac{1}{q^2}$$

$$D_{\mu\nu}(q) = D^0_{\mu\nu}(q) + D^0_{\mu\rho}(q) i T_{\rho\sigma}(q)[q^2\Pi^s(q^2;D) + \Delta^2(D)]D_{\sigma\nu}(q)$$

$$+ D^0_{\mu\rho}(q) i L_{\rho\sigma}(q) q^2 \tilde{\Pi}(q^2;D) D_{\sigma\nu}(q)$$

$$d(q^2) = \frac{1}{1 + \Pi^s(q^2;D) + (\Delta^2(D)/q^2)}$$

$$\tilde{\Pi}(q^2;D) = 0, \quad \rightarrow \quad \Delta^2(D) = 0$$

Preliminary discussion

The formal $\Delta^2(D) = 0$ limit is a real way how to preserve the color gauge invaraince/symmetry in QCD. Why does $\Delta^2(D)$ (which is nothing but the re-defined tadpole term) exist in this theory at all? There is no doubt that this symmetry should be maintained at non-zero $\Delta^2(D)$ as well.

A. The first problem is how to satisfy the ST identity but without putting $\Delta^2(D) = 0$ everywhere.

B. The second problem is how to make the relevant gluon propagator purely transversal, since the ghosts will fail to do this when $\Delta^2(D)$ will be explicitly present.

A. The spurious mechanism

$$D_{\mu\nu}(q) = D^0_{\mu\nu}(q) + D^0_{\mu\rho}(q) i T_{\rho\sigma}(q)[q^2 \Pi^s(q^2; D) + \Delta^2(D)] D_{\sigma\nu}(q)$$
$$+ D^0_{\mu\rho}(q) i L_{\rho\sigma}(q) q^2 \tilde{\Pi}(q^2; D) D_{\sigma\nu}(q)$$

$$D^0_{\mu\nu}(q) \to D^0_{\mu\nu}(q; \Delta^2(D)) = D^0_{\mu\nu}(q) + i\xi L_{\mu\nu}(q) d_0(q^2; \Delta^2(D)) \frac{1}{q^2}$$

$$D_{\mu\nu}(q) = D^0_{\mu\nu}(q) + D^0_{\mu\rho}(q) i T_{\rho\sigma}(q)[q^2 \Pi^s(q^2; D) + \Delta^2(D)] D_{\sigma\nu}(q)$$
$$+ I_{\mu\nu}(q; \Delta^2(D))$$

$$I_{\mu\nu}(q;\Delta^2(D)) = i\xi d_0(q^2;\Delta^2(D))\left[L_{\mu\nu}(q) + L_{\mu\sigma}(q) i q^2 \tilde{\Pi}(q^2;D) D_{\sigma\nu}(q)\right]\frac{1}{q^2}$$

$$+D^0_{\mu\rho}(q) i L_{\rho\sigma}(q) q^2 \tilde{\Pi}(q^2;D) D_{\sigma\nu}(q)$$

$$= i\xi L_{\mu\nu}(q)\left[d_0(q^2;\Delta^2(D))\left(1 - \xi\tilde{\Pi}(q^2;D)\right) - \xi\tilde{\Pi}(q^2;D)\right]\frac{1}{q^2}$$

$$d_0(q^2;\Delta^2(D)) = \frac{\xi\tilde{\Pi}(q^2;D)}{1 - \xi\tilde{\Pi}(q^2;D)}$$

$$I_{\mu\nu}(q;\Delta^2(D)) = 0$$

NP QCD

$$D_{\mu\nu}(q) = D^0_{\mu\nu}(q) + D^0_{\mu\rho}(q) i T_{\rho\sigma}(q)[q^2 \Pi^s(q^2; D) + \Delta^2(D)] D_{\sigma\nu}(q)$$

$$d(q^2) = \frac{1}{1 + \Pi^s(q^2; D) + (\Delta^2(D)/q^2)}.$$

PT QCD

$$D^{PT}_{\mu\nu}(q) = D^0_{\mu\nu}(q) + D^0_{\mu\rho}(q) i T_{\rho\sigma}(q) q^2 \Pi^s(q^2; D^{PT}) D^{PT}_{\sigma\nu}(q),$$

$$d^{PT}(q^2) = \frac{1}{1 + \Pi^s(q^2; D^{PT})}.$$

The Jaffe-Witten mass gap

$$\Delta^2(d) = \Delta^2 c(d)$$

$$\Delta^2 \equiv \Delta^2(\lambda, \alpha, \xi, g^2)$$

$$\Delta^2_{JW} = Z(\lambda, \alpha, \xi, g^2)\Delta^2(\lambda, \alpha, \xi, g^2).$$

