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A duality for the S-matrix

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A Duality For The S-Matrix

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Based on collaborations with N. Arkani-Hamed , J. Kaplan, C. Cheung,

J. Bourjaily and J. Trnka.

Lagrangian Description Of The S-Matrix

- Theories in 4D asymptotically flat space-time: S-matrix.
- Main examples: Yang-Mills and Gravity.
- Locality of interactions in the underlying space-time.

Main Motivation For A Dual Theory

Find a formulation equivalent to the standard one but which makes no use of local evolution through space-time. Such a reformulation might then serve as the springboard for understanding situations where a notion of space-time or locality are not at hand.

Why Search For A Dual Theory?

Consider the scattering amplitudes of gluons in Yang-Mills theory. We have a lagrangian and Feynman rules:

Color Ordering (Berends, Giele, Mangano, Parke, Xu)

$$\mathcal{A}_n(\{p_i^\mu, \epsilon_i^\mu, a_i\}) = \sum_{\sigma \in S_n/Z_n} \mathrm{Tr}(T^{a_{\sigma(1)}} \ldots T^{a_{\sigma(n)}}) A(p^\mu_{\sigma_{(1)}}, \epsilon^\mu_{\sigma_{(1)}}, \ldots, p^\mu_{\sigma_{(n)}}, \epsilon^\mu_{\sigma_{(n)}})$$

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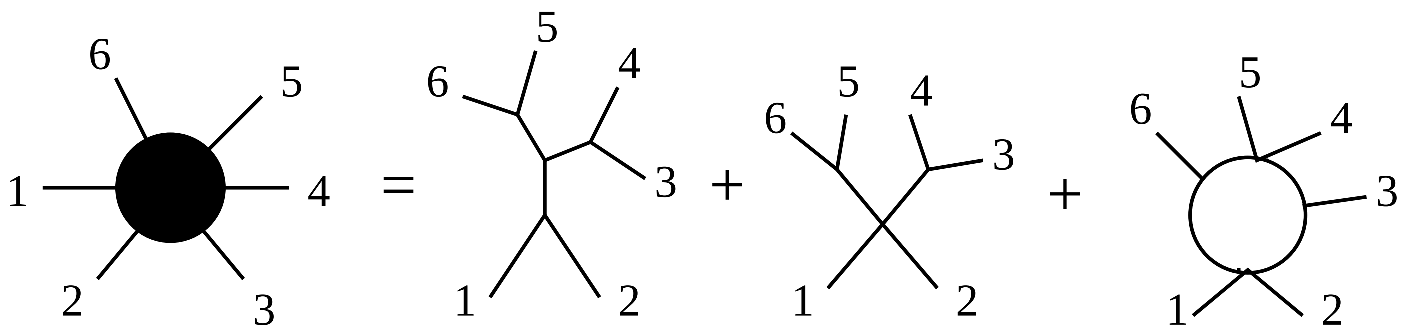


Each diagram makes: Space-time locality manifest and (can be chosen to make) Lorentz invariance manifest.

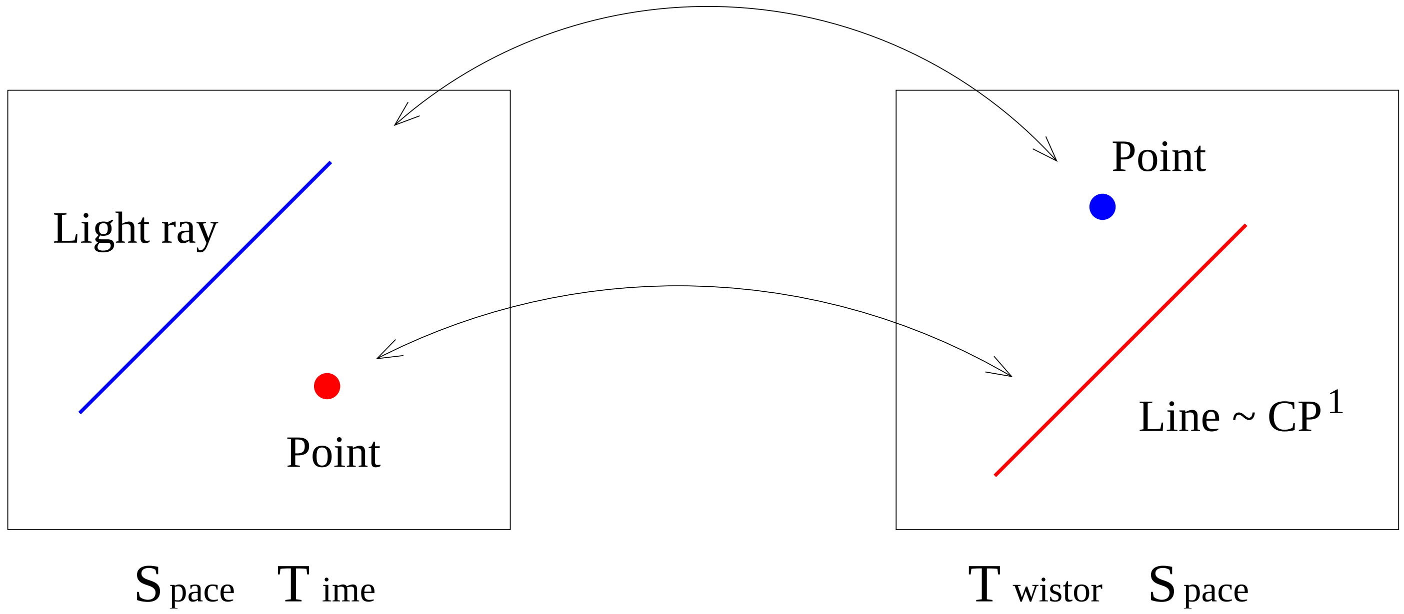
Price: Large amount of redundancy and expressions are forbiddingly long.

They also hide a new symmetry!

Remarkable properties: Tree-Level

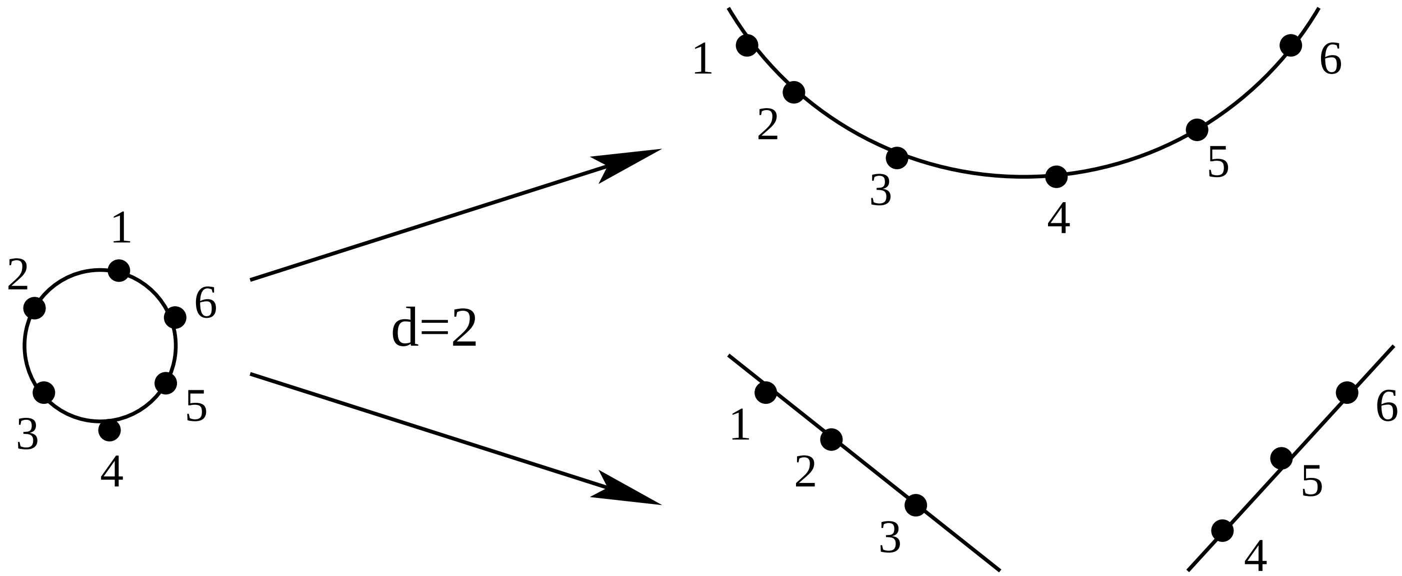
- Twistor space localization. (Witten 2003)

  Twistor space (Penrose 1960s)



Remarkable properties: Tree-Level

- Twistor space localization. (Witten 2003) (Connected vs. Disconnected)



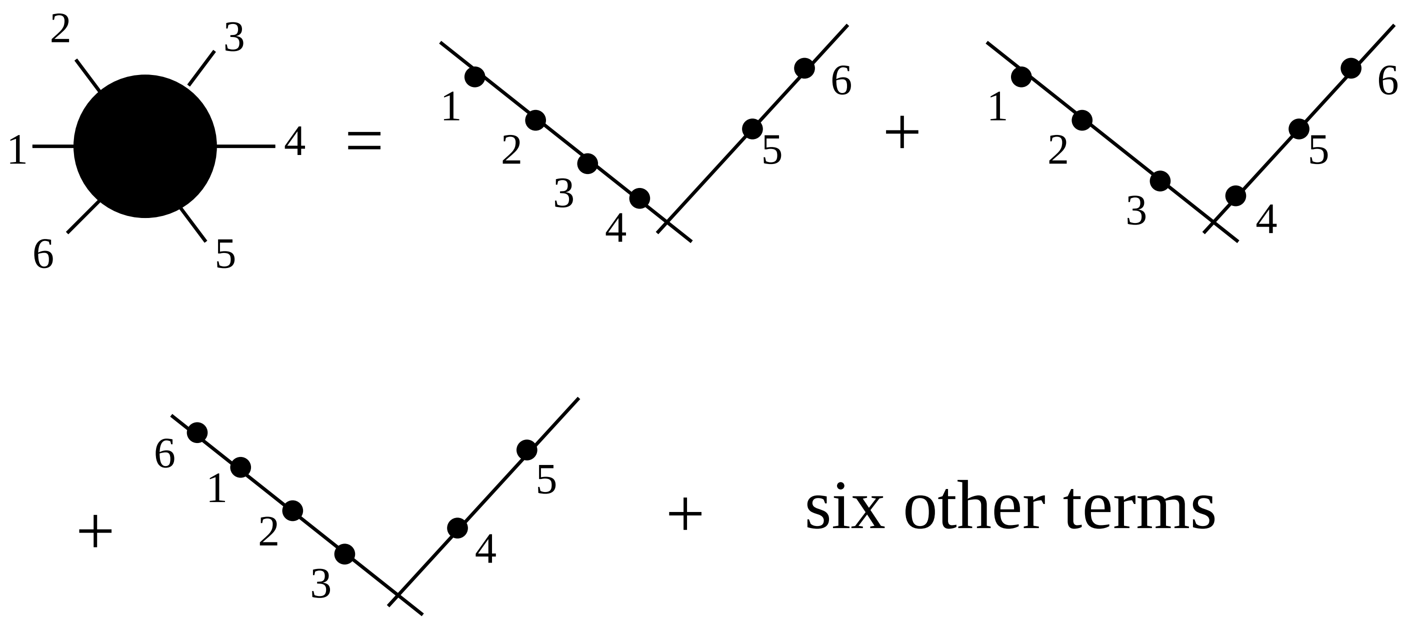
The degree of the map is related to the helicity of the gluons.

