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CP Symmetry and Phase Transitions

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CP Symmetry and Phase Transitions

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CP Symmetry

Matter-Antimatter Symmetry

- Electroweak interactions break CP symmetry
- Does QCD preserves CP symmetry?
- ϑ–Vacuum
- Strong CP problem
- Peccei-Quinn Theory: axions
- Non-perturbative breaking
- Vafa-Witten Theorem

QCD with ϑ term

- (Euclidean) partition function $\mathcal{Z}(\vartheta)$

$$Z(\vartheta) = \int \mathcal{D}A \, \mathrm{e}^{-S_{\mathrm{ef}} - i\vartheta q} \qquad S_{\mathrm{ef}} \geq 0; \; q \in \mathbb{Z}$$

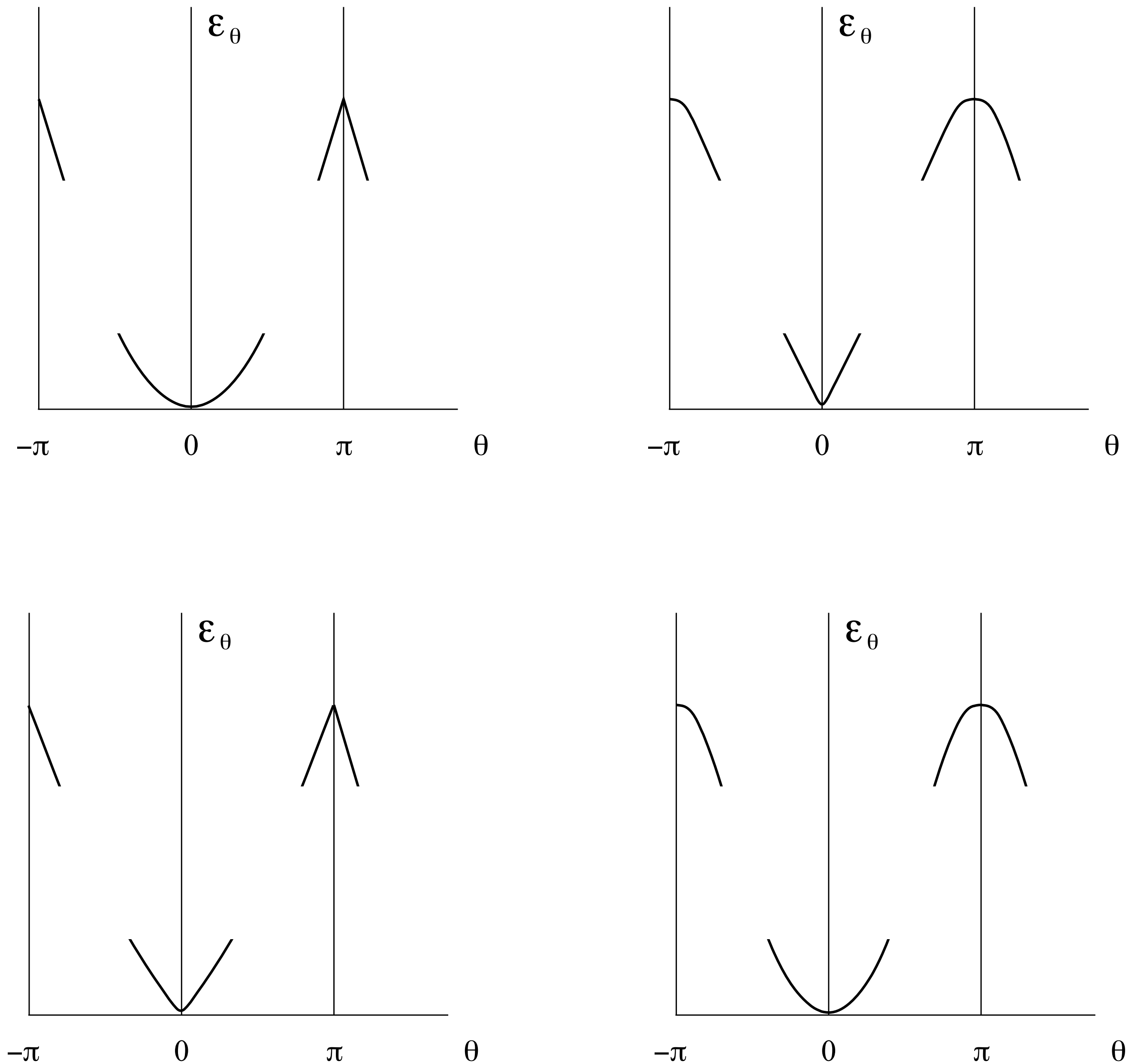
  - $\mathcal{Z}(\vartheta) = \mathcal{Z}(\vartheta + 2\pi n)$ (periodicity)
  - $\mathcal{Z}(\vartheta) = \mathcal{Z}(-\vartheta)$ (reflection symmetry)
  - $\mathcal{Z}(\pi + \vartheta) = \mathcal{Z}(\pi - \vartheta)$ (Bragg symmetry)