The PT $\Delta^2 = 0$ limit

B. Restoration of transversality of the gluon propagator

$$D^{TNP}_{\mu\nu}(q;\Delta^2) = D_{\mu\nu}(q;\Delta^2) - D_{\mu\nu}(q;\Delta^2=0) = D_{\mu\nu}(q;\Delta^2) - D^{PT}_{\mu\nu}(q)$$

$$D_{\mu\nu}(q;\Delta^2) = i\left\{T_{\mu\nu}(q)d(q^2;\Delta^2) + \xi L_{\mu\nu}(q)\right\}(1/q^2)$$

$$D^{TNP}_{\mu\nu}(q;\Delta^2) = iT_{\mu\nu}(q)d^{TNP}(q^2;\Delta^2)\frac{1}{q^2}$$

$$D^{PT}_{\mu\nu}(q) = i\left\{T_{\mu\nu}(q)d^{PT}(q^2) + \xi L_{\mu\nu}(q)\right\}(1/q^2)$$

$$D_{\mu\nu}(q;\Delta^2) = D_{\mu\nu}^{TNP}(q;\Delta^2) + D_{\mu\nu}^{PT}(q)$$

$$D_{\mu\nu}(q;\Delta^2) = i\left\{T_{\mu\nu}(q)d(q^2;\Delta^2) + \xi L_{\mu\nu}(q)\right\}(1/q^2)$$

$$= i\left\{T_{\mu\nu}(q)d(q^2;\Delta^2) + \xi L_{\mu\nu}(q)\right\}(1/q^2)$$

$$-iT_{\mu\nu}(q)d^{PT}(q^2)(1/q^2) + iT_{\mu\nu}(q)d^{PT}(q^2)(1/q^2)$$

$$= D_{\mu\nu}^{TNP}(q;\Delta^2) + D_{\mu\nu}^{PT}(q)$$

One of the important characteristics of the QCD ground state is the **Bag constant**.

$$B = VED^{PT} - VED,$$

VED is the NP but "contaminated" by the PT contributions (i.e., it is a full VED like the full gluon propagator).

$$B = VED^{PT} - VED = VED^{PT} - [VED - VED^{PT} + VED^{PT}] =$$

$$VED^{PT} - [VED^{TNP} + VED^{PT}] = -VED^{TNP} > 0,$$

Bag constant is free of the PT "contaminations".

G. G. Barnoföldi, V. Gogokhia, J. Phys. G: Nucl. Part. Phys. 37 (2010) 025003 (arXiv:0708.0163)

Preliminary conclusions

The mass gap is generated in the gluon sector of QCD mainly due to the self-interaction of massless gluon modes.

It is defined as the difference between the full gluon self-energy and its value at some point.

No any truncations/approximations, no special gauge choice, only algebraic (i.e., exact) derivations have been done.

The common belief (coming from PT) that mass gap contradicts the color gauge invariance/symmetry of QCD is false.

This fundamental symmetry is maintained/preserved at non-zero mass gap as well.

We distinguish between NP and PT QCD by the explicit presence of the mass gap, and not by the strength of the coupling constant.

Proposed subtraction makes it possible to make the relevant gluon propagator to become purely transversal in a gauge invariant way.

So unitarity of the S-matrix in TNP QCD is maintained.

No free gluons in TNP QCD.

The next step is to find a formal solution(s) for the full gluon propagator as a function of the regularized mass gap and its renormalization.