Remarkable properties: Tree-Level

- Twistor space localization. Connected formula. (Roiban, Spradlin, Volovich 2004)
- CSW expansion. (F.C.,Svrcek, Witten 2004)
  - New diagrammatic expansion. (Alternative to Feynman diagrams)
  - Local Space-time description. (Mansfield, Mason, Skinner, Boels, 2004-2006)
  - Price: One has to choose a light-like direction. Non-manifestly Lorentz invariant.

Remarkable properties: Tree-Level

- Twistor space localization.
- CSW expansion. (F.C.,Svrcek, Witten 2004)

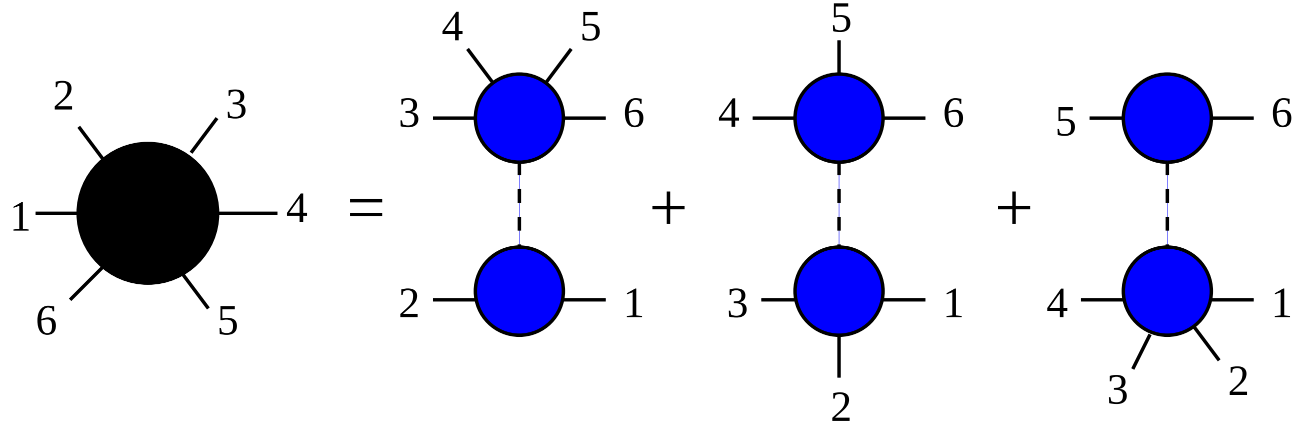


Remarkable properties: Tree-Level

- Twistor space localization.
- CSW expansion.
- BCFW Recursion Relations (Britto, F.C., Feng, Witten 2005)
  - On-Shell Recursion relations! Combine on-shell amplitudes to produce others.
  - Sum over only certain factorization channels. Very simple forms for the amplitudes.
  - Price: Non-local spurious poles.
  - The non-locality in each BCFW term asks for a description in terms of a **dual formulation** where space-time is not fundamental!

Remarkable properties: Tree-Level

- Twistor space localization.
- CSW expansion.
- BCFW Recursion Relations



Remarkable properties: Tree-Level

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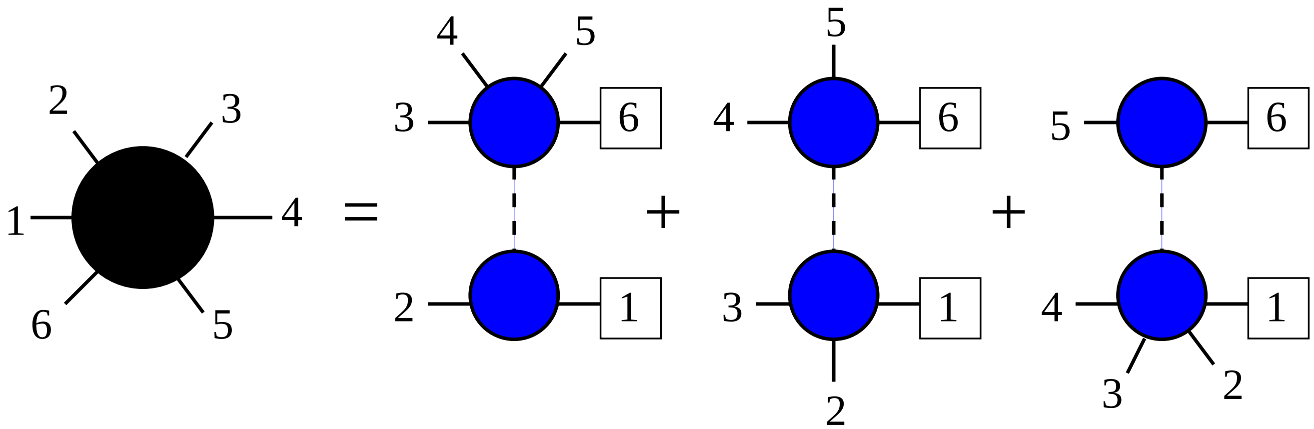
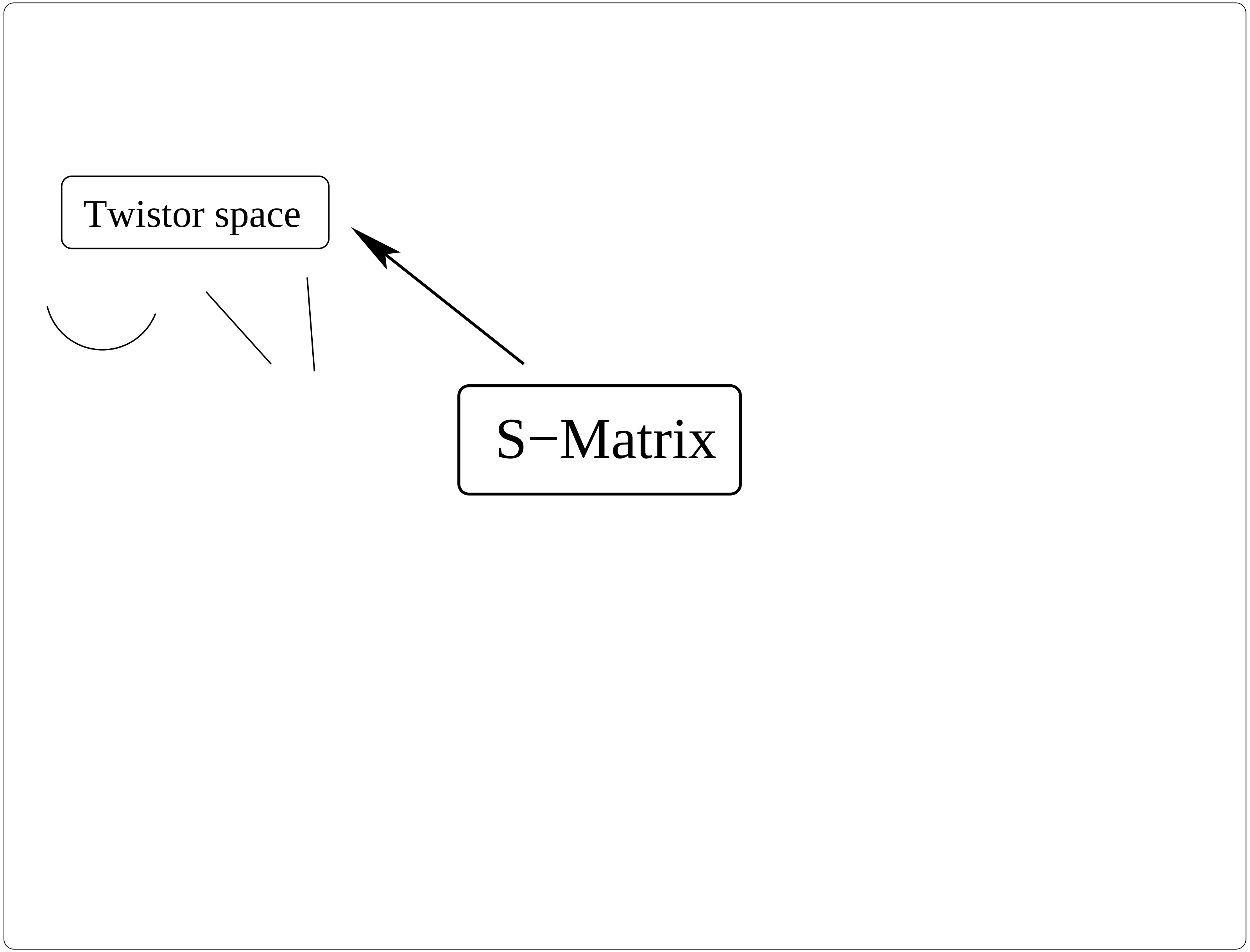
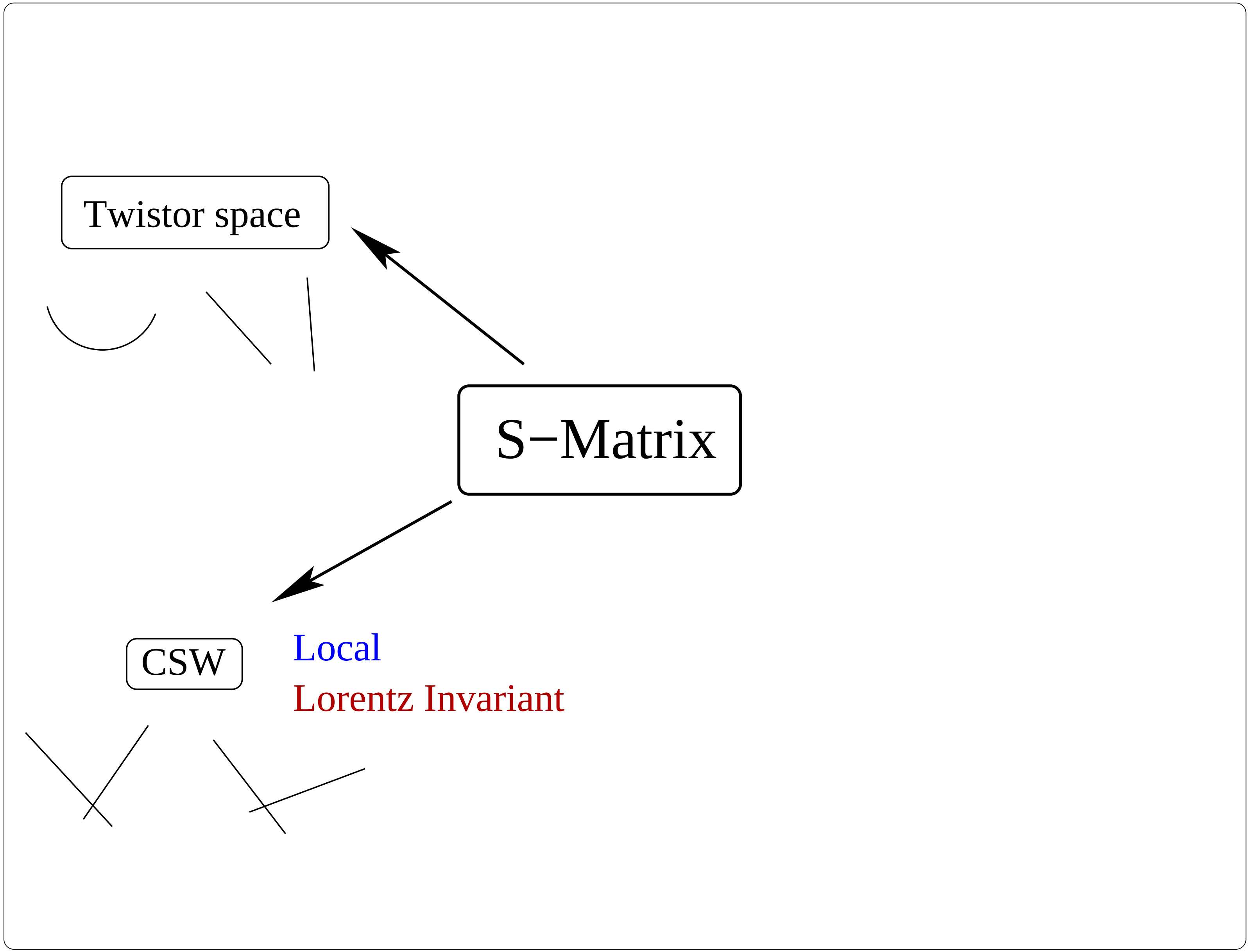
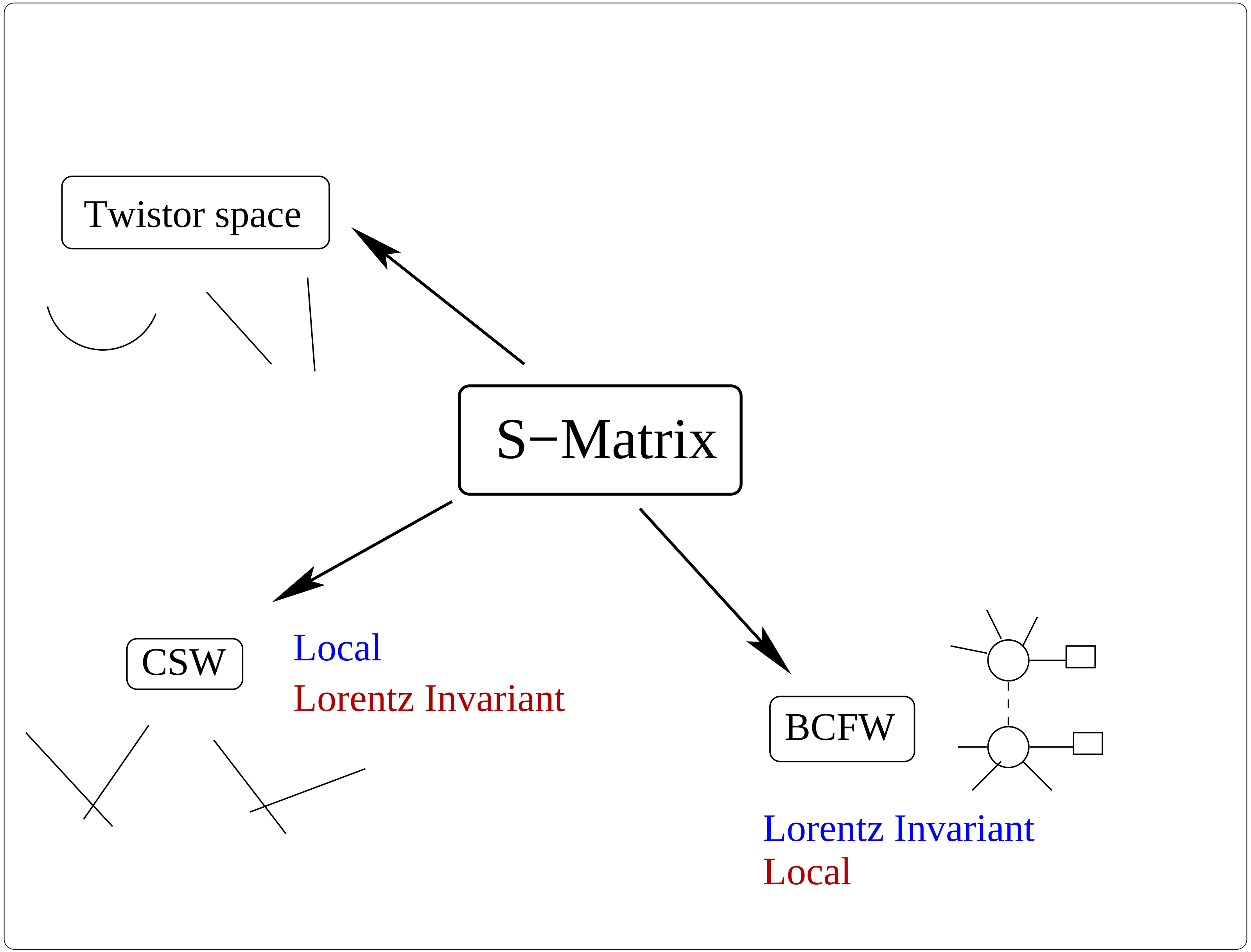