- Energy density $\mathcal{E}_0(\vartheta)$

$$Z(\vartheta) \sim \mathrm{e}^{-VT\mathcal{E}_0(\vartheta)}$$

  - $\mathcal{E}_0(\vartheta) = \mathcal{E}_0(\vartheta + 2\pi n)$ (periodicity)
  - $\mathcal{E}_0(\vartheta) = \mathcal{E}_0(-\vartheta)$ (reflection symmetry)
  - $\mathcal{E}_0(\pi + \vartheta) = \mathcal{E}_0(\pi - \vartheta)$ (Bragg symmetry)

CP symmetry and phase transitions



CP symmetry and phase transitions

1$^{\text{er}}$ order transition:

$$\left.\frac{d\mathcal{E}(\vartheta)}{d\vartheta}\right|_{\vartheta=0,\pi} \neq 0$$

- Order parameter:

$$Q = \langle F\widetilde{F}(x) \rangle \neq 0$$

Topological charge density

CP symmetry and phase transitions

2$^{\text{o}}$ order transition:

$$\left.\frac{d^2\mathcal{E}(\vartheta)}{d\vartheta_-{}^2}\right|_{\vartheta=0,\pi} \neq \left.\frac{d^2\mathcal{E}(\vartheta)}{d\vartheta_+{}^2}\right|_{\vartheta=0,\pi}$$

- Order parameter:

$$\chi_\tau = \int dx \left[\langle F\widetilde{F}(0)\ F\widetilde{F}(x)\rangle_0 - \langle F\widetilde{F}(0)\rangle_0 \langle F\widetilde{F}(x)\rangle_0\right]$$

Topological susceptibility

χ_τ diverges if $Q \neq 0$

OS positivity and topological susceptibility

- Witten-Veneziano Formula

$$\chi_\tau \sim f_\pi^2 (m_{\eta'}^2 + m_\eta^2 - 2m_K^2)/6$$

- Osterwalder-Schräder positivity

$$\langle F\widetilde{F}(0)\ F\widetilde{F}(x)\rangle \leq 0, \qquad \text{for } x \neq 0$$

- Contact term

$$\langle F\widetilde{F}(0)\ F\widetilde{F}(x)\rangle \leq a\delta(x) + K(x)$$

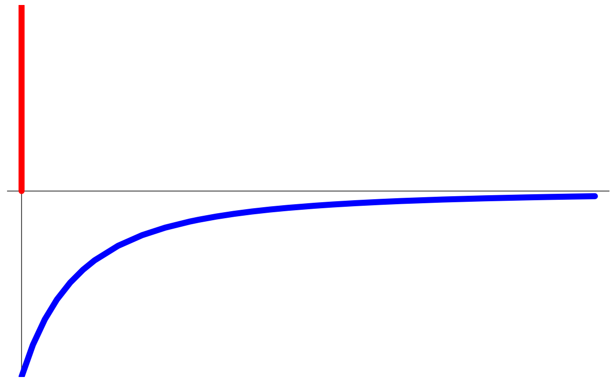
$a\delta(x) \rightarrow \geq 0$

$K(x) \rightarrow \leq 0$

- Topological susceptibility positivity

$$a \geq \int dx\, K(x)$$

OS positivity and topological susceptibility



- Cluster property

$$\lim_{x\to\infty} \langle F\widetilde{F}(0)\, F\widetilde{F}(x)\rangle = \langle F\widetilde{F}\rangle^2 \leq 0$$

- If CP symmetry is broken

$$\langle F\widetilde{F}\rangle = iQ \neq 0$$

Vafa-Witten theorem

Parity symmetry is not spontaneously broken in QCD

- Basic VW argument: Order parameter: topological charge density

$$Q = \frac{1}{32\pi^2} F^a_{\mu\nu} \widetilde{F}^a_{\mu\nu}$$

(Euclidean) partition function [with quarks]

$$\mathcal{Z}(\vartheta) = \mathrm{e}^{-VT\mathcal{E}_0(\vartheta)} = \int \mathcal{D}A \, \mathrm{e}^{-S_{\mathrm{ef}} - i\vartheta q},$$

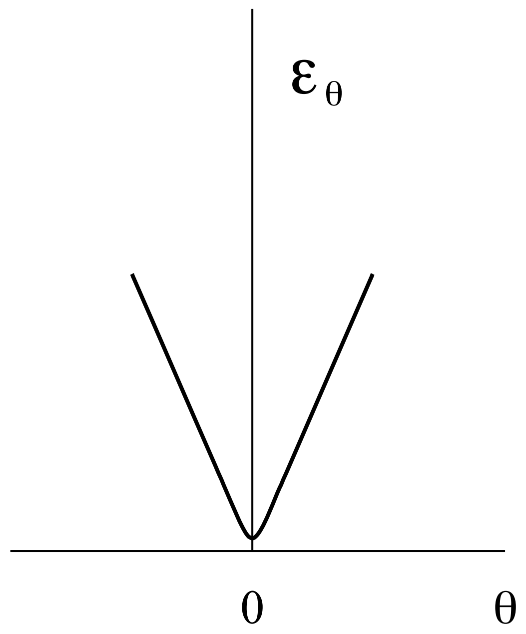
$$|\mathcal{Z}(\vartheta)| \leq \mathcal{Z}(0) \implies \mathcal{E}_0(\vartheta) \geq \mathcal{E}_0(0),$$

