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Neutrino oscillations in quantum mechanics and quantum field theory

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Neutrino oscillations in quantum mechanics and quantum field theory

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Gribov & Pontecorvo 1969 paper

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NEUTRINO ASTRONOMY AND LEPTON CHARGE

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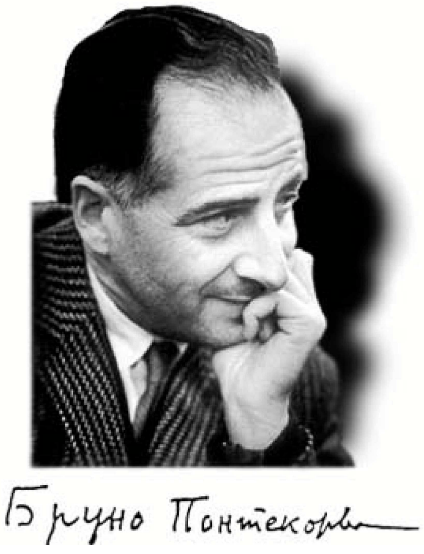
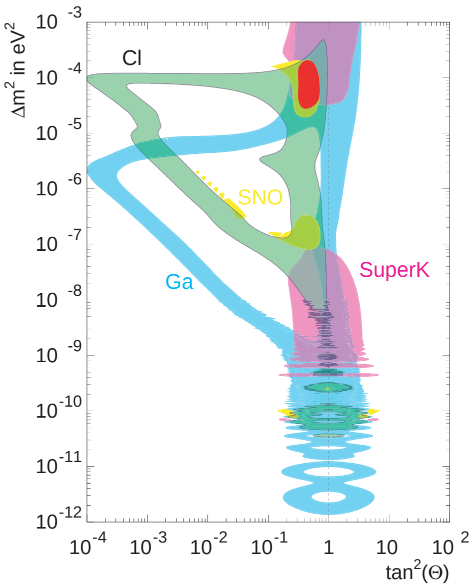
It is shown that lepton nonconservation might lead to a decrease in the number of detectable solar neutrinos at the earth surface, because of $\nu_e \rightleftarrows \nu_\mu$ oscillations, similar to $K^0 \rightleftarrows \bar{K}^0$ oscillations. Equations are presented describing such oscillations for the case when there exist only four neutrino states.

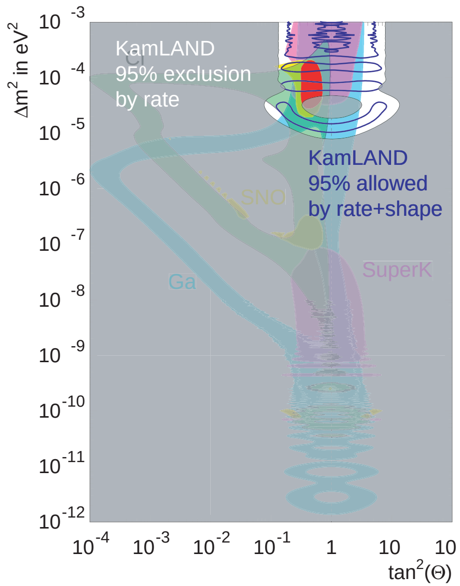


V.N. Gribov

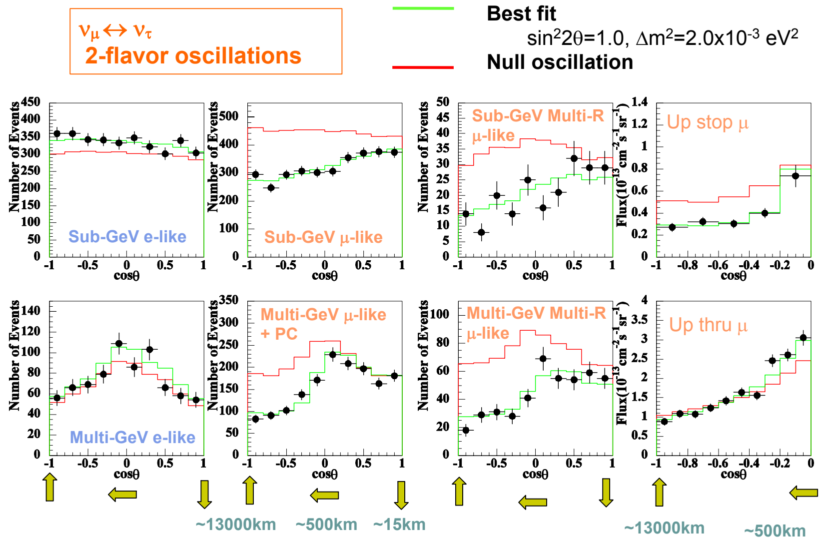
Oscillations of Majorana neutrinos considered for the first time!

Oscillations discovered experimentally !

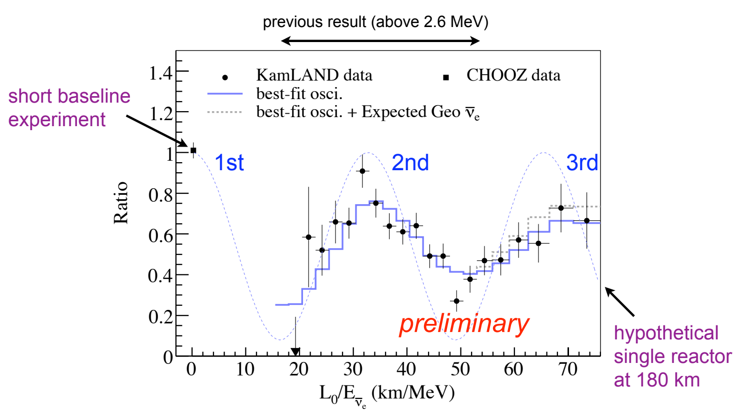




Zenith angle distributions



Neutrino Oscillation

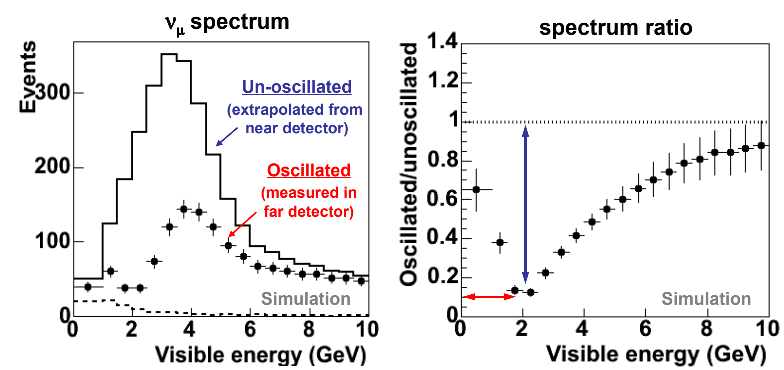


KamLAND covers the 2nd and 3rd maximum

→ characteristic of neutrino oscillation

ν_μ Disappearance Measurement

Look for ν_μ deficit : $P(\nu_\mu \to \nu_\mu) = 1 - \sin^2 2\theta \sin^2\left(\frac{\Delta m^2 L}{E}\right)$



Andy Blake, Cambridge University

The MINOS Experiment, slide 7

Neutrino oscillation theory

Neutrino oscillations appear to be a simple QM phenomenon

But: A closer look at them reveals a number of subtle and even paradoxical issues

A number of basic issues still being debated

Debating the basics of neutrino oscillations ...