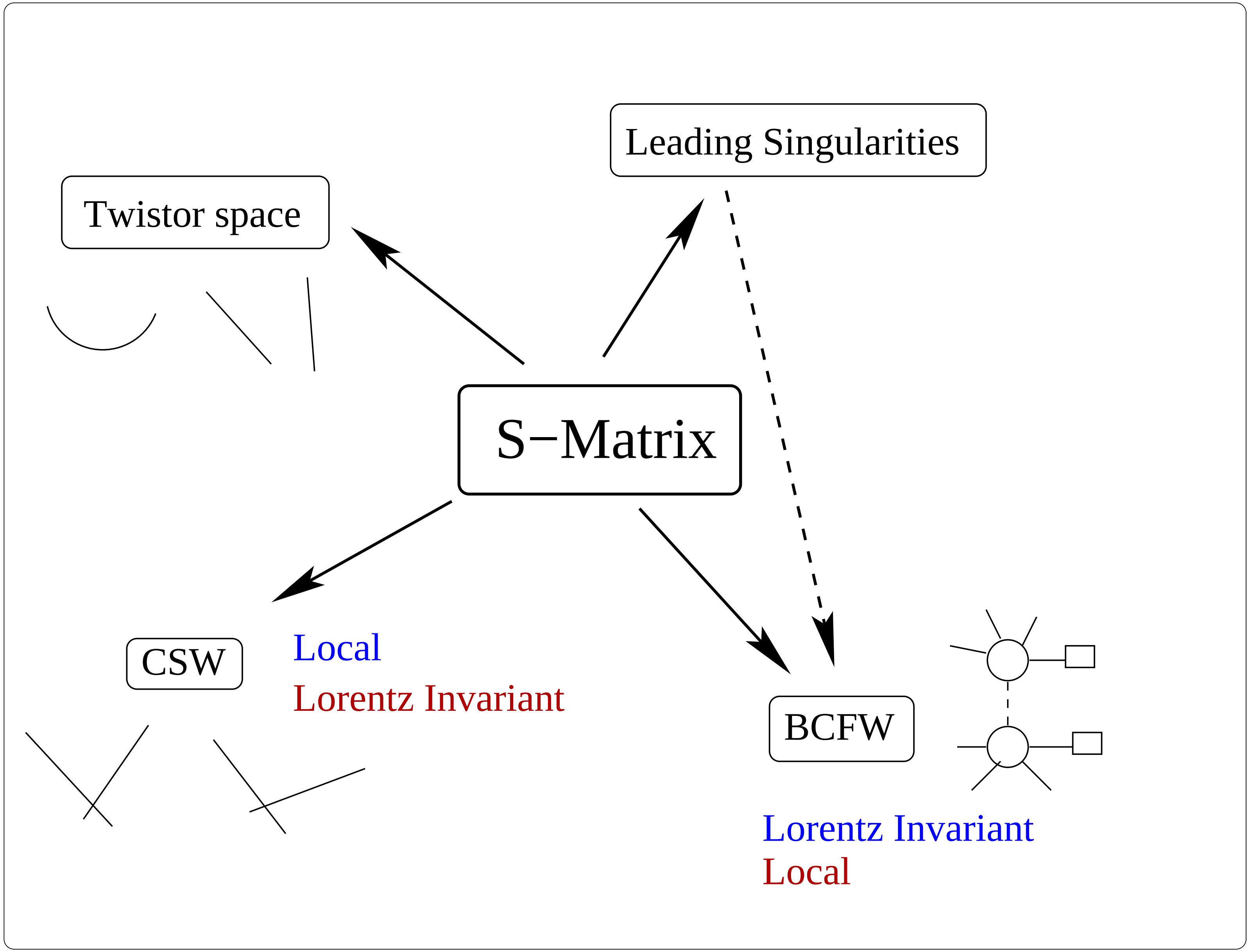
Two legs are made non-local!