Assuming that $\mathcal{E}_0(\vartheta)$ is differentiable in $\vartheta = 0$

$$\mathcal{E}_0'(0) = \frac{i}{32\pi^2} \langle F^a_{\mu\nu} \widetilde{F}^a_{\mu\nu}(0) \rangle_0 = \langle Q \rangle_{\vartheta=0} = 0.$$

Vafa-Witten theorem

- Possibility of cusp $\Leftrightarrow$ 1^{st} order phase transition



- The theorem can be proved with two additional assumptions
  - Renormalizability
  - Unitarity

Analiticity of the partition function

$$\mathcal{Z}_{\vartheta}(g) = \sum_{q=-\infty}^{\infty} \mathrm{e}^{-q\vartheta_I + iq\vartheta_R} \int_{c_2(A)=q} \mathcal{D}A \, \mathrm{e}^{-S_{\mathrm{ef}}(g,q)}$$

$$|\mathcal{Z}_{\vartheta}(g)| \leq \sum_{q=-\infty}^{\infty} \mathrm{e}^{|q\vartheta_I|} \int_{c_2(A)=q} \mathcal{D}A \, \mathrm{e}^{-S_{\mathrm{ef}}(g,q)} \leq \sum_{q=-\infty}^{\infty} \int_{c_2(A)=q} \mathcal{D}A \, \mathrm{e}^{-S_{\mathrm{ef}}(g,q) + \frac{|\vartheta_I|}{8\pi^2} S_{\mathrm{YM}}}$$

BPS bound

$$S_{\mathrm{YM}}(q) = \frac{1}{2} \int F^{\mu\nu} F_{\mu\nu} \geq \frac{1}{2} \left| \int F^{\mu\nu} \widetilde{F}_{\mu\nu} \right| = 8\pi^2 |q|$$

Analiticity of the partition function

$$\mathcal{Z}_{\vartheta}(g) = \sum_{q=-\infty}^{\infty} \mathrm{e}^{-q\vartheta_I + iq\vartheta_R} \int_{c_2(A)=q} \mathcal{D}A \, \mathrm{e}^{-S_{\mathrm{ef}}(g,q)}$$

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$$\frac{1}{\tilde{g}^2} = \frac{1}{g^2} - \frac{|\vartheta_I|}{8\pi^2}$$

$\mathcal{Z}_{\vartheta}(g)$ is an analytic function in ϑ

Analiticity of the partition function

$$\mathcal{Z}_{\vartheta}(g) = \sum_{q=-\infty}^{\infty} \mathrm{e}^{-q\vartheta_I + iq\vartheta_R} \int_{c_2(A)=q} \mathcal{D}A \, \mathrm{e}^{-S_{\mathrm{ef}}(g,q)}$$

$$|\mathcal{Z}_{\vartheta}(g)| \leq \sum_{q=-\infty}^{\infty} \mathrm{e}^{|q\vartheta_I|} \int_{c_2(A)=q} \mathcal{D}A \, \mathrm{e}^{-S_{\mathrm{ef}}(g,q)} \leq \sum_{q=-\infty}^{\infty} \int_{c_2(A)=q} \mathcal{D}A \, \mathrm{e}^{-S_{\mathrm{ef}}(\tilde{g},q)} = \mathcal{Z}_0(\tilde{g})$$

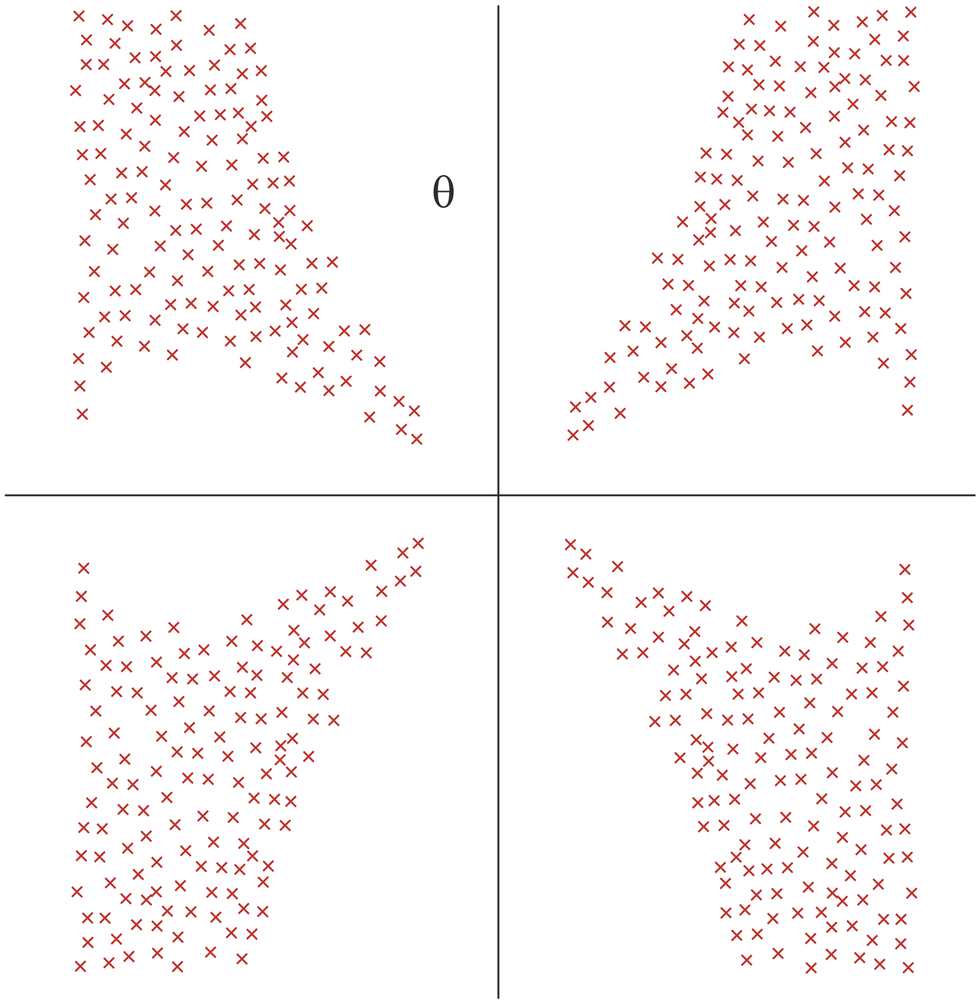
$$\frac{1}{\tilde{g}^2} = \frac{1}{g^2} - \frac{|\vartheta_I|}{8\pi^2}$$