Lipkin arXiv:0801.1465, arXiv:0905.1216, arXiv:0910.5049, Ivanov & Kienle arXiv:0909.1287, Merle arXiv:0907.3554, Peshkin arXiv:0804.4891, Faber arXiv:0801.3262, Gal arXiv:0809.1213, Giunti arXiv:0805.0431, Flambaum arXiv:0908.2039, Kienert, Kopp, Lindner & Merle arXiv:0808.2389, Walker Nature 453 (2008) 864, Giunti arXiv:0807.3818, Kleinert & Kienle ("Neutrino-pulsating vacuum") arXiv:0803.2938, Lambiase, Papini & Scarpeta arXiv:0811.2302, Burkhardt, Lowe, Stephenson, Goldman & McKellar, arXiv:0804.1099
Bilenky, v. Feilitzsch & Potzel arXiv:0804.3409, arXiv:0803.0527, J. Phys. G36 (2009) 078002, EA, Kopp & Lindner arXiv:0802.2513, arXiv:0803.1424,

Cohen, Glashow & Ligety arXiv:0810.4602, Visinelli & Gondolo arXiv:0810.4132, Keister & Polizou arXiv:0908.1404, Nishi & Guzzo arXiv:0803.1422, Lychkovskiy arXiv:0901.1198, Adhikari & Pal arXiv:0912.5266, Giunti arXiv:1001.0760, Ahluwalia & Schritt arXiv:0911.2965, Schmidt-Parzefall arXiv:0912.3620, Robertson arXiv:1004.1847 and many others.

Clarification of some of these issues and some apparent paradoxes of neutrino oscillations in:

EA & Smirnov arXiv:0905.1903, EA & Kopp arXiv:1001.4815

Unsettled issues?

- Equal energies or equal momenta?
- Evolution in space or in time?
- What is the role of QM uncertainty relations in ν oscillations?
- Is wave packet description necessary?
- What determines the size of neutrino wave packets?
- Under what conditions can oscillations be observed? (coherence issues)
- When are the oscillations described by a universal probability?
- Is the standard oscillation formula correct?
- Lorentz invariance issues
- Do charged leptons oscillate?
- Do Mössbauer neutrinos oscillate?

Neutrino flavour mixing and oscillations

Diagonalization of the mass terms of the charged leptons and neutrinos gives

$$\Delta\mathcal{L} = -\frac{g}{\sqrt{2}}(\bar{e}_{\alpha L}\gamma^{\mu}U_{\alpha i}\nu_{iL})\,W_{\mu}^{-} \;+\; \text{diag. mass terms} \;+\; h.c.$$

$$\alpha = e\,,\ \mu\,,\ \tau, \qquad i = 1\,,\ 2\,,\ 3$$

$$\nu_{\alpha} = \sum_i U_{\alpha i}\,\nu_i \qquad \Rightarrow \qquad |\nu_{\alpha}\rangle = \sum_i U_{\alpha i}^{*}\,|\nu_i\rangle$$

The standard formula for the oscillation probability of relativistic neutrinos in vacuum:

$\diamondsuit$

$$P(\nu_{\alpha}\to\nu_{\beta};L) = \left|\sum_i U_{\beta i}\,e^{-i\frac{\Delta m_{i1}^2}{2p}L}\,U_{\alpha i}^{*}\right|^2$$

How is it usually derived?

Assume at time $t = 0$ and coordinate $x = 0$ a flavour eigenstate $|\nu_a\rangle$ is produced:

$$|\nu(0,0)\rangle \;=\; |\nu_\alpha^{\rm fl}\rangle \;=\; \sum_i U_{\alpha i}^* \, |\nu_i^{\rm mass}\rangle$$

After time t at the position x, for plane-wave particles:

$$|\nu(t,\vec{x})\rangle \;=\; \sum_i U_{\alpha i}^* \, e^{-ip_i x} |\nu_i^{\rm mass}\rangle$$

Mass eigenstates pick up the phase factors $e^{-i\phi_i}$ with

$$\phi_i \;\equiv\; p_i \, x \;=\; Et \;-\; \vec{p}\,\vec{x}$$

$$P(\nu_\alpha \to \nu_\beta) \;=\; \left|\langle \nu_\beta^{\rm fl} | \nu(t,x)\rangle\right|^2$$

How is it usually derived?

Consider $\vec{x} \,||\, \vec{p} \quad \Rightarrow \quad \vec{p}\vec{x} = \mathrm{px} \quad (\mathrm{p} = |\vec{p}|, \quad \mathrm{x} = |\vec{x}|)$

Phase differences between different mass eigenstates:

$$\Delta\phi = \Delta E \cdot t - \Delta \mathrm{p} \cdot \mathrm{x}$$

<u>Shortcuts to the standard formula</u>

1. Assume the emitted neutrino state has a well defined momentum (same momentum prescription) $\Rightarrow \quad \Delta \mathrm{p} = 0.$

   For ultra-relativistic neutrinos $E_i = \sqrt{\mathrm{p}^2 + m_i^2} \simeq p + \frac{m_i^2}{2\mathrm{p}} \quad \Rightarrow$

$$\Delta E \simeq \frac{m_2^2 - m_1^2}{2E} \equiv \frac{\Delta m^2}{2E}; \qquad t \approx x \qquad (\hbar = c = 1)$$

$\Rightarrow$ The standard formula is obtained

How is it usually derived?

2. Assume the emitted neutrino state has a well defined energy (same energy prescription) $\Rightarrow \quad \Delta E = 0.$

$$\Delta\phi = \Delta E \cdot t - \Delta \mathrm{p} \cdot \mathrm{x} \quad \Rightarrow \quad - \Delta \mathrm{p} \cdot \mathrm{x}$$

For ultra-relativistic neutrinos $\mathrm{p}_i = \sqrt{E^2 - m_i^2} \simeq E - \frac{m_i^2}{2\mathrm{p}} \quad \Rightarrow$

$$-\Delta \mathrm{p} \equiv \mathrm{p}_1 - \mathrm{p}_2 \approx \frac{\Delta m^2}{2E};$$

$\Rightarrow$ The standard formula is obtained

Stand. phase $\Rightarrow$ $(l_{\mathrm{osc}})_{ik} = \frac{4\pi E}{\Delta m_{ik}^2} \simeq 2.5\ m\, \frac{E\,(\mathrm{MeV})}{\Delta m_{ik}^2\ \mathrm{eV}^2}$

Same E and same p approaches

Very simple and transparent

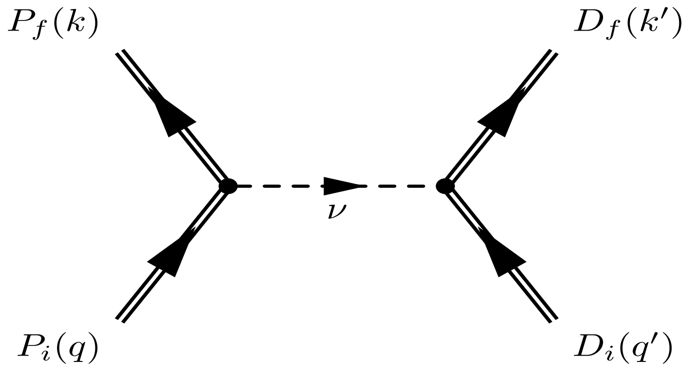
Allow one to quickly arrive at the desired result

Trouble: they are both wrong

◇ Plane wave approach: plagued with inconsistencies. If applied correctly, does not lead to neutrino oscillations at all !