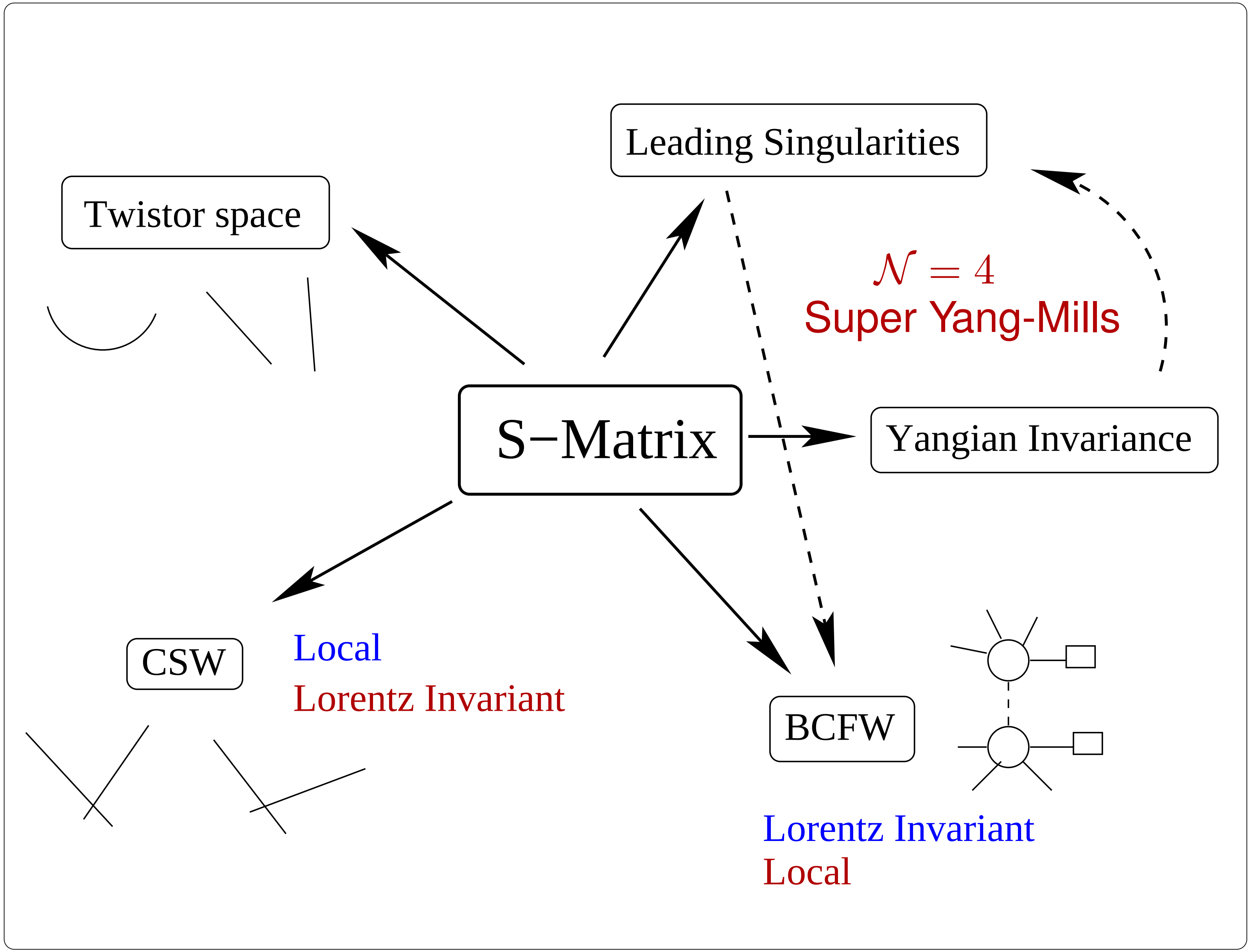
S−Matrix


Twistor space
S−Matrix


Twistor space
S−Matrix
CSW
Local
Lorentz Invariant


Twistor space
S−Matrix
CSW
Local
Lorentz Invariant
BCFW
Lorentz Invariant
Local


Leading Singularities
Twistor space
S−Matrix
CSW
Local
Lorentz Invariant
BCFW
Lorentz Invariant
Local



Leading Singularities
Twistor space
$\mathcal{N} = 4$
Super Yang-Mills
S−Matrix
Yangian Invariance
CSW
Local
Lorentz Invariant
BCFW
Lorentz Invariant
Local

Classification Of Amplitudes

Scattering Amplitudes of n gluons are naturally classified by the number of minus helicity gluons k. Since gluons can only carry helicity ± 1, there are $n-k$ plus helicity gluons.

This means that we can talk about: $A_{n,k}$

If we want to include more information we can write

$$A_{n,k}^{i_1,i_2,\ldots,i_k}$$

to indicate an amplitude where gluons $\{i_1, i_2, \ldots, i_k\}$ have $h=-1$ while the remaining $n-k$ gluons have $h=+1$.

Classification Of Amplitudes

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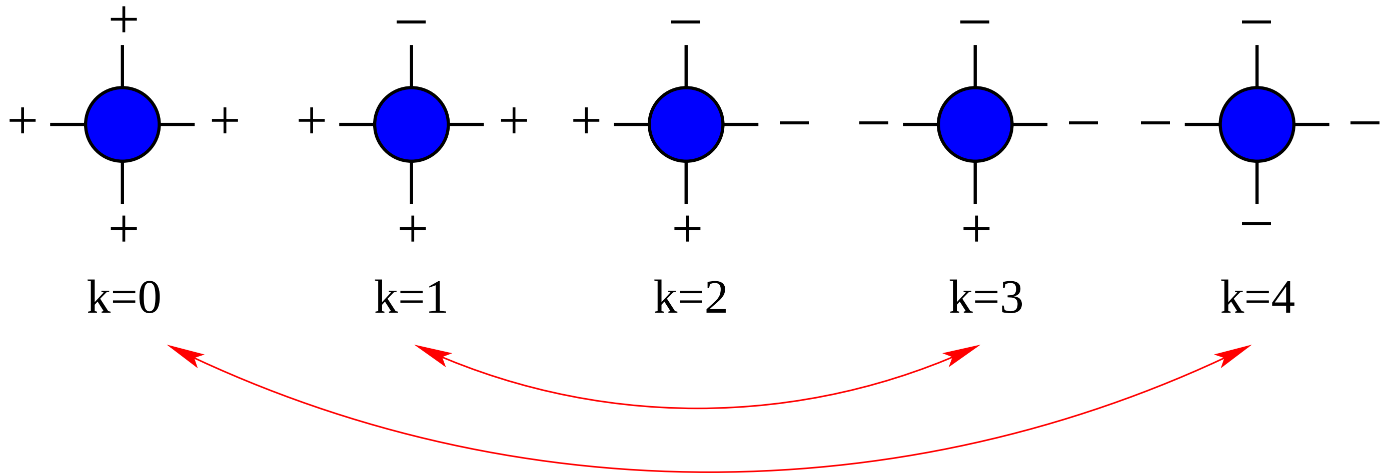
E.g: $n = 4$



Parity! $k \leftrightarrow n - k$

The Proposal

The Grassmannian Theory $\mathcal{L}_{n,k}$

The Grassmannian $G(k,n)$: Space of k-planes containing the origin in $\mathbb{C}^n$.

$$C = \begin{pmatrix} c_{11} & c_{12} & c_{13} & \cdots & c_{1k} & c_{1k+1} & \cdots & c_{1n-1} & c_{1n} \\ c_{21} & c_{22} & c_{23} & \cdots & c_{2k} & c_{2k+1} & \cdots & c_{2n-1} & c_{2n} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ c_{k1} & c_{k2} & c_{k3} & \cdots & c_{kk} & c_{kk+1} & \cdots & c_{kn-1} & c_{kn} \end{pmatrix}$$

Modulo the action of $GL(k)$ on the right.

Plucker coordinates: (Invariant under $SL(k)$)

$$(1,2,\ldots,k) = \det M_{12\ldots k}$$

The Grassmannian Theory $\mathcal{L}_{n,k}$

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Claim: $A_{n,k}$ is related to $G(k,n)$!

Precise Claim:

(Arkani-Hamed, F.C., Cheung and Kaplan, 2009)

$$\mathcal{L}_{n,k}^{i_1,i_2,\ldots,i_k} = \frac{1}{\mathrm{Vol(Gl(k))}} \int \frac{d^{nk}c_{ij}\ \delta(C\cdot\Sigma)\delta(C^{\perp}\cdot\Gamma)\ (i_1,i_2,\ldots,i_k)^4}{(1,2,\ldots,k)(2,3,\ldots,k+1)\ldots(n,1,\ldots,k-1)}$$

where Γ and Σ are given 2-planes in $\mathbb{C}^n$.

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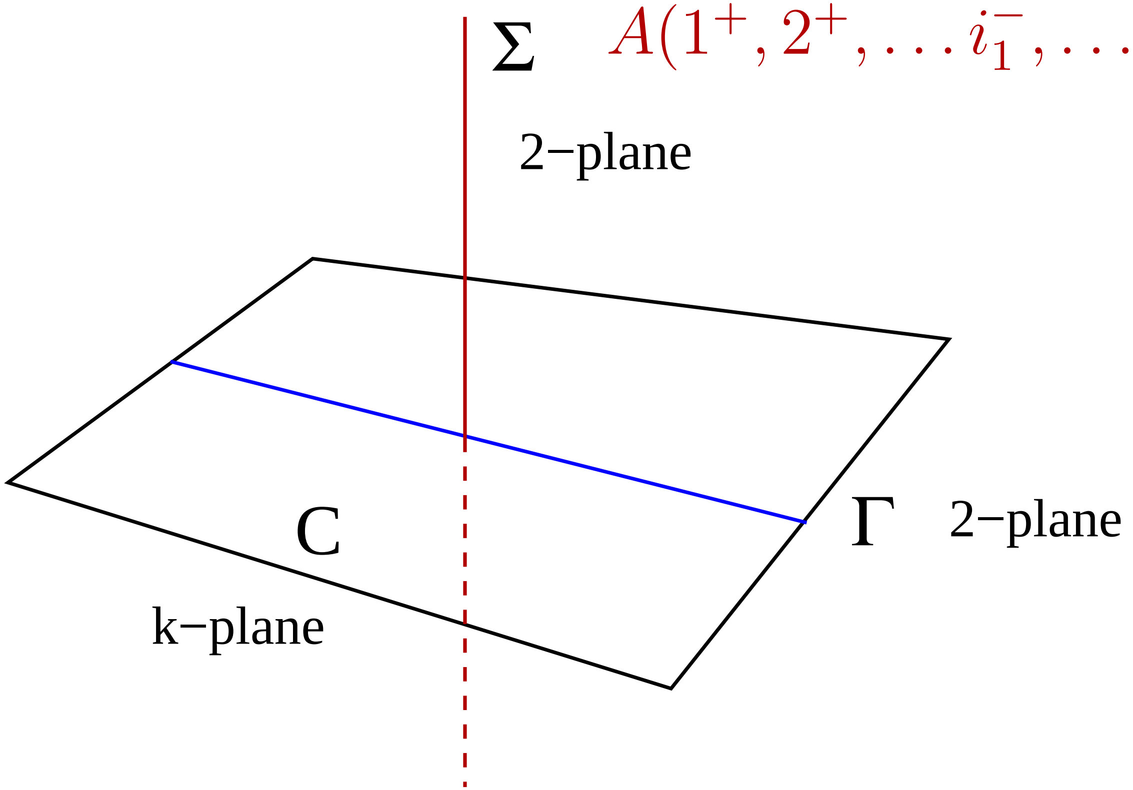
QCD

$$A(1^+,2^+,\ldots i_1^-,\ldots i_2^-,\ldots,i_k^-,\ldots n^+)$$



Precise Claim:

$$\mathcal{L}_{n,k} \qquad = \frac{1}{\mathrm{Vol(Gl(k))}} \int \frac{d^{nk}c_{ij}\; \delta(C\cdot\Sigma)\delta(C^{\perp}\cdot\Gamma)}{(1,2,\ldots,k)(2,3,\ldots,k+1)\ldots(n,1,\ldots,k-1)}$$