[Lee-Yang Zeros in the partition function]

Lee-Yang Zeros

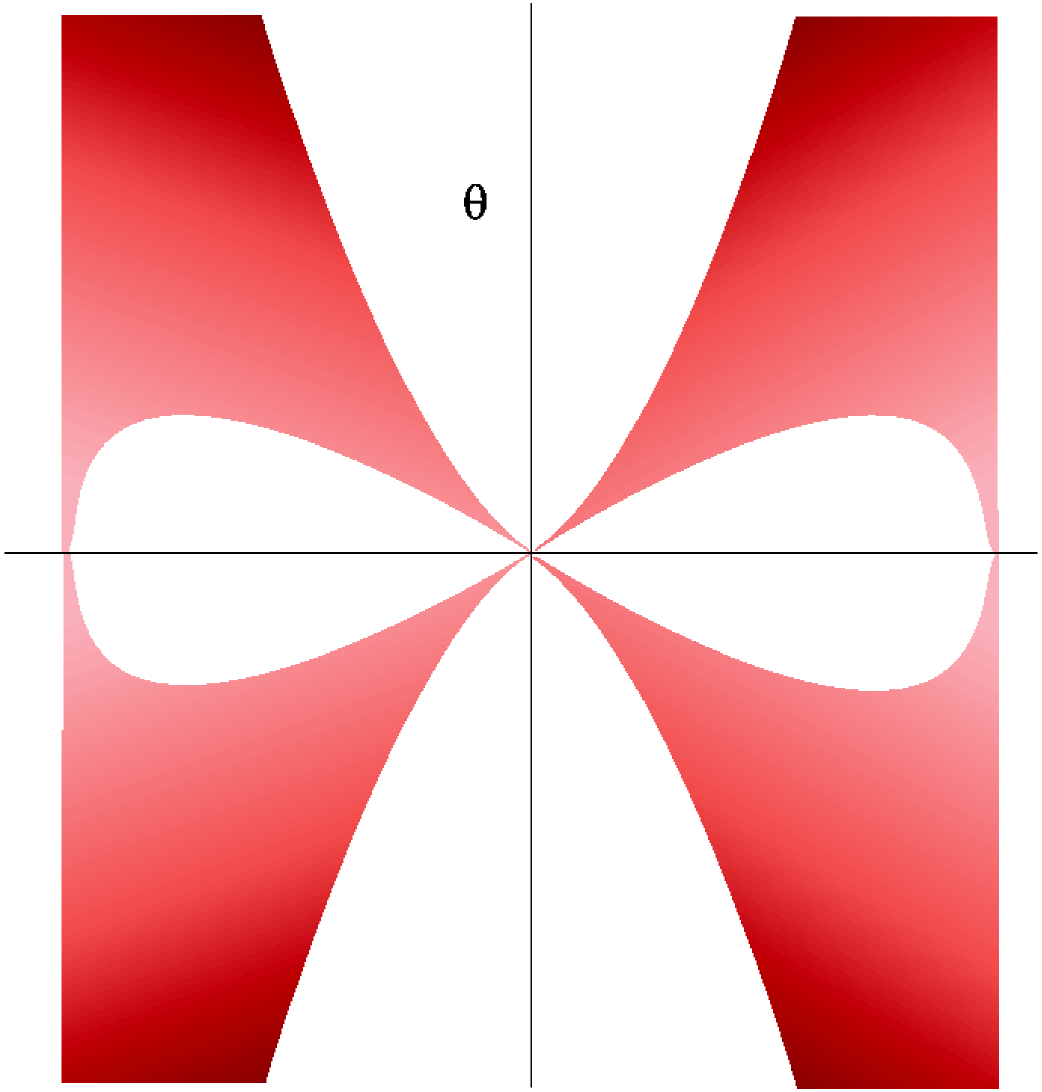
$$\mathcal{E}_V(\vartheta) = \frac{-2}{V}\,\mathrm{Re}\sum_{n=1}^{\infty}\rho_n \ln\left(1 - \frac{\sin^2\frac{\vartheta}{2}}{\sin^2\frac{\vartheta_n}{2}}\right) + \sum_{n=1}^{\infty} h_n(\vartheta, V)$$