Consistent approaches:

- QM wave packet approach – neutrinos described by wave packets rather than by plane waves
- QFT approach: neutrino production and detection explicitly taken into account. Neutrinos are intermediate particles described by propagators



QM wave packet approach

The evolved produced state:

$$|\nu_\alpha^{\rm fl}(\vec{x},t)\rangle \;=\; \sum_i U_{\alpha i}^*\,|\nu_i^{\rm mass}(\vec{x},t)\rangle \;=\; \sum_i U_{\alpha i}^*\,\Psi_i^P(\vec{x},t)|\nu_i^{\rm mass}\rangle$$

The coordinate-space wave function of the ith mass eigenstate (w. packet):

$$\Psi_i^P(\vec{x},t) \;=\; \int \frac{d^3p}{(2\pi)^3}\, f_i^P(\vec{p})\, e^{i\vec{p}\vec{x}-iE_i(p)t}$$

Momentum distribution function $f_i^S(\vec{p})$: sharp maximum at $\vec{p}=\vec{P}$ (width of the peak $\sigma_{pP} \ll P$).

$$E_i(p) \;=\; E_i(P) \;+\; \left.\frac{\partial E_i(p)}{\partial\vec{p}}\right|_{\vec{P}} (\vec{p}-\vec{P}) \;+\; \frac{1}{2}\left.\frac{\partial^2 E_i(p)}{\partial\vec{p}^2}\right|_{\vec{p}_0} (\vec{p}-\vec{P})^2 \;+\; \ldots$$

$$\vec{v}_i \;=\; \frac{\partial E_i(p)}{\partial\vec{p}} \;=\; \frac{\vec{p}}{E_i}\,, \qquad\qquad \alpha \;\equiv\; \frac{\partial^2 E_i(p)}{\partial\vec{p}^2} \;=\; \frac{m_i^2}{E_i^2}$$

Evolved neutrino state

$$\boxed{\Psi_i^P(\vec{x},\,t)\;\simeq\;e^{-iE_i(P)t+i\vec{P}\vec{x}}\,g_i^P(\vec{x}-\vec{v}_i t)}\qquad (\alpha\;\to\;0)$$

$$g_i^P(\vec{x}-\vec{v}_i t)\;\equiv\;\int\frac{d^3p_1}{(2\pi)^3}\,f_i^P(\vec{p}_1)\,e^{i\vec{p}_1(\vec{x}-\vec{v}_g t)}$$

Center of the wave packet: $\vec{x}-\vec{v}_i t=0$. Spatial length: $\sigma_{xP}\;\sim\;1/\sigma_{pP}$
(g_i^S decreases quickly for $|\vec{x}-\vec{v}_i t|\gtrsim\sigma_{xP}$).

Detected state (centered at $\vec{x}=\vec{L}$):

$$|\nu_\beta^{\rm fl}(\vec{x})\rangle\;=\;\sum_k U_{\beta k}^*\,\Psi_k^D(\vec{x})|\nu_i^{\rm mass}\rangle$$

The coordinate-space wave function of the ith mass eigenstate (w. packet):

$$\boxed{\Psi_k^D(\vec{x})\;=\;\int\frac{d^3p}{(2\pi)^3}\,f_k^D(\vec{p})\,e^{i\vec{p}(\vec{x}-\vec{L})}}$$

Oscillation probability

Transition amplitude:

$$\mathcal{A}_{\alpha\beta}(T,\vec{L}) \;=\; \langle \nu_\beta^{\rm fl}|\nu_\alpha^{\rm fl}(T)\rangle \;=\; \sum_i U_{\alpha i}^* U_{\beta i}\,\mathcal{A}_i(T,\vec{L})$$

$$\boxed{\mathcal{A}_i(T,\vec{L}) \;=\; \int \frac{d^3p}{(2\pi)^3}\, f_i^P(\vec{p})\, f_i^{D*}(\vec{p})\, e^{-iE_i(p)T+i\vec{p}\vec{L}}}$$

Strongly suppressed unless $|\vec{L}-\vec{v}_i T| \lesssim \sigma_x$. E.g., for Gaussian wave packets:

$$\mathcal{A}_i(T,\vec{L}) \;\propto\; \exp\left[-\frac{(\vec{L}-\vec{v}_i T)^2}{4\sigma_x^2}\right], \qquad \sigma_x^2 \;\equiv\; \sigma_{xP}^2 + \sigma_{xD}^2$$

Oscillation probability:

♢ $$P(\nu_\alpha \to \nu_\beta; T,\vec{L}) \;=\; |\mathcal{A}_{\alpha\beta}|^2 = \sum_{i,k} U_{\alpha i}^* U_{\beta i} U_{\alpha k} U_{\beta k}^*\, \mathcal{A}_i(T,\vec{L})\mathcal{A}_k^*(T,\vec{L})$$

Oscillation phase

Oscillations are due to phase differences of different mass eigenstates:

$$\Delta\phi = \Delta E \cdot t - \Delta p \cdot x \qquad (E_i = \sqrt{p_i^2 + m_i^2})$$

Consider the case $\Delta E \ll E$ (relativistic or quasi-degenerate neutrinos) $\Rightarrow$

$$\Delta E = \frac{\partial E}{\partial p}\Delta p + \frac{\partial E}{\partial m^2}\Delta m^2 = v\Delta p + \frac{1}{2E}\Delta m^2$$

$$\Delta\phi = (v\Delta p + \frac{1}{2E}\Delta m^2)\, t - \Delta p \cdot x$$

$$= -(x - vt)\Delta p + \frac{\Delta m^2}{2E}t$$

In the center of wave packet $(x - vt) = 0$. In general, $|x - vt| \lesssim \sigma_x$;
if $\sigma_x \Delta p \ll 1$ (i,e, $\Delta p \ll \sigma_p$), $|x - vt|\Delta p \ll 1 \quad \Rightarrow$

$$\Delta\phi = \frac{\Delta m^2}{2E}\,t\,, \qquad x \simeq vt \simeq t$$

– the result of the “same momentum” approach recovered!

Now instead of expressing ΔE through Δp and Δm^2 express Δp through ΔE and Δm^2:

$$\diamondsuit \qquad \Delta\phi = -\frac{1}{v}(x - vt)\Delta E + \frac{\Delta m^2}{2p}\,x \qquad \Rightarrow \qquad \frac{\Delta m^2}{2p}\,x$$

– for $\Delta E \sigma_x / v \ll 1$ (i.e. $\Delta E \ll \sigma_E$) – “same energy” result recovered.

The reasons why wrong assumptions give the correct result:

- Neutrinos are relativistic or quasi-degenerate with $\Delta E \ll E$
- Neutrno energy uncertainty $\sigma_E \gg \Delta E$ (typically this means $\sigma_x \ll l_{\rm osc}$)

When are neutrino oscillations observable?

Keyword: *Coherence*

Neutrino flavour eigenstates ν_e, ν_μ and ν_τ are coherent superpositions of mass eigenstates ν_1, ν_2 and ν_3 $\Rightarrow$ oscillations are only observable if

- neutrino production and detection are coherent
- coherence is not (irreversibly) lost during neutrino propagation.

Possible decoherence at production (detection): If by accurate E and p measurements one can tell (through $E=\sqrt{p^2+m^2}$) which mass eigenstate is emitted, the coherence is lost and oscillations disappear! $\Rightarrow$ Coherent production/detection conditions $\Delta E \ll \sigma_E$, $\Delta p \ll \sigma_p$. Equivalent to loclization conditions: $L_S, L_D \ll l_{\rm osc}$.