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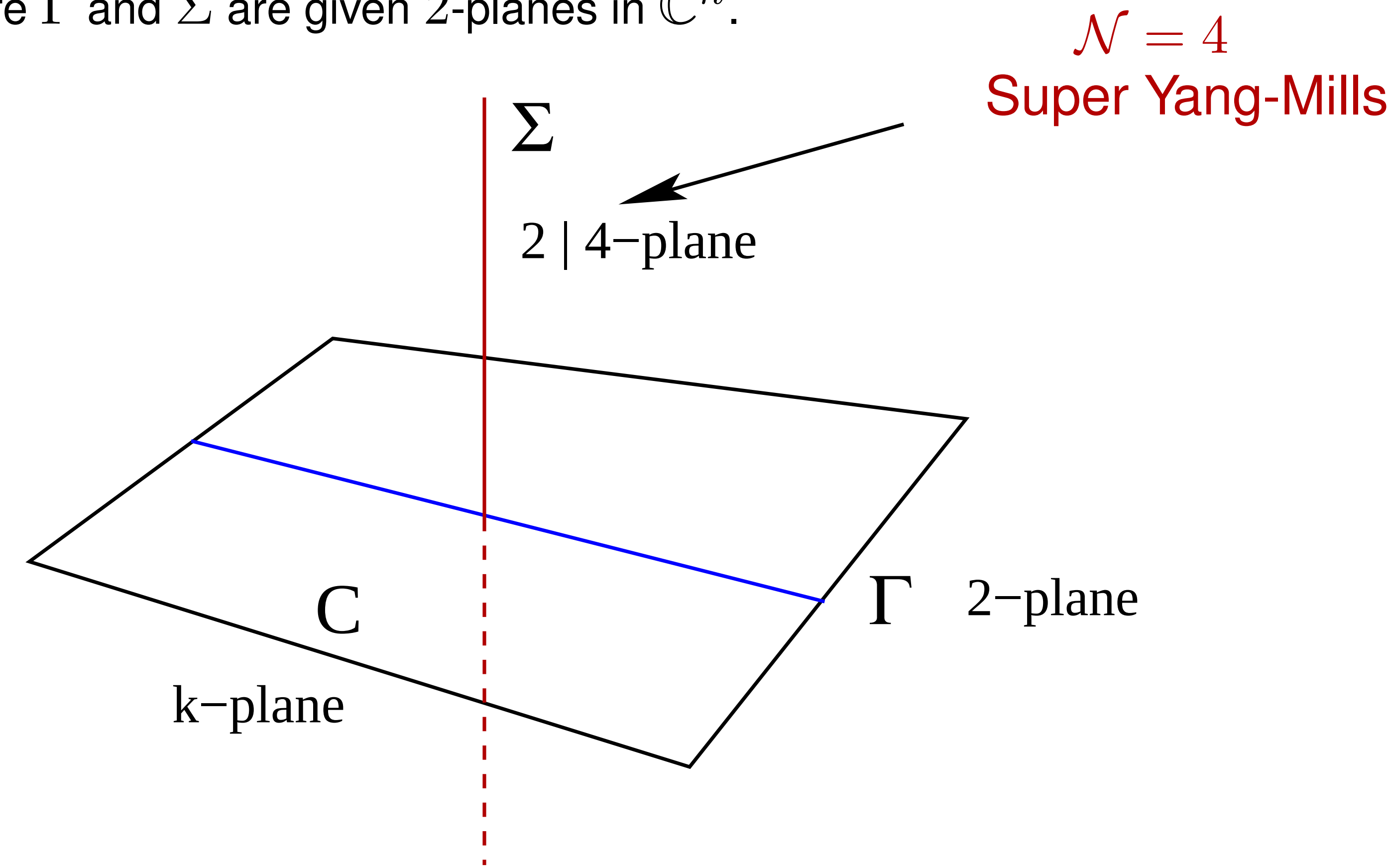

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- All kinematical information contained in $\{p_i^\mu, \epsilon_i^\mu\}$ is encoded in the two planes Σ and Γ.
- The complexified Lorentz group $SL(2,\mathbb{C})\times SL(2,\mathbb{C})$ acts on each of the 2-planes naturally as the subgroups of the corresponding $GL(2,\mathbb{C})$.
- Momentum conservation $\delta^4(p_1+p_2+\ldots+p_n)$ is equivalent to $\Sigma\cdot\Gamma=0$.

What does one do with it?

In practice one uses the $GL(k)$ invariance to pick a “gauge” and solve the delta functions.

$GL(k)$ removes k^2 variables $\Rightarrow$

$k \times n - k^2 = k(n-k) = \mathrm{dim} G(k,n)$

$\delta(C \cdot \Sigma)\delta(C^{\perp} \cdot \Gamma)$ imply

$k(n-k) - (2n-4) = (k-2)(n-k-2)$

Therefore the integral is only over:

$(k-2)(n-k-2)$ variables.

Proposal:

$\mathcal{L}_{n,k}$ is to be interpreted as a multidimensional **contour** integral in $\mathbb{C}^{(k-2)(n-k-2)}$ and its residues are the building blocks of the amplitudes.

Example I:

Consider the case $k = 0$ and $k = 1$ and any n. $(\delta(C^{\perp} \cdot \Gamma))$

$$\mathcal{L}_{n,k} = 0$$

It turns out that

$$A_{n,0} = A(1^+, 2^+, ..., n-1^+, n^+) = 0$$

$$A_{n,1} = A(1^-, 2^+, ..., n-1^+, n^+) = 0$$

Example II:

Consider $k = 2$ and any n. The number of integrations left after the delta functions is zero $(k-2)(n-k-2)$. This means that the answer is just the jacobian.

$$\mathcal{L}_{n,2} = A^{\rm tree}_{\rm Parke-Taylor}$$

The precise form was conjectured by PT and then proven by Berends and Giele in the 1980s

At the time it came as a surprise that the sum over a large number of Feynman diagrams could be rewritten as a one-term expression! For us this one-term object is nothing but a jacobian.

Simple observation:

How about $k = n$, $k = n - 1$ and $k = n - 2$?

Recall that $G(k, n)$ is isomorphic to $G(n - k, n)$.

This is nothing but parity in terms of the amplitudes.

Example III:

Consider $k = 3$ and $n = 6$. Therefore $(k-2)(n-k-2) = 1$.

$$\mathcal{L}_{6,3} = \int \frac{(135)^4 d\tau}{(123)(234)(345)(456)(561)(612)}$$

where each of the minors is linear in τ. In other words, $(i, i+1, i+2) = a_i \tau + b_i$ where a_i and b_i depend on the data of the external particles.

$$A(1^-, 2^+, 3^-, 4^+, 5^-, 6^+)$$

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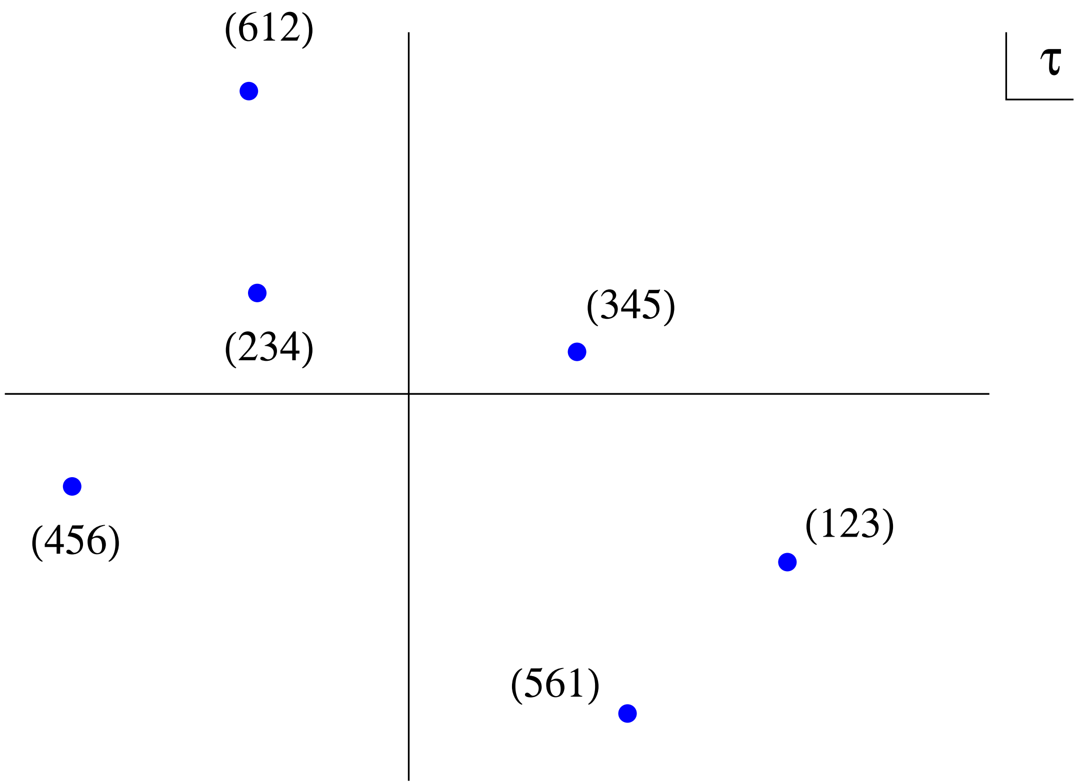