ρ_n degree of degeneracy



Lee-Yang Zeros

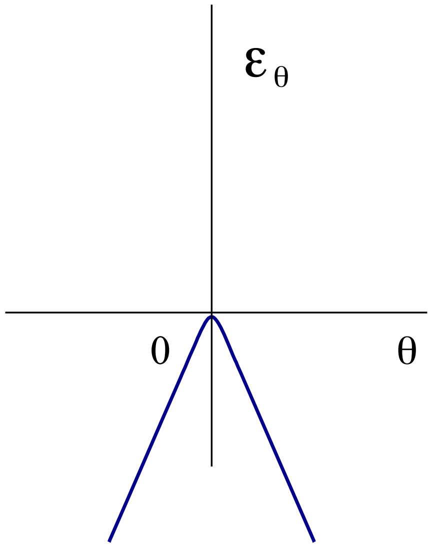
$$\mathcal{E}(\vartheta) = -2\,\mathrm{Re}\int_0^\infty dt\,\rho(t)\,\ln\left(1 - \frac{\sin^2\frac{\vartheta}{2}}{\sin^2\frac{\vartheta(t)}{2}}\right) + \int_0^\infty dt\,h(t,\vartheta)$$



Vafa-Witten theorem

- Energy density $\mathcal{E}(\vartheta) \Leftrightarrow$ Electrostatic potential with negative charge density $-\rho > 0$

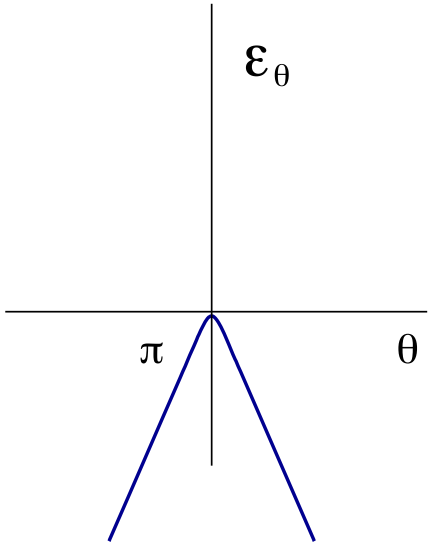
$$\mathcal{E}_V(\vartheta) \approx c|\vartheta|$$
$$c < 0$$



- Violation of Vafa-Witten inequality
  - CP symmetry is not preserved
  - There is no phase transition at $\vartheta = 0$

Vafa-Witten theorem

- The argument does not exclude the existence of a 1^{er} order phase transition at $\vartheta = \pi$



  - CP symmetry can be spontaneously broken $\vartheta = \pi$
  - 1^{er} order phase transition at $\vartheta = \pi$

- Even a 2^{nd} order phase transition can appear at $\vartheta = 0$ if

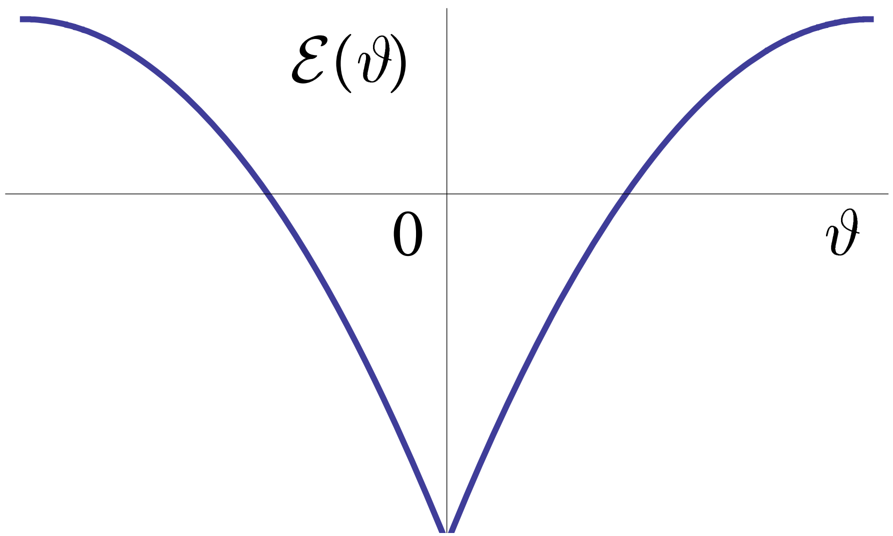
$$\rho(z) \sim \rho_1 \, z_2^{\alpha} \delta(z_1 - m z_2^{\beta}) \qquad \text{with} \qquad 0 < \alpha \leq 1$$