Massive solution

$$\frac{1}{q^2} d(q^2) = \frac{1}{q^2 + q^2 \Pi^s(q^2;\xi) + \Delta^2 c(\xi)}$$

If the denominator has a zero at point $q^2 = -m_g^2$ (Eucl. sign.), where $m_g^2 \equiv m_g^2(\xi)$ is an effective gluon mass

$$-m_g^2 - m_g^2 \Pi^s(-m_g^2;\xi) + \Delta^2 c(\xi) = 0$$

$$q^2 + q^2 \Pi^s(q^2;\xi) + \Delta^2 c(\xi) = q^2 + m_g^2 + q^2 \Pi^s(q^2;\xi) + m_g^2 \Pi^s(-m_g^2;\xi)$$

$$\Pi^s(q^2;\xi) = \Pi^s(-m_g^2;\xi) + (q^2 + m_g^2)\Pi'^s(-m_g^2;\xi) + O\Big((q^2 + m_g^2)^2\Big).$$

$$q^2 + m_g^2 + q^2\Pi^s(q^2;\xi) + m_g^2\Pi^s(-m_g^2;\xi)$$

$$= (q^2 + m_g^2)[1 + \Pi^s(-m_g^2;\xi) - m_g^2\Pi'^s(-m_g^2;\xi)][1 + \Pi^{s,R}(q^2;\xi)],$$

where $\Pi^{s,R}(q^2;\xi) = 0$ at $q^2 = -m_g^2$ and regular at small q^2.

$$D_{\mu\nu}(q;m_g^2) = iT_{\mu\nu}(q)\frac{Z_3(m_g^2)}{(q^2 + m_g^2)[1 + \Pi^{s,R}(q^2;m_g^2)]} + i\xi L_{\mu\nu}(q)\frac{1}{q^2}$$

$$Z_3(m_g^2) = \frac{1}{1 + \Pi^s(-m_g^2;\xi) - m_g^2\Pi'^s(-m_g^2;\xi)}.$$

$$Z_3 = \frac{1}{1 + \Pi^s(0;\xi)}.$$

Nonlinear iteration solution

$$d(q^2;\Delta^2) = \frac{1}{1+\Pi^s(q^2;d)+c(d)(\Delta^2/q^2)}, \quad z = \frac{\Delta^2}{q^2}$$

$$d(q^2;\Delta^2) = d(q^2;z) = \sum_{k=0}^{\infty} z^k f_k(q^2),$$

$$f_k(q^2) = (-1)^k d^{PT}(q^2)[d^{PT}(q^2)c(d^{PT})]^k.$$

$$f_k(q^2) = \sum_{n=0}^{k} x^n f_{kn}(0) + x^{k+1} B_k(x), \quad x = \frac{q^2}{M^2}$$

$$d(q^2) = \sum_{k=0}^{\infty} z^k f_k(x) = \sum_{k=0}^{\infty} z^k \Big(\sum_{n=0}^{k} x^n f_{kn}(0) + x^{k+1} B_k(x) \Big).$$

$$d(q^2) = z \sum_{k=0}^{\infty} z^k \sum_{m=0}^{\infty} \Phi_{km}(0) + a \sum_{k=0}^{\infty} a^k \sum_{m=0}^{\infty} A_{km}(x) + d^{PT}(q^2),$$

where the constant $a = xz = \Delta^2/M^2$ and the dimensionless functions $A_{km}(x)$ are regular functions of x:

$$d(q^2; \Delta^2) = 1 - [\Pi^s(q^2; d) + c(d)(\Delta^2/q^2)]d(q^2)$$

The exact structure of the general nonlinear iteration solution

$$D_{\mu\nu}(q;\Delta^2) = D_{\mu\nu}^{INP}(q;\Delta^2) + D_{\mu\nu}^{MPT}(q;\Delta^2),$$

$$D_{\mu\nu}^{INP}(q;\Delta^2) = iT_{\mu\nu}(q)\frac{\Delta^2}{(q^2)^2}L(q^2;\Delta^2),$$

$$L(q^2;\Delta^2) = \sum_{k=0}^{\infty}\Big(\frac{\Delta^2}{q^2}\Big)^k \Phi_k(\lambda,\alpha,\xi,g^2) = \sum_{k=0}^{\infty}\Big(\frac{\Delta^2}{q^2}\Big)^k \sum_{m=0}^{\infty}\Phi_{k,m}(\lambda,\alpha,\xi,g^2),$$

$$D_{\mu\nu}^{MPT}(q;\Delta^2) = i\Big[T_{\mu\nu}(q)d^{MPT}(q^2;\Delta^2) + \xi L_{\mu\nu}(q)\Big]\frac{1}{q^2}.$$