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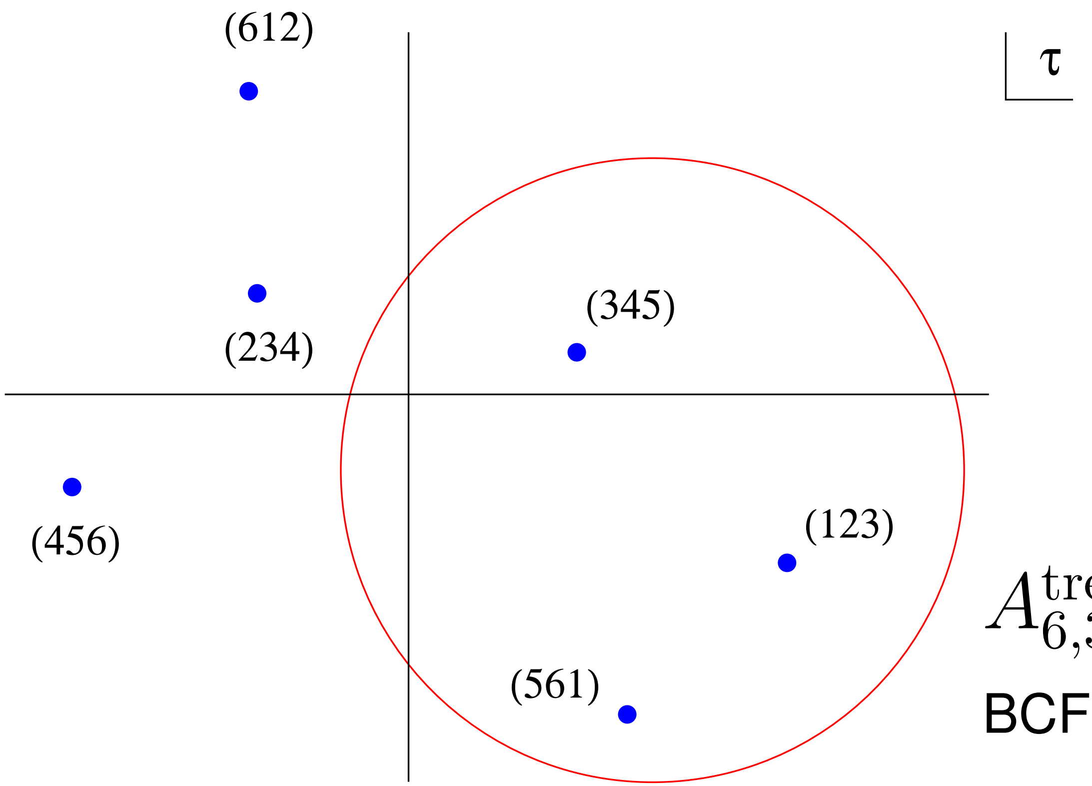

$A_{6,3}^{\text{tree}}$

BCFW Form

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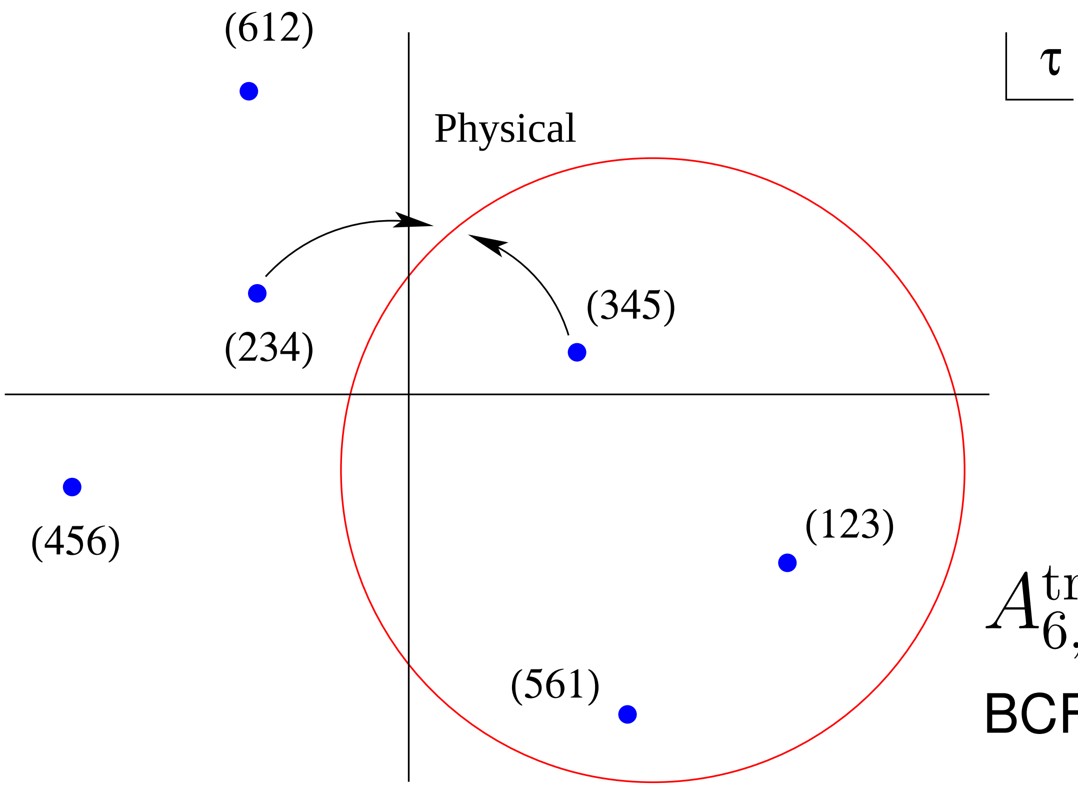

$A^{\text{tree}}_{6,3}$

BCFW Form

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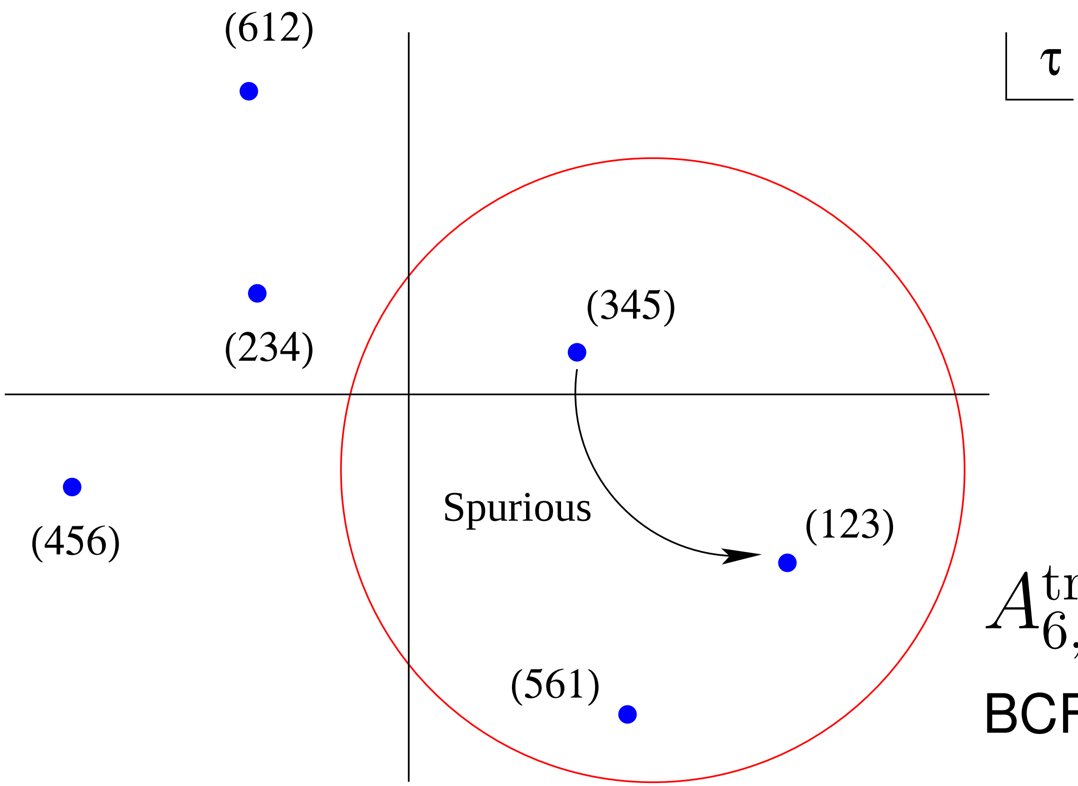

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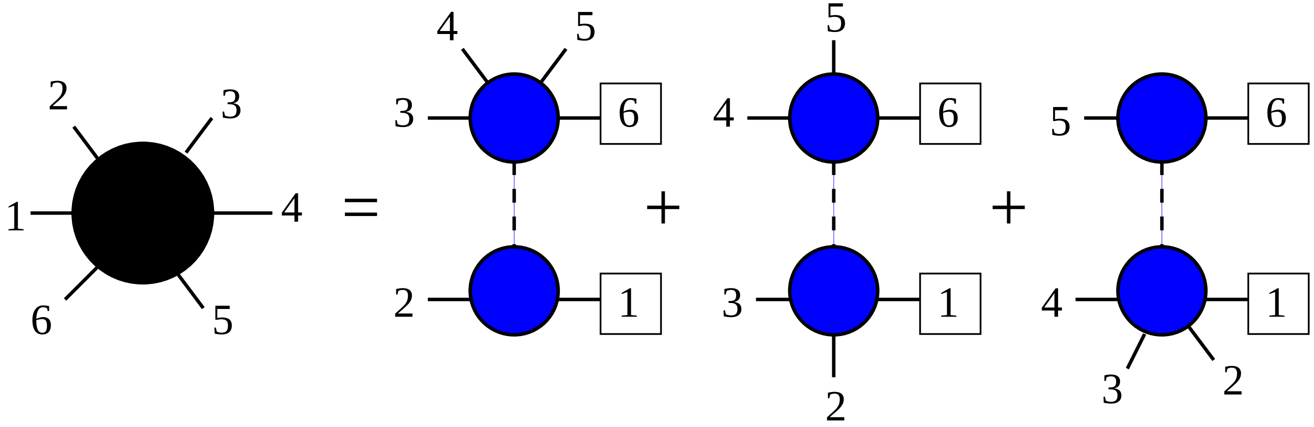
Conclusion: Absence of spurious poles is guaranteed by a residue theorem! First hint that local space-time physics is arising in a novel way.

For more than six particles one needs many relations and all of them have been shown to be residue theorems!

A Short Technical Aside

Q: Why are BCFW terms appearing as residues of $\mathcal{L}_{n,k}$?