Vafa-Witten theorem

- Basic ingredients of the proof
  - Diamagnetic inequality
  - BPS bound

SCALAR FIELD THEORY IN 1+1 DIMENSIONS

$$S_\vartheta(\phi) = \frac{1}{2}\int d^2x\; \partial^\mu\phi^*\partial_\mu\phi \;-\; i\vartheta\int d^2x\; \varepsilon^{\mu\nu}\partial_\nu\phi^*\partial_\mu\phi$$



$\mathbb{C}P^{N-1}$ Sigma Models

2-D sigma model with $\mathbb{C}P^{N-1}$ target space

$$S[\Psi, \Psi^\dagger] = \frac{N}{2g^2} \int \mathrm{d}^2x \, (\mathrm{D}^\mu \Psi)^\dagger \mathrm{D}_\mu \Psi$$

$$\Psi^\dagger \Psi = 1; \quad A_\mu = \frac{i}{2} \left(\Psi^\dagger \partial_\mu \Psi - \partial_\mu \Psi^\dagger \Psi \right)$$

- U(1) gauge symmetry
- 1+1 D Poincaré symmetry
- SU(N) global symmetry
- Conformal symmetry
- ϑ Vacuum

$\mathbb{C}P^{N-1}$ Sigma Models

2-D sigma model with $\mathbb{C}P^{N-1}$ target space

$$S[\Psi, \Psi^\dagger] = \frac{N}{2g^2} \int d^2x \, (D^\mu \Psi)^\dagger D_\mu \Psi$$

$$- i \frac{\vartheta}{4\pi} \int \varepsilon_{\mu\nu} F_{\mu\nu}$$

- U(1) gauge symmetry ✓
- 1+1 D Poincaré symmetry ✓
- SU(N) global symmetry ✓
- Conformal symmetry ✗
- CP Invariance for $\vartheta = 0$ and $\vartheta = \pi$?

Exact Results

$\mathbb{CP}^1$ Sigma Model

1. $\vartheta = 0$ [Polyakov-Weigmann]

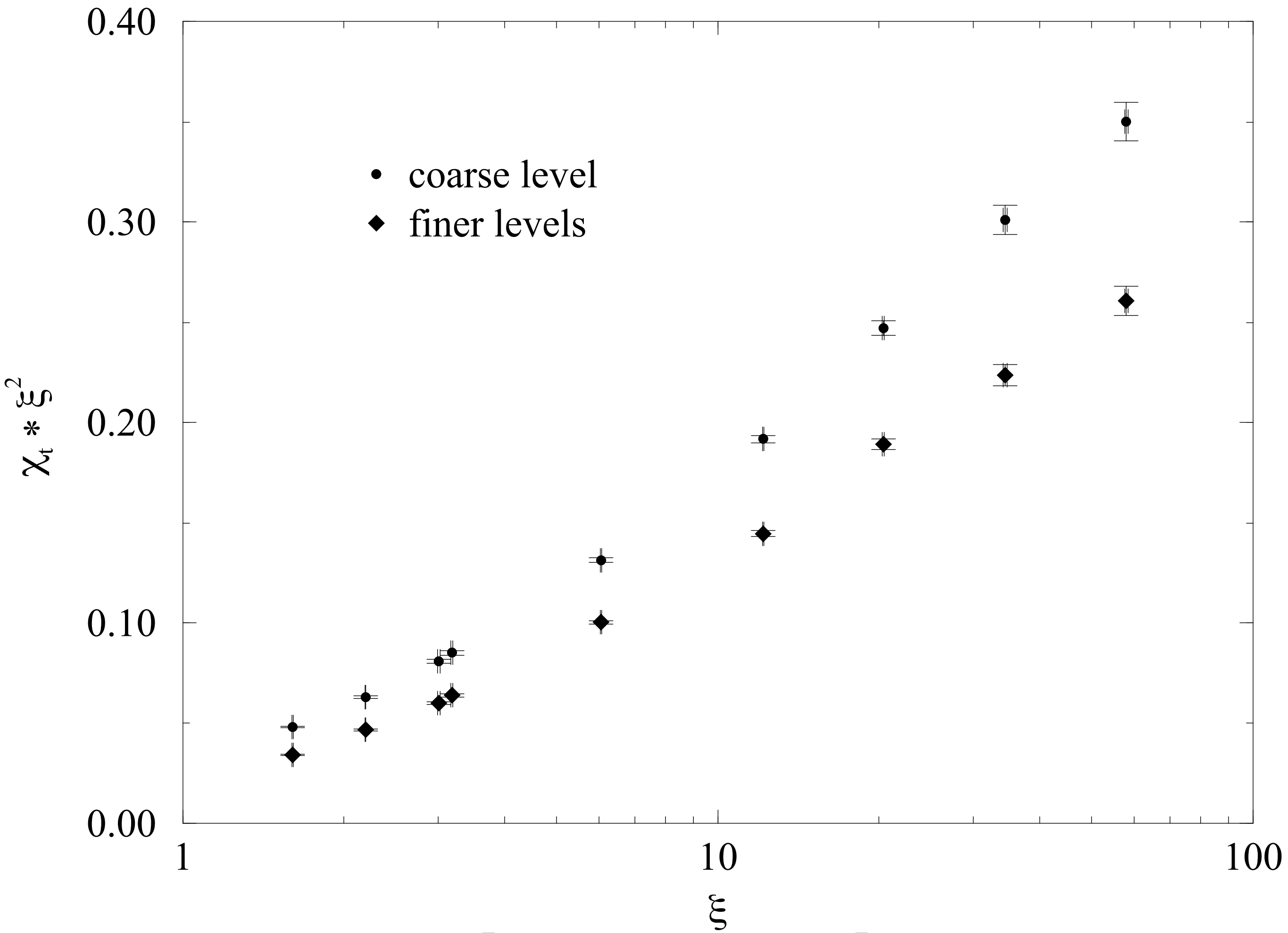
- Mass gap
- The lightless particle is in the adjoint SU(2) representation
- No spontaneous CP symmetry breaking