Coherent propagation: no wave packet separation due to $\Delta v \neq 0$ $\Rightarrow$

$$L \ll l_{\rm coh} = \frac{v}{\Delta v}\sigma_x$$

Oscillation probability

Neutrino emission and detection times are not measured (or not accurately measured) in most experiments $\Rightarrow$ integration over T:

$$P(\nu_\alpha \to \nu_\beta; L) \;=\; \int dT\, P(\nu_\alpha \to \nu_\beta; T, L) \;=\; \sum_{i,k} U_{\alpha i}^* U_{\beta i} U_{\alpha k} U_{\beta k}^*\, e^{-i\frac{\Delta m_{ik}^2}{2P}L}\, F_{ik}$$

$$F_{ik} = \int \frac{dq}{2\pi v} f_i^P(r_k q - \Delta E_{ik}/2v + P_i) f_i^{D*}(r_k q - \Delta E_{ik}/2v + P_i)$$
$$\times f_k^{P*}(r_i q + \Delta E_{ik}/2v + P_k) f_k^D(r_i q + \Delta E_{ik}/2v + P_k)\, e^{i\frac{\Delta v}{v}qL}$$

Here: $v \equiv \frac{v_i+v_k}{2}\,, \quad \Delta v \equiv v_k - v_i\,, \quad r_{i,k} \equiv \frac{v_{i,k}}{v}$

- For $(\Delta v/v)\sigma_p L \ll 1$ (i.e. $L \ll l_{\rm coh} = (v/\Delta v)\sigma_x$) F is approximately independent of L; in the opposite case F is strongly suppressed
- F is also strongly suppressed unless $\Delta E_{ik}/v \ll \sigma_p$, i.e. $\Delta E_{ik} \ll \sigma_E$ – coherent production/detection condition

Normalization prescription

Oscillation probability calculated in QM w. packet approach is not automatically normalized ! Can be normalized “by hand” by imposing the unitarity condition:

$$\sum_\beta P_{\alpha\beta}(L) \;=\; 1\,.$$

This gives

$$F_{ii} = \int \frac{dp}{2\pi v} |f_i^P(p)|^2\, |f_i^D(p)|^2 \;=\; 1$$

– important for proving Lorentz invariance of the oscillation probability.

Depends on the overlap of $f_i^P(p)$ and $f_i^D(p)$ $\Rightarrow$ no independent normalization of the produced and detected neutrino wave function would do!

In QFT approach the correctly normalized $P_{\alpha\beta}(L)$ is automatically obtained and the meaning of the normalization procedure adopted in the w. packet approach clarified

Shortcomings of the QM w. packet approach

- Neutrino wave packet postulated rather than derived, widths estimated
- Production and detection processes are not considered
- Inadequate normalization procedure. Normalization "by hand" is unavoidable.

Advantage: simplicity

Calc. from 1st principles – QFT approach

Production - propagation - detection treated as a single inseparable process.
External particles are described by wave packets, neutrinos – by propagators

One-particle states of external particles:

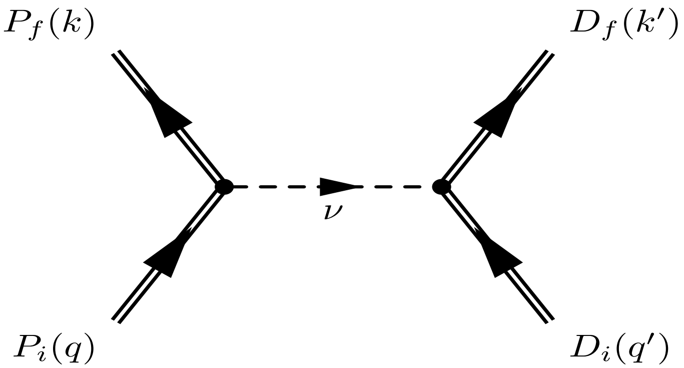
$$|A\rangle = \int [dp]\, f_A(\vec{p}, \vec{P})\, |A, \vec{p}\rangle\,, \qquad [dp] \equiv \frac{d^3p}{(2\pi)^3\sqrt{2E_A(\vec{p})}}$$

$|A, \vec{p}\rangle$ – one-particle momentum eigenstate corresponding to momentum $\vec{p}$ and energy $E_A(\vec{p})$ (free particles: $E_A(\vec{p}) = \sqrt{\vec{p}^2 + m_A^2}$).
$f_A(\vec{p}, \vec{P})$ – momentum distribution function with the mean momentum $\vec{P}$.
Normalization condition: $\langle A|A\rangle = 1 \quad \Rightarrow \quad \int d^3p\, |f_A(\vec{p})|^2/(2\pi)^3 = 1$.

Coordinate-space wave packet with maximum at $\vec{x} = \vec{x}_0$ at the time $t - t_0$:

$$\Psi_A(x) = \int \frac{d^3p}{(2\pi)^3}\, f_A(\vec{p}) e^{-iE_A(\vec{p})(t-t_0)+i\vec{p}(\vec{x}-\vec{x}_0)}$$

QFT approach – contd.



$$|P_i\rangle = \int [dq]\, f_{Pi}(\vec{q}, \vec{Q})\, |P_i, \vec{q}\rangle\,, \qquad |P_f\rangle = \int [dk]\, f_{Pf}(\vec{k}, \vec{K})\, |P_f, \vec{k}\rangle\,,$$

$$|D_i\rangle = \int [dq']\, f_{Di}(\vec{q}\,', \vec{Q}')\, |D_i, \vec{q}\,'\rangle\,, \qquad |D_f\rangle = \int [dk']\, f_{Df}(\vec{k}', \vec{K}')\, |D_f, \vec{k}'\rangle\,.$$

The transition amplitude:

$$i\mathcal{A}_{\alpha\beta} = \langle P_f\, D_f | \hat{T} \exp\Big[-i \int d^4x\, \mathcal{H}_I(x) \Big] - \mathbb{1} | P_i\, D_i \rangle\,,$$

QFT approach – contd.

In the second order in weak interaction:

$$i\mathcal{A}_{\alpha\beta} \;=\; \sum_j U^*_{\alpha j} U_{\beta j} \int [dq]\, f_{Pi}(\vec{q}, \vec{Q}) \int [dk]\, f^*_{Pf}(\vec{k}, \vec{K})$$

$$\times \int [dq']\, f_{Di}(\vec{q}\,', \vec{Q}') \int [dk']\, f^*_{Df}(\vec{k}', \vec{K}')\, i\mathcal{A}_j^{p.w.}(q, k; q', k')\,.$$

Plane-wave amplitude:

$$i\mathcal{A}_j^{p.w.}(q, k; q', k') = \int d^4x_1 \int d^4x_2\, \tilde{M}_D(q', k')\, e^{-i(q'-k')(x_2 - x_D)}$$

$$\times\, i \int \frac{d^4p}{(2\pi)^4} \frac{\not{p} + m_j}{p^2 - m_j^2 + i\epsilon}\, e^{-ip(x_2 - x_1)}\, \tilde{M}_P(q, k)\, e^{-i(q-k)(x_1 - x_P)}$$

$\tilde{M}_{jP}$, $\tilde{M}_{jD}$ – production and detection amplitudes with neutrino spinors exluded. Full amplitudes:

$$M_{jP}(q, k) \equiv \frac{\bar{u}_{jL}(p)}{\sqrt{2p_0}} \tilde{M}_P(q, k)\,, \qquad M_{jD}(q', k') \equiv \tilde{M}_D(q', k') \frac{u_{jL}(p)}{\sqrt{2p_0}}$$

QFT approach – contd.

$$i\mathcal{A}_{\alpha\beta} = i\sum_j U^*_{\alpha j} U_{\beta j} \int \frac{d^4p}{(2\pi)^4}\, \Phi_{jP}(p^0,\vec{p})\Phi_{jD}(p^0,\vec{p})\, \frac{2p_0\, e^{-ip^0T+i\vec{p}\vec{L}}}{p^2-m_j^2+i\epsilon}\,.$$

$$\Phi_{jP}(p^0,\vec{p}) = \int d^4x_1' e^{ipx_1'} \int [dq] \int [dk]\, f_{Pi}(\vec{q},\vec{Q})\, f^*_{Pf}(\vec{k},\vec{K})\, e^{-i(q-k)x_1'} M_{jP}(q,k)$$

$$\Phi_{jD}(p^0,\vec{p}) = \int d^4x_2' e^{-ipx_2'} \int [dq'] \int [dk']\, f_{Di}(\vec{q}',\vec{Q}') f^*_{Df}(\vec{k}',\vec{K}')\, e^{-i(q'-k')x_2'} M_{jD}(q',k')$$

For $L,T \gg 1/p$ – fast oscillating factor in $i\mathcal{A}_{\alpha\beta}$ $\Rightarrow$ main contribution to integral over p^0 from the pole at $p^0 = E_j(\vec{p}) - i\epsilon$ (on-shell neutrinos).

$$\Downarrow$$

$$i\mathcal{A}_{\alpha\beta} = \Theta(T)\sum_j U^*_{\alpha j} U_{\beta j} \int \frac{d^3p}{(2\pi)^3}\, \Phi_{jP}(E_j(\vec{p}),\vec{p})\Phi_{jD}(E_j(\vec{p}),\vec{p})\, e^{-iE_j(\vec{p})T+i\vec{p}\vec{L}}$$

QFT approach – contd.