A Short Technical Aside

Q: Why are BCFW terms appearing as residues of $\mathcal{L}_{n,k}$?

Part I: Take $\mathcal{L}_{n,k}$ to (dual) super twistor space. A dual super twistor is given by a vector $\mathcal{W} \in \mathbb{C}^{4|4}$. Each external particle is associated to one $\mathcal{W}$ and one finds that if we make a $4|4$-plane in $\mathbb{C}^{n|4}$ by

$$\Phi_W = (\mathcal{W}_1, \mathcal{W}_2, \ldots, \mathcal{W}_n)$$

$$\tilde{\mathcal{L}}_{n,k} = \frac{1}{\mathrm{Vol(GL(k))}} \int \frac{d^{nk} c_{ij} \prod_{\alpha=1}^{k} \delta\left(C \cdot \Phi_W\right)}{(12\ldots k)(23\ldots k+1)\ldots(n1\ldots k-1)}$$

This form implies that all residues of this object are $SL(4|4)$ invariant.

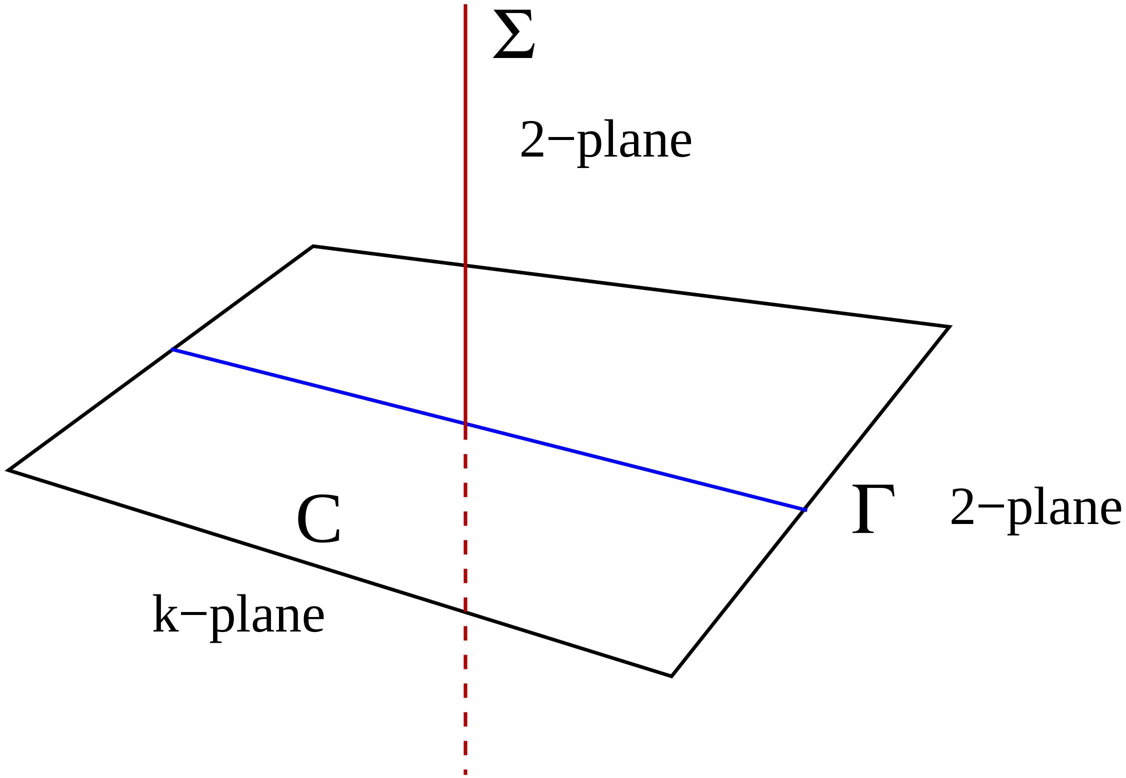
What is the physical meaning of this group?

It is nothing but the group of superconformal transformations!

This means that each residue is **superconformal invariant**.

Part II:

Recall that also the physical integration in ordinary momentum space is over a $k-2$ plane that is orthogonal to a given 4-plane!



Part II:

Q: Could this new 4-plane be interpreted as some sort of twistor space?

A:

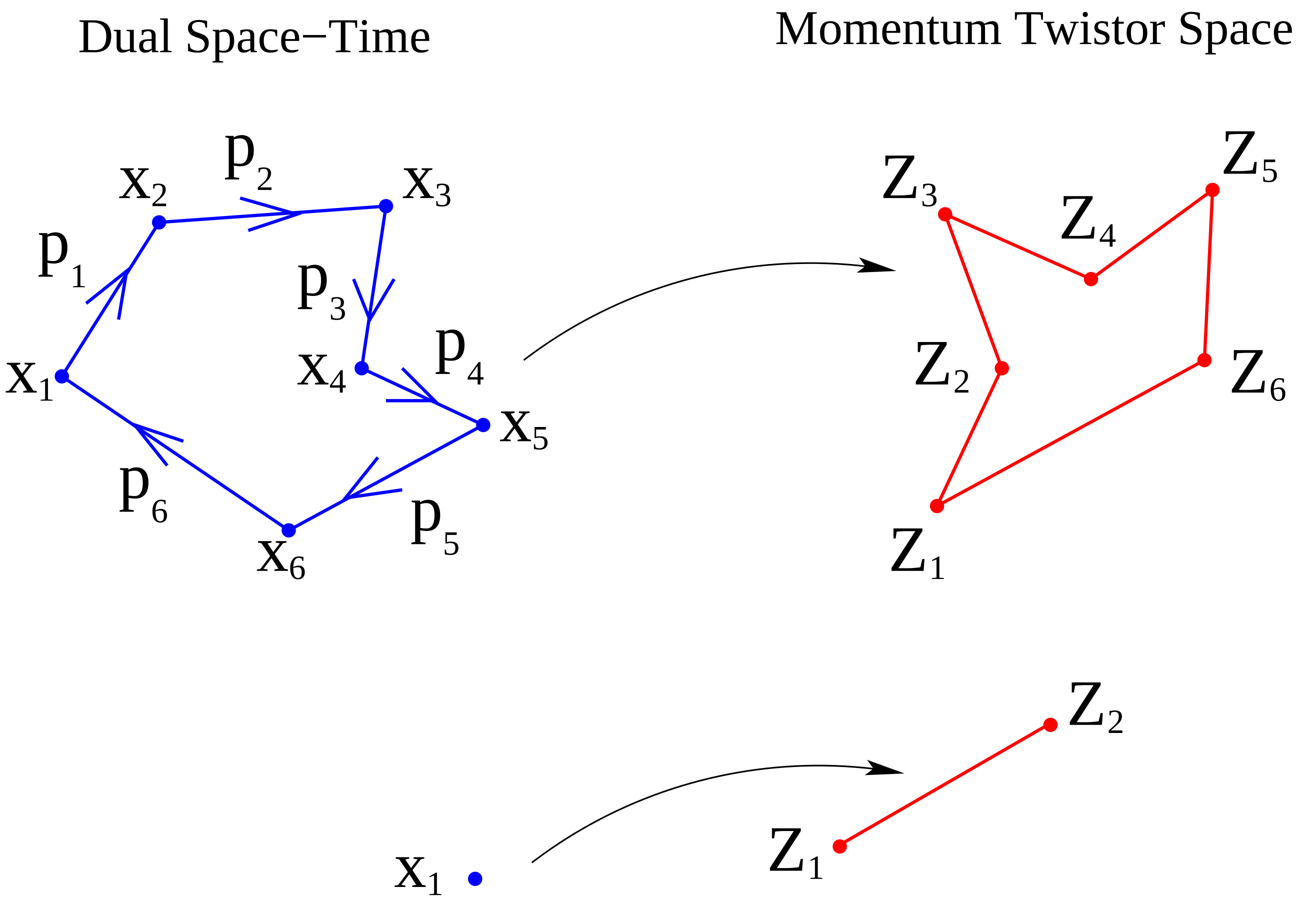
(Maldacena, Alday, Drummond, Korchensky, Sokatchev)

$$+$$

Hodges

$$=$$

Momentun Twistor Space!



Dual Space−Time
Momentum Twistor Space
x_1
x_2
x_3
x_4
x_5
x_6
p_1
p_2
p_3
p_4
p_5
p_6
Z_1
Z_2
Z_3
Z_4
Z_5
Z_6
x_1
Z_1
Z_2

Part II:

Quite surprisingly, one can show that our original $\mathcal{L}_{n,k}$ in momentum space is given by

$$\mathcal{L}_{n,k} = \mathcal{L}_{n,2} \times \tilde{\mathcal{L}}_{n,k-2}(\mathcal{Z}_1, \mathcal{Z}_2, \ldots, \mathcal{Z}_n)$$

where

$$\tilde{\mathcal{L}}_{n,k-2}(\mathcal{Z}) = \frac{1}{\mathrm{Vol(GL(k-2))}} \int \frac{d^{n(k-2)} D_{ij} \prod_{\alpha=1}^{k} \delta\left(D \cdot \Phi_Z\right)}{(12 \ldots k-2)(23 \ldots k-1) \ldots (n1 \ldots k-3)}$$

where $\mathcal{Z}_i$ are momentum twistors! (Mason, Skinner, Arkani-Hamed, F.C. Cheung 2009)

Obs: This makes a second $SL(4|4)$ symmetry manifest.

This is dual super-conformal invariance!

Conclusion:

Residues of $\mathcal{L}_{n,k}$ are invariant under an infinite dimensional symmetry algebra called: **A Yangian!**

Conjecture:

All Yangian invariants (with the correct little group properties) are generated as residues of $\mathcal{L}_{n,k}$.

Corollary:

Since BCFW terms are Yangian invariants then they coincide with some residue of $\mathcal{L}_{n,k}$.