2. $\vartheta = \pi$ [Zamolodchikov^2]

- Massless spectrum
- The basic massless particle is in the fundamental SU(2) representation
- No spontaneous CP symmetry breaking

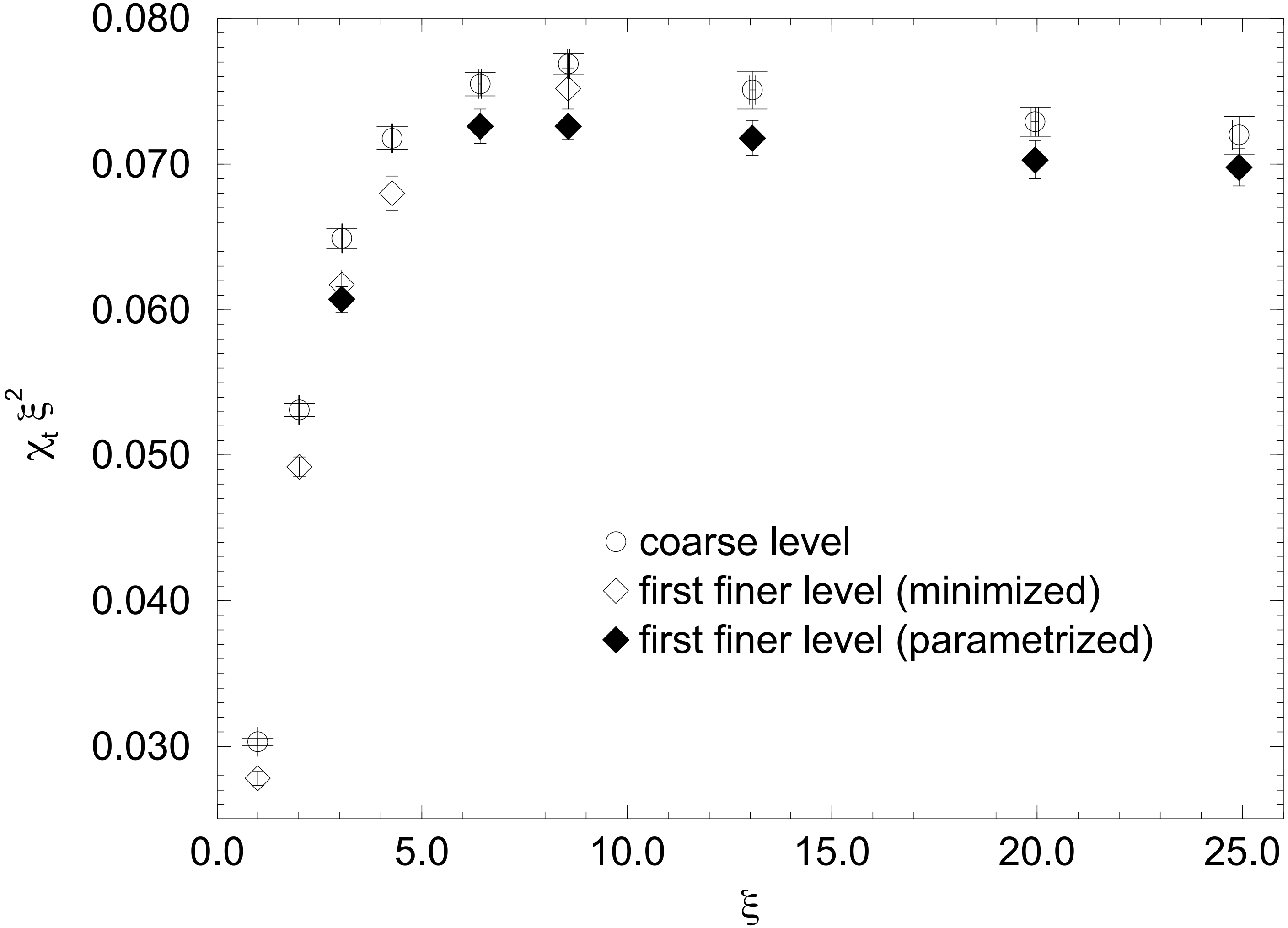
Topological Susceptibility in $\mathbb{CP}^1$