Compare with $\mathcal{A}_{ab}(T, \vec{L})$ obtained in the QM w. packet approach: the two amplitudes coincide if

$$f_j^P(\vec{p}) = \Phi_{jP}(E_j(\vec{p}), \vec{p})\,, \qquad f_j^D(\vec{p}) = \Phi^*_{jD}(E_j(\vec{p}), \vec{p})\,,$$

Easy to understand: $\Phi_{jP}(E_j(p), \vec{p})$ is the probability amplitude of ν production process in which ν_j is emitted with momentum $\vec{p}$
$\Rightarrow$ Φ_{jP} is momentum distribution function of the produced neutrino, i.e. the momentum-state wave packet $f_j^P(\vec{p})$. Similarly for neutrino detection.
N.B.: $f_j^P(\vec{p})$ and $f_j^D(\vec{p})$ are not "canonically" normalized.

Alternative approaches:

- $|P_f\nu_j\rangle \;=\; (S - \mathbb{1})|P_i\rangle\,, \qquad |\nu_j\rangle \;=\; \langle P_f|P_f\nu_j\rangle$
- In coord. space: ψ_{ν_j} = convolution of the ν source (prod. amplitude) and retarded propagator

All three approaches give the same results.

General properties of ν w. packets in QFT

$$f_j^P(\vec{p}) \simeq M_{jP}(Q,K)\int d^4x\, e^{iE_j(\vec{p})t-i\vec{p}\vec{x}}\int[dq]\int[dk] f_{Pi}(\vec{q},\vec{Q}) f^*_{Pf}(\vec{k},\vec{K}) e^{-i(q-k)x}$$

Integral over $\vec{x}$ gives $\sim \delta^{(3)}(\vec{q}-\vec{k}-\vec{p})$. Since $f_{Pi}(\vec{q},\vec{Q})$, $f_{Pf}(\vec{k},\vec{K})$ are sharply peaked at $\vec{Q}$ and $\vec{K}$ $\Rightarrow$ $f_j^P(\vec{p})$ is sharply peaked at

$\vec{P} \equiv \vec{Q} - \vec{K}.$ Width of the peak: $\sigma_{pP} \simeq \max\{\sigma_{P_i}, \sigma_{P_f}\}$

For external particles described by plane waves:

$$f_j^P(\vec{p}) = \frac{M_{jP}(Q,K)}{\sqrt{2E_{Pi}V \cdot 2E_{Pf}V}}\,\delta^{(4)}(Q-K-P)$$

In general: $f_j^P(\vec{p})$ $\Rightarrow$ $M_{jP}(Q,K) \times$ ("smeared δ-functions") representing approx. conservation of mean energies and mean momenta.

Matching QM & QFT expressions for ν w. p.

Example – Gaussian wave packets for external particles. QFT gives

$$f_j^P(\vec{p}) \;\propto\; [M_{jP}(Q,K)]/(\sigma_{eP}\sigma_{pP}^3)\,\exp\big[-g_P(E_j(\vec{p}),\,\vec{p})\big],$$

$$g_P(E_j(\vec{p}),\,\vec{p}) = \frac{(\vec{p}-\vec{P})^2}{4\sigma_{pP}^2} + \frac{[E_j(\vec{p}) - E_P - \vec{v}_P(\vec{p}-\vec{P})]^2}{4\sigma_{eP}^2}\,.$$

Here $\vec{P} \equiv \vec{Q} - \vec{K}\,, \quad E_P \equiv E_{Pi}(\vec{Q}) - E_{Pf}(\vec{K})\,,$

$$\sigma_{pP}^2 = \sigma_{pPi}^2 + \sigma_{pPf}^2\,, \qquad \sigma_{xP}\sigma_{pP} = \frac{1}{2}\,,$$

$$\vec{v}_P \equiv \sigma_{xP}^2\left(\frac{\vec{v}_{Pi}}{\sigma_{xPi}^2} + \frac{\vec{v}_{Pf}}{\sigma_{xPf}^2}\right), \qquad \Sigma_P \equiv \sigma_{xP}^2\left(\frac{\vec{v}_{Pi}^2}{\sigma_{xPi}^2} + \frac{\vec{v}_{Pf}^2}{\sigma_{xPf}^2}\right),$$

$$\sigma_{eP}^2 = \sigma_{pP}^2(\Sigma_P - \vec{v}_P^2) \equiv \sigma_{pP}^2\,\lambda_P\,, \qquad 0 \le \lambda_P \le 1\,.$$

Matching QM & QFT expressions for ν w. p.

Compare with Gaussian wave packet in QM approach:

$$f_j^P(\vec{p},\vec{P}) = \left(\frac{2\pi}{\sigma_{pP}^2}\right)^{3/4} \exp\left[-\frac{(\vec{p}-\vec{P})^2}{4\sigma_{pP}^2}\right]$$

To match the QM and QFT expression: expand $E_j(\vec{p})$ around $\vec{p}=\vec{P}$ and subst. into $g_P(E_j(\vec{p}),\vec{p})$:

$$\diamondsuit \qquad \boxed{g_P(E_j(\vec{p}),\vec{p}) = (p-P)^k\,\alpha^{kl}\,(p-P)^l - \beta^k(p-P)^k + \gamma_j}$$

$$\alpha^{kl} = \frac{1}{4\sigma_{eP}^2}\left[\lambda_P\,\delta^{kl} + (v_j - v_P)^k\,(v_j - v_P)^l + \frac{E_j - E_P}{E_j}(\delta^{kl} - v_j^k v_j^l)\right],$$

$$\beta^k = -\frac{1}{2\sigma_{eP}^2}(E_j - E_P)(v_j - v_P)^k\,, \qquad \gamma_j = \frac{(E_j - E_P)^2}{4\sigma_{eP}^2}\,.$$

Try to represent $g_P(E_j(\vec{p}),\vec{p})$ in the form

$$\diamondsuit \qquad \boxed{g_P(E_j(\vec{p}),\vec{p}) = (p-P_{\rm eff})^k\,\alpha^{kl}\,(p-P_{\rm eff})^l + \tilde{\gamma}_j}\,, \qquad \vec{P}_{\rm eff} \equiv \vec{P} + \vec{\delta}$$

Matching QM & QFT expressions for ν w. p.

$$\delta^k = -\frac{(E_j - E_P)(v_j - v_P)^k}{\lambda_P + (\vec{v}_j - \vec{v}_P)^2}\,, \qquad \tilde{\gamma}_j = \frac{(E_j - E_P)^2}{4\sigma_{eP}^2}\frac{\lambda_P}{\lambda_P + (\vec{v}_j - \vec{v}_P)^2}\,.$$