A. Power-type NP (severe) IR singularities

$$(q^2)^{-2-k}, \; k = 0, 1, 2, 3, ... \quad q^2 \to 0$$

B. Power-type PT IR singulary

$$(q^2)^{-1}, \quad q^2 \to 0$$

$$d^{MPT}(q^2; \Delta^2) = \sum_{k=0}^{\infty} \left(\frac{\Delta^2}{M^2}\right)^k \sum_{m=0}^{\infty} A_{k,m}(q^2)$$

The separation is not only exact but is unique as well. It is **unique** because there exists a special regularization expansion for severe (i.e., NP) IR singularities, while for the PT IR singularity such kind of expansion does not exist at all.

Two general-type of solutions for NP QCD

Non-linear iteration solution

$$D_{\mu\nu}(q;\Delta^2) = D_{\mu\nu}^{INP}(q;\Delta^2) + D_{\mu\nu}^{MPT}(q;\Delta^2)$$

$$D_{\mu\nu}^{INP}(q;\Delta^2) = iT_{\mu\nu}(q)\frac{\Delta^2}{(q^2)^2}L(q^2;\Delta^2)$$

Massive solution

$$D_{\mu\nu}(q;m_g^2) = iT_{\mu\nu}(q)\frac{Z_3(m_g^2)}{(q^2+m_g^2)[1+\Pi^{s,R}(q^2;m_g^2)]} + i\xi L_{\mu\nu}(q)\frac{1}{q^2}.$$

INP QCD

$$D_{\mu\nu}(q;\Delta^2) \to D^{INP}_{\mu\nu}(q;\Delta^2)$$

$$D^{INP}_{\mu\nu}(q;\Delta^2) = iT_{\mu\nu}(q)\frac{\Delta^2}{(q^2)^2}L(q^2)$$

$$L(q^2) \equiv L(q^2;\Delta^2) = \sum_{k=0}^{\infty}\left(\frac{\Delta^2}{q^2}\right)^k \Phi_k(\lambda,\alpha,\xi,g^2)$$

$$\Phi_k(\lambda,\alpha,\xi,g^2) = \sum_{m=0}^{\infty}\Phi_{k,m}(\lambda,\alpha,\xi,g^2).$$

Weierstrass-Sokhatsky-Casorati (WSK) theorem

describes the behavior of meromorphic functions near essential singularities.

If z_0 is an essential singularity of the function $f(z)$, then for any complex number Z there exists the sequence of points $z_k \to z_0$, such that

$$\lim_{k\to\infty} f(z_k) = Z, \quad z_k \to z_0.$$

So this theorem tells us that the behavior of the function $f(z)$ near its essential singularity z_0 is not determined, i.e, in fact, it remains arbitrary. It depends on the chosen sequence of points z_k along which z goes to z_0, that's $Z \equiv Z(z_k)$.

$$f(z) = e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!z^n}, \quad 0 < z < \infty$$

$$(\mathbf{i}).\ \lim_{k\to\infty} f(z_k) = \lim_{k\to\infty} e^k = \infty, \quad z_k = 1/k,\ k = 1, 2, 3, ...$$

$$(\mathbf{ii}).\ \lim_{k\to\infty} f(z_k) = \lim_{k\to\infty} e^{-k} = 0, \quad z_k = -1/k,\ k = 1, 2, 3, ...$$

$$z_k = 1/\ln A + 2k\pi i,\ k = 0, 1, 2, 3, ...$$

$$(\mathbf{iii}).\ \lim_{k\to\infty} f(z_k) = \lim_{k\to\infty} e^{\ln A + 2k\pi i} = A.$$