Lüscher's problem



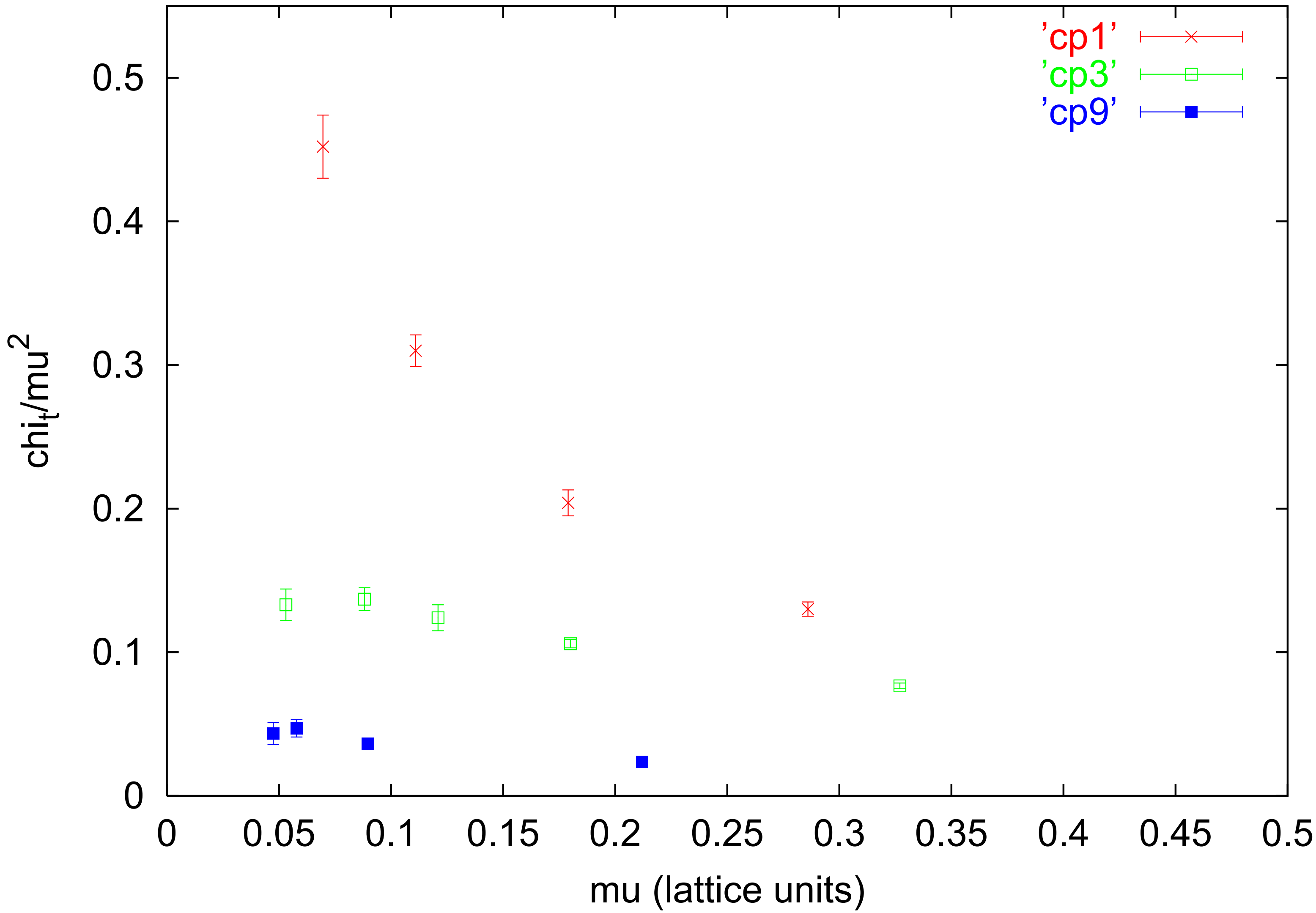
[Blatter et al]

Topological Susceptibility in $\mathbb{CP}^3$



[Burkhalter et al]

Topological Susceptibility in $\mathbb{C}P^{N-1}$



[Ahmad et al]

CONCLUSIONS

- Vafa-Witten conjecture is correct [Theorem]
- CP symmetry is preserved at $\vartheta = 0$
- No 1^{er} order phase transitions $\vartheta = 0$
- Topological susceptibility is positive thanks to a contact term
- Analytic results are compatible with exact known results in the $\mathbb{C}\mathrm{P}^1$ model. Phase transition at $\vartheta = 0$
- They are also compatible with $N \gg 1$ limit results in $\mathbb{C}\mathrm{P}^{\mathrm{N}-1}$ models. CP symmetry is spontaneously broken at $\vartheta = \pi$

OPEN QUESTIONS

- Is topological susceptibility well defined in the $\mathbb{C}P^1$ sigma model at $\vartheta = 0$?
- Does the theory confine in the $\vartheta = \pi$ vacuum?
- There is a critical value N_c such that the $\mathbb{C}P^{N-1}$ model
  - undergoes a second order phase transition at $\vartheta = \pi$ for $N < N_c$ and
  - a first order phase transition for $N > N_c$
- Is $N_c = 4$, or $N_c = 3$?