Diagonalization of α^{kl} gives $(OZ||(\vec{v}_j - \vec{v}_P))$:

$$(\sigma^x_{pP\,\mathrm{eff}})^2 = (\sigma^y_{pP\,\mathrm{eff}})^2 = \sigma_{pP}^2\,, \qquad \frac{1}{(\sigma^z_{pP\,\mathrm{eff}})^2} = \frac{1}{\sigma_{pP}^2} + \frac{(\vec{v}_j - \vec{v}_P)^2}{\sigma_{eP}^2}\,,$$

$\Rightarrow$ QM neutrino wave packets can match those obtained QFT if

- Momentum uncertainties of the neutrino mass eigenstates are replaced (anisotropic) effective ones: $-(\vec{p} - \vec{P})^2/(4\sigma_{pP}^2) \to$
  $-[(p^x - P^x_{\mathrm{eff}})^2/4(\sigma^x_{pP})^2 + (p^y - P^y_{\mathrm{eff}})^2/4(\sigma^y_{pP})^2 + (p^z - P^z_{\mathrm{eff}})^2/4(\sigma^z_{pP})^2]$.
- The mean momentum $\vec{P}$ is shifted according to $\vec{P} \to \vec{P}_{\mathrm{eff}} = \vec{P} + \vec{\delta}$.
- The wave packet of each neutrino mass eigenstate gets an extra factor $N_j = \exp[-\tilde{\gamma}_j]$.

Matching QM & QFT expressions for ν w. p.

If $|E_i - E_j| \ll \sigma_{eP} \quad \Rightarrow$

factors N_j are the same for all ν mass eigenstates, can be included in common normalization factor. In the opposite case – coherence of different neutrino mass eigenstates is lost.

$\sigma_{eP} \le \sigma_{pP} \quad \Rightarrow$ except for $\vec{v}_j \approx \vec{v}_P$ momentum uncertainty along $(\vec{v}_j - \vec{v}_P)$ is dominated by σ_{eP}.

In the stationary neutrino source limit $(\sigma_{eP}, \vec{v}_P \to 0)$, effective longitudinal mom. uncertainty $\sigma^z_{pP\,\mathrm{eff}} = 0$ even though the true mom. uncertainty $\sigma_{pP} \neq 0$.

$$\Downarrow$$

Coherence length $l_{\mathrm{coh}} \to \infty$

Oscillation probability in QFT

What is calculated in QFT is the probability of the overall production-propagation-detection process. How to extract from it the oscillation probability $P_{\alpha\beta}(L)$?

1. Recall the operational definition of $P_{\alpha\beta}(L)$. Detection rate for ν_β:

$$\Gamma_\beta^{\rm det} = \int dE\, j_\beta(E)\sigma_\beta(E)\,,$$

If a source at a distance L from the detector emits ν_α with the energy spectrum $d\Gamma_\alpha^{\rm prod}(E)/dE$:

$$j_\beta(E) = \frac{1}{4\pi L^2}\frac{d\Gamma_\alpha^{\rm prod}(E)}{dE}P_{\alpha\beta}(L,E)\,,$$

$\Rightarrow$ substititute into $\Gamma_\beta^{\rm det}$:

Oscillation probability in QFT

$$\Gamma^{\rm tot}_{\alpha\beta} \equiv \int dE\, \frac{d\Gamma^{\rm tot}_{\alpha\beta}(E)}{dE} = \frac{1}{4\pi L^2}\int dE\, \frac{d\Gamma^{\rm prod}_{\alpha}(E)}{dE}\, P_{\alpha\beta}(L,E)\, \sigma_\beta(E)$$

$$P_{\alpha\beta}(L,E) = \frac{d\Gamma^{\rm tot}_{\alpha\beta}(E)/dE}{\frac{1}{4\pi L^2}\,[d\Gamma^{\rm prod}_{\alpha}(E)/dE]\,\sigma_\beta(E)}\,.$$

An important ingredient: the assumption that the overall rate factorizes into the production rate, propagation (oscillation) probability and detection cross section.

If this does not hold, oscillation probability is undefined $\Rightarrow$

Need to deal instead with the overall rate of neutrino production, propagation and detection.

Oscillation probability in QFT

Try to cast $P^{\rm tot}_{\alpha\beta}$ in the same form (check if the factorization condition holds !)

$$i\mathcal{A}_{\alpha\beta} = i\sum_j U^*_{\alpha j}U_{\beta j}\int\frac{d^4p}{(2\pi)^4}\,\Phi_{jP}(p^0,\vec{p})\Phi_{jD}(p^0,\vec{p})\,\frac{2p_0\,e^{-ip^0T+i\vec{p}\vec{L}}}{p^2-m_j^2+i\epsilon}$$

Integrate first over $\vec{p}$, then over $p^0\equiv E$. Make use of Grimus-Stockinger theorem: for a large L, $A>0$ and a sufficiently smooth function $\psi(\vec{p})$,

$$\int d^3p\frac{\psi(\vec{p})\,e^{i\vec{p}\vec{L}}}{A-\vec{p}^2+i\epsilon} = -\frac{2\pi^2}{L}\psi(\sqrt{A}\tfrac{\vec{L}}{L})e^{i\sqrt{A}L}+\mathcal{O}(L^{-\frac{3}{2}}) \quad \Rightarrow$$

$$i\mathcal{A}_{\alpha\beta}(T,\vec{L}) = \frac{-i}{8\pi^2L}\sum_j U^*_{\alpha j}U_{\beta j}\int dE\,\Phi_P(E,p_j\vec{l})\Phi_D(E,p_j\vec{l})\;2E\,e^{-iE\,T+ip_jL}$$

where

$$p_j\equiv\sqrt{E^2-m_j^2}\,,\qquad \vec{l}\equiv\frac{\vec{L}}{L}\,,$$

Oscillation probability in QFT

Introduce

$$\tilde{P}^{\rm tot}_{\alpha\beta}(\vec{L}) = \int dT\, P_{\alpha\beta}(T,\vec{L}) = \frac{1}{8\pi^2}\frac{1}{4\pi L^2}\sum_{j,k} U^*_{\alpha j}U_{\beta j}U_{\alpha k}U^*_{\beta k}$$

$$\times \int dE\, \Phi_P(E,p_j\vec{l})\Phi_D(E,p_j\vec{l})\,\Phi^*_P(E,p_k\vec{l})\Phi^*_D(E,p_k\vec{l})\;(2E)^2\, e^{i(p_j-p_k)L}$$

Neutrino production probability:

$$P^{\rm prod}_\alpha = \sum_j |U_{\alpha j}|^2 \int \frac{d^3p_j}{(2\pi)^3}\,\left|\Phi_P(E,p_j)\right|^2 = \sum_j |U_{\alpha j}|^2 \frac{1}{8\pi^2}\int dE\,\left|\Phi_P(E,p_j)\right|^2 4Ep_j$$

Detection probability:

$$P^{\rm det}_\beta(E) = \sum_k |U_{\beta k}|^2 |\Phi_D(E,p_k)|^2 \frac{1}{V}\,,$$

Oscillation probability in QFT

Let the number of particles P_i entering the production region during time interval T_0 be N_P and number of D_i entering the detection region be N_D. Probability of neutrino emission during the finite interval of time t:

$$\mathcal{P}_\alpha^{\rm prod}(t) = N_P \int_0^t \frac{dt_P}{T_0}\, P_\alpha^{\rm prod} = N_P\, P_\alpha^{\rm prod} \frac{t}{T_0}\,, \quad \text{rate:} \quad \Gamma_\alpha^{\rm prod} = N_P\, P_\alpha^{\rm prod} \frac{1}{T_0}$$