Renormaliztion of the regularized mass gap

$$D^{INP}_{\mu\nu}(q;\Delta^2) = iT_{\mu\nu}(q)\frac{\Delta^2}{(q^2)^2} \times \sum_{k=0}^{\infty}\Big(\frac{\Delta^2}{q^2}\Big)^k \Phi_k(\lambda,\alpha,\xi,g^2)$$

$$L(q^2) \to Z(\lambda,\alpha,\xi,g^2), \quad q^2 \to 0.$$

where Z is any complex number, and $\{q_n^2\}$ is a sequence of points $q_1^2, q_2^2, ..., q_n^2$ along which $q^2 \to 0$, so that $Z = Z(\{q_n^2\})$.

$$D^{INP}_{\mu\nu}(q;\Delta^2) = iT_{\mu\nu}(q)\frac{1}{(q^2)^2} Z(\lambda,\alpha,\xi,g^2)\Delta^2(\lambda,\alpha,\xi,g^2)$$

$$\Delta_R^2 = Z(\lambda,\alpha,\xi,g^2)\Delta^2(\lambda,\alpha,\xi,g^2)$$

Δ_R^2 is the physical mass gap. Due to the WSK theorem, we can always choose such Z in order to make Δ_R^2 positive, finite, gauge-independent, it should exist when $\lambda \to \infty$ and $\alpha \to 0$, etc.

$$Z = Z_1 \times Z_2 \times Z_3 \times ...., \quad Z_m = Z(\{q_n^2\}_m), \; m = 1, 2, 3....$$

$$iT_{\mu\nu}(q)\frac{\Delta_R^2}{(q^2)^2} = iT_{\mu\nu}(q)\frac{\Delta^2}{(q^2)^2} \times \sum_{k=0}^{\infty}\Big(\frac{\Delta^2}{q^2}\Big)^k \Phi_k(\lambda, \alpha, \xi, g^2)$$

$$0 \lesssim q^2 < \infty$$

IR multiplicative renormalization

$$D^{INP}_{\mu\nu}(q;\Delta^2_R) = iT_{\mu\nu}(q)\frac{\Delta^2_R}{(q^2)^2}.$$

$$DT + DRM \longrightarrow (q^2)^{-2} = \frac{1}{\epsilon}\Big[\pi^2\delta^4(q) + O(\epsilon)\Big], \quad \epsilon \longrightarrow 0^+$$

$$(q^2)^{-2-k} = \frac{1}{\epsilon}\Big[a(k)[\delta^4(q)]^{(k)} + O_k(\epsilon)\Big], \quad \epsilon \longrightarrow 0^+$$

$$D^{INP}_{\mu\nu}(q;\Delta^2_R) = \frac{1}{\epsilon}iT_{\mu\nu}(q)\Delta^2_R\delta^4(q).$$

Let us define the IR renormalized mass gap as follows:

$$\Delta_R^2 = X(\epsilon)\bar{\Delta}_R^2 = \epsilon\bar{\Delta}_R^2, \quad \epsilon \to 0^+,$$

where $X(\epsilon)$ is the IRMR constant for the mass gap, and the IR renormalized mass gap $\bar{\Delta}_R^2$ exists as $\epsilon \to 0^+$, by definition.

$$D_{\mu\nu}^{INP}(q) = X(\epsilon)\bar{D}_{\mu\nu}^{INP}(q)$$

The IR and UV renormalized gluon propagator becomes

$$\bar{D}_{\mu\nu}^{INP}(q) \equiv D_{\mu\nu}^{INP}(q; \bar{\Delta}_R^2) = iT_{\mu\nu}(q)\bar{\Delta}_R^2\delta^4(q),$$

The general criterion of gluon confinement

$$(\mathbf{A}). \quad D^{INP}_{\mu\nu}(q; \bar{\Delta}^2_R) = \epsilon \times iT_{\mu\nu}(q) \frac{\bar{\Delta}^2_R}{(q^2)^2}, \quad \epsilon \to 0^+$$

1). $q^2 \to 0$ as independent skeleton loop variable

$$D^{INP}_{\mu\nu}(q; \bar{\Delta}^2_R) = iT_{\mu\nu}(q) \bar{\Delta}^2_R \delta^4(q),$$

2). $q^2 \to 0$ as a free particle momentum

$$D^{INP}_{\mu\nu}(q; \bar{\Delta}^2_R) = \epsilon \times iT_{\mu\nu}(q) \frac{\bar{\Delta}^2_R}{(q^2)^2} \sim \epsilon, \quad \epsilon \to 0^+$$