(Drummond, Henn, Plefka, Korchemsky, Sokatchev, Brandhuber, Heslop, Travaglini, Alday, Maldacena, Elvang, Freedman, Kiermaier 2007-2010)

End of Technical Aside

Emergence of Local Space-Time

(Arkani-Hamed, Bourjaily, F.C. and Trnka 2009)

Emergence of Local Space-Time

Blowing Residues Apart

Consider the six particle example $n = 6, k = 3$:

$$\mathcal{L}_{6,3} = \frac{1}{\mathrm{Vol}(\mathrm{GL}(3))} \int \frac{d^{18}c_{ij}\delta^6(C \cdot \Sigma)\delta^6(C^T \cdot \Gamma)}{(123)(234)(345)(456)(561)(612)}$$

Counting: $18 - 9 - (2 \times 6 - 4) = 1$. We want to think about one of the delta functions as a pole. The only way to do this is to break manifest Lorentz invariance by introducing a null direction ζ.

$$\mathcal{L}_{6,3} = \int \frac{d^2\tau}{(123)(234)(345)(456)(561)(612)(N)_{(\zeta)}}$$

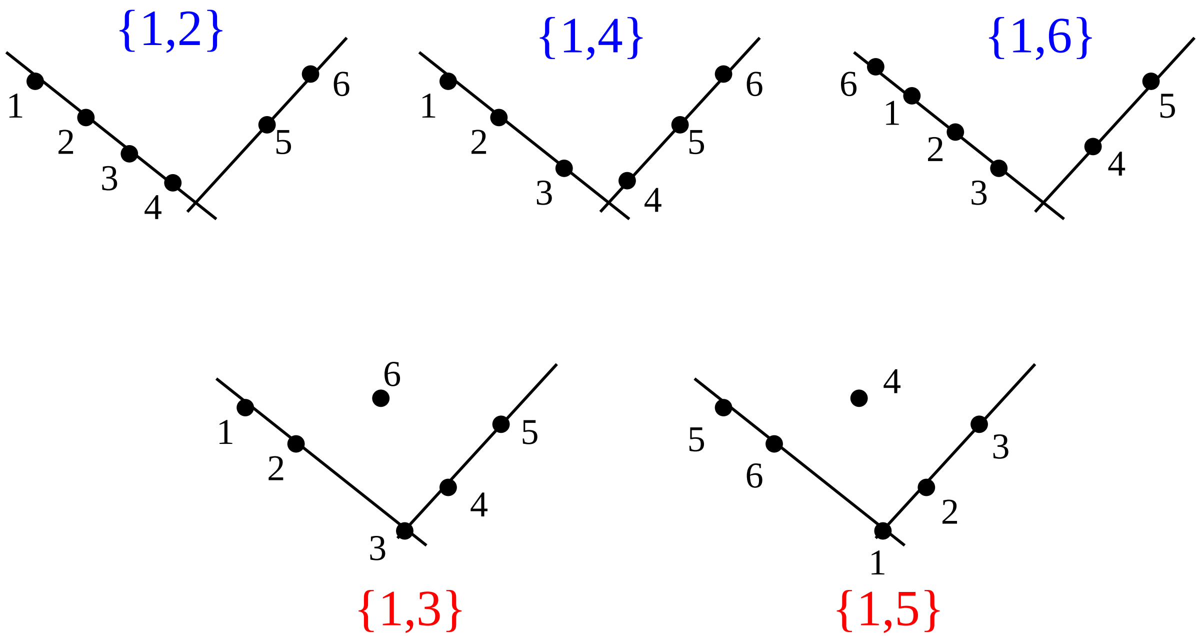
This is a contour integral in two complex variables!

$$\mathcal{L}_{6,3} = \int \frac{d^2\tau}{(123)(234)(345)(456)(561)(612)(N)_{(\zeta)}}$$

Recall that $A^{\text{tree}}_{6,3} = \{(123)\} + \{(345)\} + \{(561)\}$. Now we have:

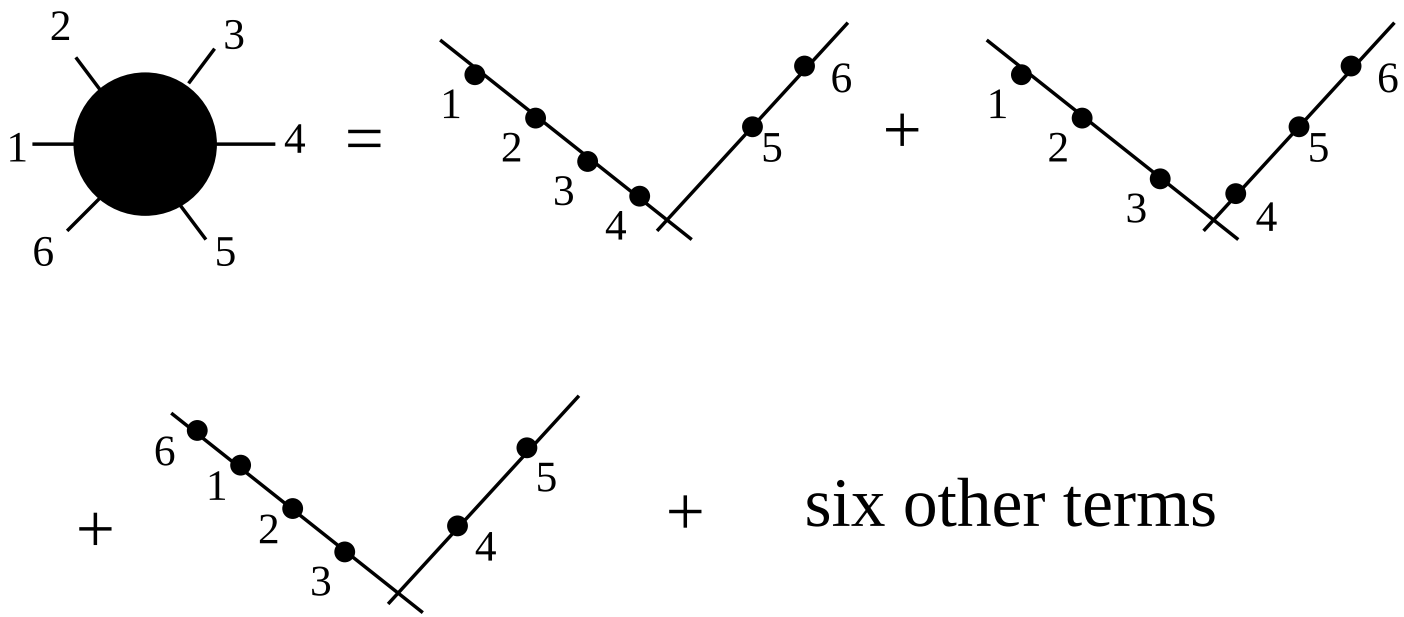
$$A^{\text{tree}}_{6,3} = \{(123), N\} + \{(345), N\} + \{(561), N\}$$

Let's use a residue theorem to blow $\{(123), N\}$ apart into pieces!



Blowing apart all three residues one finds that all non-local poles cancel in pairs and only 9 terms survive.

These 9 terms are precisely the 9 CSW diagrams!



Blowing apart all three residues one finds that all non-local poles cancel in pairs and only 9 terms survive.

These 9 terms are precisely the 9 CSW diagrams.

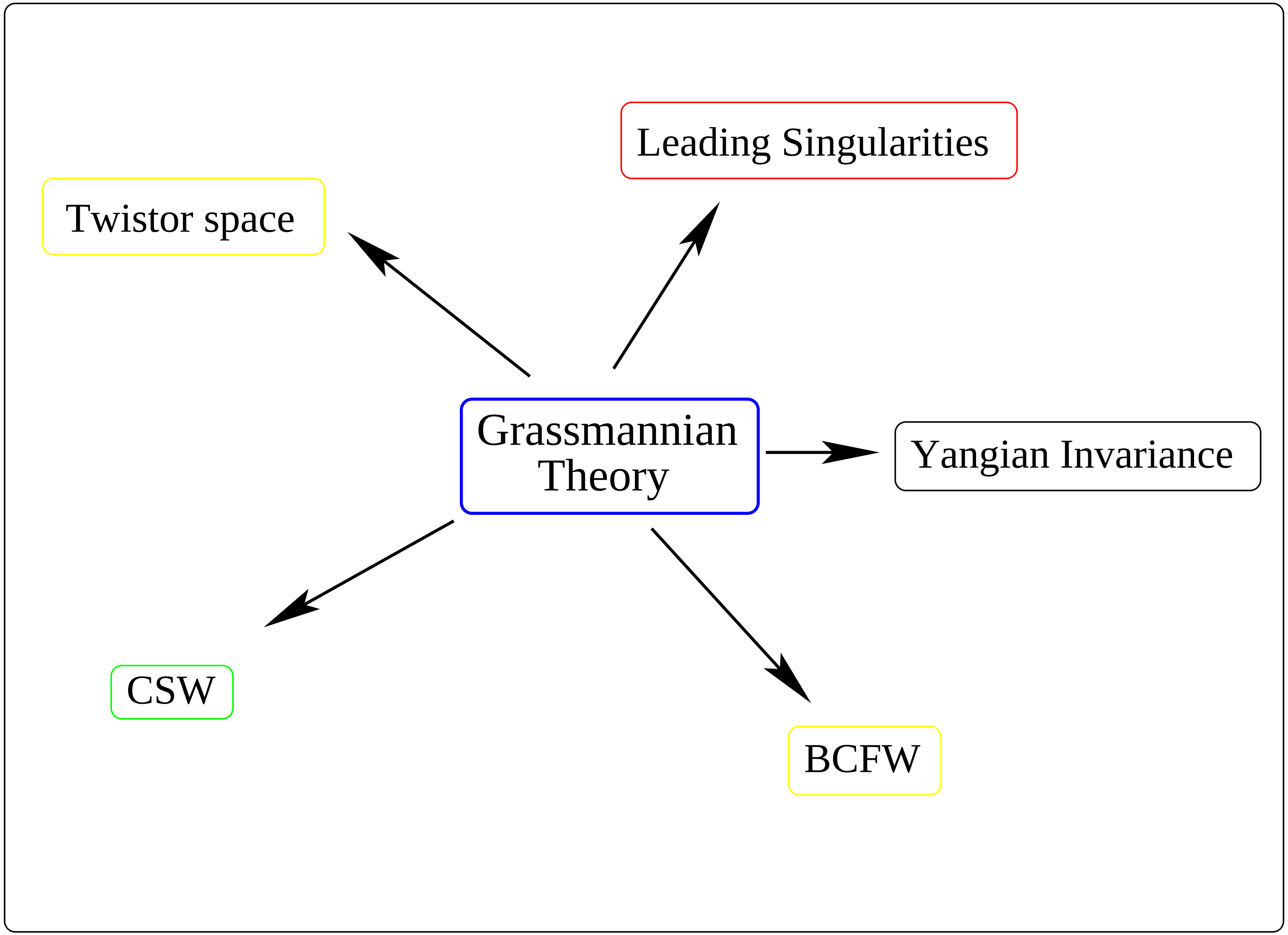
Since CSW diagrams can be generated from a lagrangian we have recovered space-time!

It turns out that there is an operation which instead of blowing the residues apart puts them together.

Unifying the residues in this way leads to

The twistor string formula in its connected form!

(Arkani-Hamed, Bourjaily, F.C.,Trnka 2009, Nandan, Volovich and Wen 2009)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