Detection cross section:

$$\sigma_\beta(E) = \frac{N_D}{T_0} \sum_k |U_{\beta k}|^2 |\Phi_{kD}(E)|^2 \frac{E}{p_k}$$

Probability of the overall production-propagation-detection process:

$$\mathcal{P}_{\alpha\beta}^{\rm tot}(t,L) = \frac{N_P N_D}{T_0^2} \int_0^t dt_D \int_0^t dt_P\, P_{\alpha\beta}^{\rm tot}(T,L) \quad \Rightarrow$$

Oscillation probability in QFT

New integration variables $\tilde{T} \equiv (t_P + t_D)/2$ and $T = t_D - t_P \quad \Rightarrow$

$$\begin{aligned}\mathcal{P}^{\rm tot}_{\alpha\beta}(t, L) &= \frac{N_P N_D}{T_0^2}\Big[\int_0^t dT\, P^{\rm tot}_{\alpha\beta}(T, L)(t-T) + \int_{-t}^0 dT\, P^{\rm tot}_{\alpha\beta}(T, L)(t+T)\Big] \\ &= \frac{N_P N_D}{T_0^2}\Big[t\int_{-t}^t dT\, P^{\rm tot}_{\alpha\beta}(T, L) - \int_0^t dT\, T P^{\rm tot}_{\alpha\beta}(T, L) + \int_{-t}^0 dT\, T P^{\rm tot}_{\alpha\beta}(T, L)\Big] \\ &\equiv \frac{N_P N_D}{T_0^2}\Big[t I_1(t) - I_2(t) + I_3(t)\Big]\,.\end{aligned}$$

For large t (much larger than the time scales of the neutrino production and detection processes) $I_1 = \tilde{P}^{\rm tot}_{\alpha\beta}(L)$ whereas $I_2 = I_3 = 0 \quad \Rightarrow$

$$\mathcal{P}^{\rm tot}_{\alpha\beta}(t, L) = \frac{N_P N_D}{T_0^2}\, t\, \tilde{P}^{\rm tot}_{\alpha\beta}(L)\,, \qquad \Gamma^{\rm tot}_{\alpha\beta}(L) = \frac{N_P N_D}{T_0^2}\, \tilde{P}^{\rm tot}_{\alpha\beta}$$

Oscillation probability in QFT

$$\text{“}P_{\alpha\beta}(L,E)\text{”} = \frac{\sum_{j,k} U^*_{\alpha j} U_{\beta j} U_{\alpha k} U^*_{\beta k} \Phi_P(E,p_j)\Phi_D(E,p_j)\Phi^*_P(E,p_k)\Phi^*_D(E,p_k) e^{i(p_j-p_k)L}}{\sum_j |U_{\alpha j}|^2 \, |\Phi_P(E,p_j)|^2 \, p_j \; \sum_k |U_{\beta k}|^2 \, |\Phi_D(E,p_k)|^2 p_k^{-1}}$$

For $|p_j - p_k| \ll p_j, p_k$ (ultra-relativistic or quasi-degenerate in mass ν's):
In expressions for $\Gamma^{\rm prod}_\alpha$ and σ_β can replace

$$p_j \to p\,, \qquad \Phi_P(E,p_j) \to \Phi_P(E,p) \qquad (p\ -\ \text{average momentum})$$

$\Rightarrow$ in the denominator of “$P_{\alpha\beta}(L,E)$”:

$$\sum_j |U_{\alpha j}|^2 \, |\Phi_P(E,p_j)|^2 \, p_j \to |\Phi_P(E,p)|^2 \, p \sum_j |U_{\alpha j}|^2 = |\Phi_P(E,p)|^2 \, p\,,$$

$$\sum_k |U_{\beta j}|^2 \, |\Phi_D(E,p_k)|^2 \, p_k^{-1} \to |\Phi_D(E,p)|^2 \, p^{-1} \sum_k |U_{\beta k}|^2 = |\Phi_D(E,p)|^2 \, p^{-1}\,,$$

Cannot in general be done in the numerator of “$P_{\alpha\beta}(L,E)$” !

Oscillation probability in QFT

For $|p_j - p_k| \ll p_j, p_k$ $\Gamma_\alpha^{\rm prod}$ and σ_β do not depend on the elements of the mixing matrix $\Rightarrow$ factorization holds. $P_{\alpha\beta}(E, L)$ can be defined as a sensible quantity:

$$P_{\alpha\beta}(L,E) = \frac{\sum_{j,k} U_{\alpha j}^* U_{\beta j} U_{\alpha k} U_{\beta k}^* \Phi_P(E,p_j)\Phi_D(E,p_j)\Phi_P^*(E,p_k)\Phi_D^*(E,p_k) e^{i(p_j-p_k)L}}{|\Phi_P(E,p)|^2\,|\Phi_D(E,p)|^2}$$

Automatically satisfies unitarity, i.e. is properly normalized.

For $|p_j - p_k| \gg \sigma_p$ $(\Leftrightarrow \Delta m_{jk}^2/(2p) \gg \sigma_p)$ – $P_{\alpha\beta}(L,E)$ strongly suppressed.

In the opposite case

$$\frac{\Delta m_{jk}^2}{2p} \ll \sigma_p\,,$$

(prodution & detection coherence cond. satisfied) – $\Phi_P(E, p_{j,k})$, $\Phi_D(E, p_{j,k})$ can be pulled out of the sums in the numerator $\Rightarrow$ stand. osc. probabilty:

$$P_{\alpha\beta}(L,E) \,=\, \sum_{j,k} U_{\alpha j}^* U_{\beta j} U_{\alpha k} U_{\beta k}^*\, e^{-i\frac{\Delta m_{jk}^2}{2p}L}$$

Summary

- QM and QFT wave packet formalisms provide consistent approaches to neutrino oscillations.
- QFT approach is superior to the QM one:
  - Consistently takes into account neutrino production and detection mechanisms
  - Allows to obtain the neutrino wave packets used in the QM approach (instead of postulating them)
  - Automatically produces correctly normalized oscillation probability and clarifies the normalization prescription of QM approach
- $\Rightarrow$ the simplistic QM wave packet approach may need QFT-motivated modifications; however, once they have been done, one can still work within the QM framework without losing any essential physics content.

Photo: http://alexandermigdal.com/prose/paradise1.shtml

Backup slides

Problems with the plane-wave approach

- Same momentum $\Rightarrow$ oscillation probabilities depend only on time. Leads to a paradoxical result – no need for a far detector ! "Time-to-space conversion" ($x = vt \simeq t$) – assumes neutrinos to be point-like particles (notion opposite to plane waves).
- Same energy – oscillation probabilities depend only on coordinate. Does not explain how neutrinos are produced and detected at certain times. Correspponds to a stationary situation.

Plane wave approach $\Leftrightarrow$ exact energy-momentum conservation. Neutrino energy and momentum are fully determined by those of external particles $\Rightarrow$ only one mass eigenstate can be emitted!

Kinematic constraints

Same momentum and same energy assumptions: contradict kinematics!