(B). No free gluons in INP QCD

Physical limits

$$(\mathbf{1}).\ Z(\lambda,\alpha_s(\lambda))\Delta^2(\lambda,\alpha_s(\lambda)) = \Delta_R^2, \quad \lambda\to\infty, \quad \alpha_s(\lambda)\to\infty.$$

$$(\mathbf{2}).\ Z(\lambda,\alpha_s(\lambda))\Delta^2(\lambda,\alpha_s(\lambda)) = 0, \quad \lambda\to\infty, \quad \alpha_s(\lambda)\to 1.$$

$$(\mathbf{3}).\ Z(\lambda,\alpha_s(\lambda))\Delta^2(\lambda,\alpha_s(\lambda)) = \Lambda_{PT}^2, \quad \lambda\to\infty, \quad \alpha_s(\lambda)\to 0.$$

AF without renormalization group

$$d^{PT}(q^2) = \frac{1}{1 + \Pi^s(q^2)}$$

$$\alpha_s(q^2; \Lambda^2) = \frac{\alpha_s(\lambda)}{1 + b\alpha_s(\lambda)\ln(q^2/\Lambda^2)},$$

$$\Lambda^2 = f(\lambda)\Delta^2(\lambda, \alpha_s(\lambda)),$$

$$\Lambda^2 = f(\lambda)Z(\lambda, \alpha_s(\lambda))\Lambda^2_{QCD}, \quad \lambda \to \infty, \quad \alpha_s(\lambda) \to 0.$$

$$\alpha_s(q^2) = \frac{\alpha_s}{1 + b\alpha_s \ln(q^2/\Lambda^2_{QCD})},$$

$$\alpha_s = \frac{\alpha_s(\lambda)}{1 - b\alpha_s(\lambda)\ln(fZ)}, \quad \lambda \to \infty, \quad \alpha_s(\lambda) \to 0.$$

$$\ln(fZ) = \frac{\alpha_s - \alpha_s(\lambda)}{\alpha_s b\alpha_s(\lambda)} \to \frac{1}{b\alpha_s(\lambda)}, \quad \lambda \to \infty, \quad \alpha_s(\lambda) \to 0,$$

$$\lim_{(\Lambda,\lambda)\to\infty} \Lambda^2 \exp\left(-\frac{1}{b\alpha_s(\lambda)}\right) = \Lambda^2_{QCD}, \quad \alpha_s(\lambda) \to 0,$$

$$\alpha_s(q^2) = \frac{1}{b\ln(q^2/\Lambda^2_{QCD})},$$

General conclusions

$$\Lambda^2_{INP} \xleftarrow[\infty \leftarrow \lambda]{\infty \leftarrow \alpha_s(\lambda)} \Delta^2(\lambda, \alpha_s(\lambda)) \xrightarrow[\lambda \to \infty]{\alpha_s(\lambda) \to 0} \Lambda^2_{PT},$$

$$INP\ QCD \Longleftarrow QCD \Longrightarrow PT\ QCD,$$

Yang-Mills Existence And Mass Gap: Prove that for any compact simple gauge group G, quantum Yang-Mills theory on $\mathbf{R^4}$ exists and has a mass gap $\Delta > 0$.

Mass Gap Existence And Gluon Confinement: If quantum Yang-Mills theory with compact simple gauge group $G = SU(3)$ exists on $\mathbf{R^4}$, then undergoing the phase transition in the strong coupling regime it becomes INP YM, which has a physical mass gap and confines gluons.

Some important features of INP QCD are:

1. Its full gluon propagator is exactly fixed.
2. It has a physical mass gap.
3. It confines "dressed" gluons in asymptotic states.
4. It has no free gluons.
5. It requires subtractions at all levels.

The INP gluon propagator is generalized function, i.e., it is distribution. So all the relations involving distributions should be considered under corresponding integrals, taking into account the smoothness properties of the corresponding space of test functions. The space in which our generalized functions are continuous linear functionals is K, that's the space of infinitely differentiable functions having compact support, i.e., they are zero outside some finite region (different for each test function).

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