Pion decay at rest $(\pi^+ \to \mu^+ + \nu_\mu, \;\; \pi^- \to \mu^- + \bar{\nu}_\mu)$:
For decay with emission of a massive neutrino of mass m_i:

$$E_i^2 = \frac{m_\pi^2}{4}\left(1 - \frac{m_\mu^2}{m_\pi^2}\right)^2 + \frac{m_i^2}{2}\left(1 - \frac{m_\mu^2}{m_\pi^2}\right) + \frac{m_i^4}{4m_\pi^2}$$

$$p_i^2 = \frac{m_\pi^2}{4}\left(1 - \frac{m_\mu^2}{m_\pi^2}\right)^2 - \frac{m_i^2}{2}\left(1 + \frac{m_\mu^2}{m_\pi^2}\right) + \frac{m_i^4}{4m_\pi^2}$$

For massless neutrinos: $E_i = p_i = E \equiv \frac{m_\pi}{2}\left(1 - \frac{m_\mu^2}{m_\pi^2}\right) \simeq 30 \;\; \mathrm{MeV}$

To first order in m_i^2:

$$E_i \simeq E + \xi \frac{m_i^2}{2E}\,, \qquad p_i \simeq E - (1-\xi)\frac{m_i^2}{2E}\,, \qquad \xi = \frac{1}{2}\left(1 - \frac{m_\mu^2}{m_\pi^2}\right) \approx 0.2$$

Kinematic constraints

Same momentum or same energy would require
$\xi = 1$ or $\xi = 0$ – not the case!

Also: would violate Lorentz invariance of the oscillation probability

How can wrong assumptions lead to the correct oscillation formula ?

When are neutrino oscillations observable?

Keyword: *Coherence*

Neutrino flavour eigenstates ν_e, ν_μ and ν_τ are coherent superpositions of mass eigenstates ν_1, ν_2 and ν_3 $\Rightarrow$ oscillations are only observable if

- neutrino production and detection are coherent
- coherence is not (irreversibly) lost during neutrino propagation.

Possible decoherence at production (detection): If by accurate E and p measurements one can tell (through $E = \sqrt{p^2 + m^2}$) which mass eigenstate is emitted, the coherence is lost and oscillations disappear!

Full analogy with electron interference in double slit experiments: if one can establish which slit the detected electron has passed through, the interference fringes are washed out.

When are neutrino oscillations observable?

Another source of decoherence: wave packet separation due to the difference of group velocities Δv of different mass eigenstates.

If coherence is lost: Flavour transition can still occur, but in a non-oscillatory way. E.g. for $\pi \to \mu\nu_i$ decay with a subsequent detection of ν_i with the emission of e:

$$P \,\propto\, \sum_i P_{\rm prod}(\mu\,\nu_i)P_{\rm det}(e\,\nu_i) \,\propto\, \sum_i |U_{\mu i}|^2|U_{ei}|^2$$

– the same result as for averaged oscillations.

How are the oscillations destroyed? Suppose by measuring momenta and energies of particles at neutrino production (or detection) we can determine its energy E and momentum p with uncertainties σ_E and σ_p. From $E_i = \sqrt{p_i^2+m_i^2}$:

$$\sigma_{m^2} \;=\; \left[(2E\sigma_E)^2 \,+\, (2p\sigma_p)^2\right]^{1/2}$$

When are neutrino oscillations observable?

If $\sigma_{m^2} < \Delta m^2 = |m_i^2 - m_k^2|$ – one can tell which mass eigenstate is emitted.

$\sigma_{m^2} < \Delta m^2$ implies $2p\sigma_p < \Delta m^2$, or $\sigma_p < \Delta m^2/2p \simeq l_{\rm osc}^{-1}$.

But: To measure p with the accuracy σ_p one needs to measure the momenta of particles at production with (at least) the same accuracy $\Rightarrow$ uncertainty of their coordinates (and the coordinate of ν production point) will be

$$\sigma_{\rm x,\,prod} \gtrsim \sigma_p^{-1} \sim l_{\rm osc}$$

$\Rightarrow$ Oscillations washed out. Similarly for neutrino detection.

Natural necessary condition for coherence (observability of oscillations):

$$L_{\rm source} \ll l_{\rm osc}\,, \qquad L_{\rm det} \ll l_{\rm osc}$$

No averaging of oscillations in the source and detector

Satisfied with very large margins in most cases of practical interest

Wave packet separation

Wave packets representing different mass eiegenstate components have different group velocities v_{gi} $\Rightarrow$ after time $t_{\rm coh}$ (coherence time) they separate $\Rightarrow$ Neutrinos stop oscillating! (Only averaged effect observable).

Coherence time and length:

$$\Delta v \cdot t_{\rm coh} \simeq \sigma_x\,; \qquad l_{\rm coh} \simeq v t_{\rm coh}$$

$$\Delta v = \frac{p_i}{E_i} - \frac{p_k}{E_k} \simeq \frac{\Delta m^2}{2E^2}$$

$$\boxed{l_{\rm coh} \simeq \frac{2E^2}{\Delta m^2}\, v\sigma_x}$$

The standard formula for $P_{\rm osc}$ completely neglects decoherence effects. How should it be modified when decoherence is present?

Oscillations and QM uncertainty relations

Neutrino oscillations – a QM interference phenomenon, owe their existence to QM uncertainty relations

Neutrino energy and momentum are characterized by uncertainties σ_E and σ_p related to the spatial localization and time scale of the production and detection processes. These uncertainties

- allow the emiited/absorbed neutrino state to be a coherent superposition of different mass eigenstates
- determine the size of the neutrino wave packets $\Rightarrow$ govern decoherence due to wave packet separation

σ_E – the effective energy uncertainty, dominated by the smaller one between the energy uncertainties at production and detection. Similarly for σ_p.

The paradox of σ_E and σ_p

QM uncertainty relations: σ_p is related to the spatial localization of the production (detection) process, while σ_E to its time scale $\Rightarrow$ independent quantities.

On the other hand: Neutrinos propagting macroscopic distances are on the mass shell. For on-shell mass eigenstates $E^2 = p^2 + m_i^2$ means

$$E\sigma_E \;=\; p\sigma_p$$

How can this be understood?

The solution: At production, neutrinos are *not* on the mass shell. They go on shell only after they propagate $x \sim$ (a few)$\times$ De Broglie wavelengths. After that their energy and momentum get related by $E^2 = p^2 + m_i^2 \quad \Rightarrow$ the larger uncertainty shrinks towards the smaller one to satisfy $E\sigma_E \;=\; p\sigma_p$.

On-shell relation between E and p allows to determine the less certain of the two through the more certain one, reducing the error of the latter.

What determines the length of ν w. packets?

The length of ν w. packets: $\sigma_x \sim 1/\sigma_p$. For propagating on-shell neutrinos:

$$\sigma_p \simeq \min\{\sigma_p^{\rm prod},\, (E/p)\sigma_E^{\rm prod}\} = \min\{\sigma_p^{\rm prod},\, (1/v_g)\sigma_E^{\rm prod}\}$$

Which uncertainty is smaller at production, $\sigma_p^{\rm prod}$ or $\sigma_E^{\rm prod}$?

Consider neutrino production in decays of an unstable particle localized in a box of size L_S. Time between two collisions with the walls of the box: T_S.

- If $T_S < \tau$ (τ – lifetime of the parent unstable particle) $\Rightarrow$ $\sigma_E \simeq T_S^{-1}$ (collisional broadening). Mom. uncertainty: $\sigma_p \simeq L_S^{-1}$.

  But: $L_S = v_S T_S \quad \Rightarrow \quad \sigma_E < \sigma_p$ (a consequence of $v_S < 1$)

- If $T_S > \tau$ (quasi-free parent particle) $\Rightarrow$ $\sigma_E \simeq \tau^{-1} = \Gamma$.

  $\sigma_p \simeq [(p/E)\tau]^{-1} \simeq [(p/E)\sigma_E]^{-1}$, i.e. $\sigma_E \simeq (p/E)\sigma_p < \sigma_p$.

The length of ν w. packets – contd.

In both cases $\boxed{\sigma_E < \sigma_p}$ $\Leftarrow$ also when $\nu' s$ are produced in collisions.

$$\Longrightarrow \qquad \sigma_{p\ \mathrm{eff}} \simeq \frac{\sigma_E}{v_g}\,, \qquad\qquad \boxed{\sigma_x \simeq \frac{v_g}{\sigma_E}}$$

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