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# Gribov-80 Memorial Workshop on Quantum Chromodynamics and Beyond'

*26 - 28 May 2010*

## Integrability in Planar N=4 Gauge Theory


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# Integrability in Planar $\mathcal{N} = 4$ Gauge Theory

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ICTP Trieste
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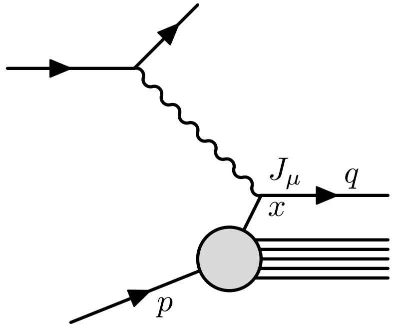
# Deep Inelastic Scattering

Correlator of currents. OPE:




$$J_\mu(x)J_\nu(y) \sim |y-x|^{T_\mathcal{O}-6}\big[(y-x)^{S_\mathcal{O}} \cdot \mathcal{O}(x)\big].$$

- Operators of lowest twist $T_\mathcal{O} = D_\mathcal{O} - S_\mathcal{O}$ are dominant.
- Scaling violations: anomalous dimensions $\delta D_\mathcal{O}$ of twist-two operators.
- Classical application for DGLAP evolution equation.

This talk: **Anomalous dimensions** (twist-two/any local operator)

- in conformal $\mathcal{N} = 4$ supersymmetric gauge theory (instead of QCD),
- of single-trace ops in the 't Hooft limit $N_\mathrm{c} = \infty$,
- at arbitrary 't Hooft coupling $\lambda = g_\mathrm{YM}^2 N_\mathrm{c}$,
- how they can be obtained from an integrable spin chain,
- how this is related to scattering amplitudes.

# I. Maximally Supersymmetric Yang–Mills Theory

# $\mathcal{N} = 4$ Super Yang–Mills Theory

We all know and like gauge field theories.

- QCD/Standard Model: 4D Yang–Mills theory coupled to matter.

$D = 4$ **$\mathcal{N} = 4$ super Yang–Mills** ($\mathcal{N} = 4$ SYM) very similar. Take:

- 4D Yang–Mills theory with $\mathrm{U}(N_\mathrm{c})$ gauge group: gauge field $\mathcal{A}_\mu$,
- more matter fields: 4 fermions $\Psi^a_\alpha$, 6 scalars $\Phi_m$,
- all fields massless and adjoint: $N_\mathrm{c} \times \bar{N}_\mathrm{c}$ matrices,
- various interactions: more Feynman diagrams,
- single coupling constant $\lambda = g^2_{\mathrm{YM}} N_\mathrm{c}$ (plus top. $\theta$-angle).

Unique fine-tuned action due to supersymmetry:

$$S_{\mathcal{N}=4} = \frac{16\pi^2 N_\mathrm{c}}{\lambda} \int \frac{d^4x}{4\pi^2}\, \mathrm{Tr}\Big(\tfrac{1}{4}(\mathcal{F}_{\mu\nu})^2 + \tfrac{1}{2}(\mathcal{D}_\mu \Phi_m)^2 - \tfrac{1}{4}[\Phi_m, \Phi_n]^2 + \ldots\Big).$$

Extra fields and interactions in $\mathcal{N} = 4$ SYM conspire to yield:

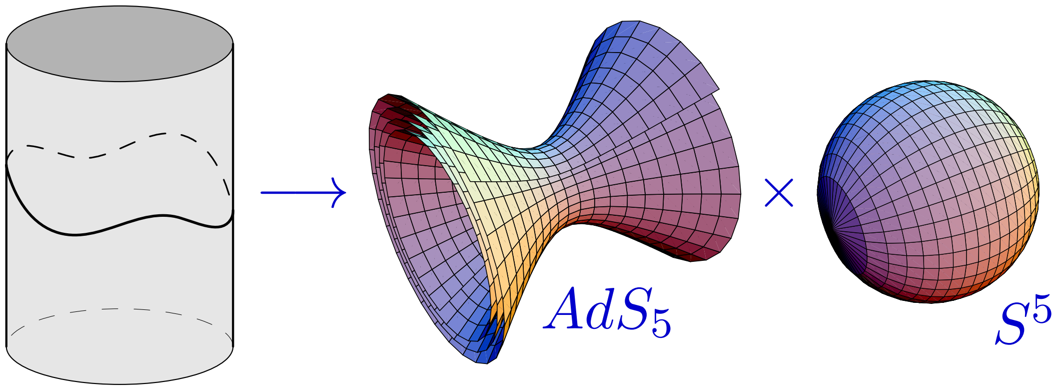
- drastical cancellations in calculations,
- remarkably simple final results.

# Remarkable $\mathcal{N}=4$ SYM

$\mathcal{N}=4$ SYM has some remarkable properties:

- "Finite" theory: beta-function exactly zero, no running coupling.
- Unbroken conformal symmetry: superconformal symmetry $\widetilde{\mathrm{PSU}}(2,2|4)$.
- AdS/CFT conjecture: exact duality to a string theory. [Maldacena hep-th/9711200]
- And some more mysterious ones. . .

**IIB Superstrings** on the curved $AdS_5 \times S^5$ superspace:




$\times$ fermi $= \dfrac{\widetilde{\mathrm{PSU}}(2,2|4)}{\mathrm{Sp}(1,1)\times\mathrm{Sp}(2)}$

Coset space sigma model with worldsheet and string couplings: $\lambda$, $g_\mathrm{s}$.

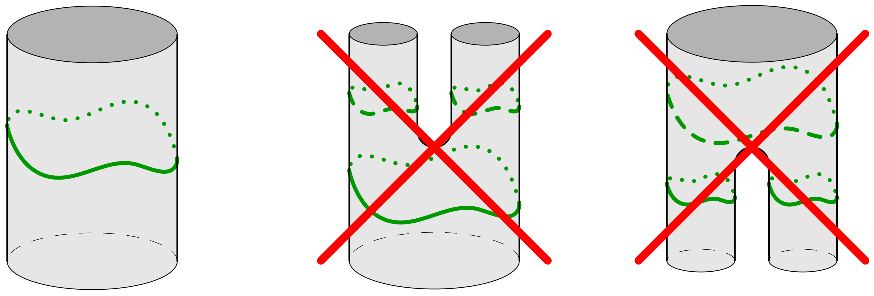
## $\mathcal{N}=4$ SYM and AdS/CFT:

- Towards a better understanding of gauge theory!

# Planar Limit

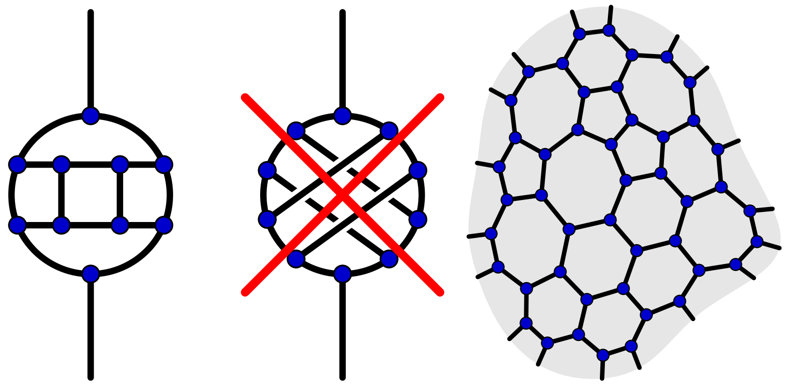
**Large-$N_c$ Expansion:** Formal insight into string/gauge similarity. ['t Hooft Nucl. Phys. B72, 461]

**String Theory:**



- No string coupling $g_s = 0$, no string splitting or joining.
- Strictly cylindrical worldsheet.
- Sigma model coupling $\lambda$ remains.

**Gauge Theory:**

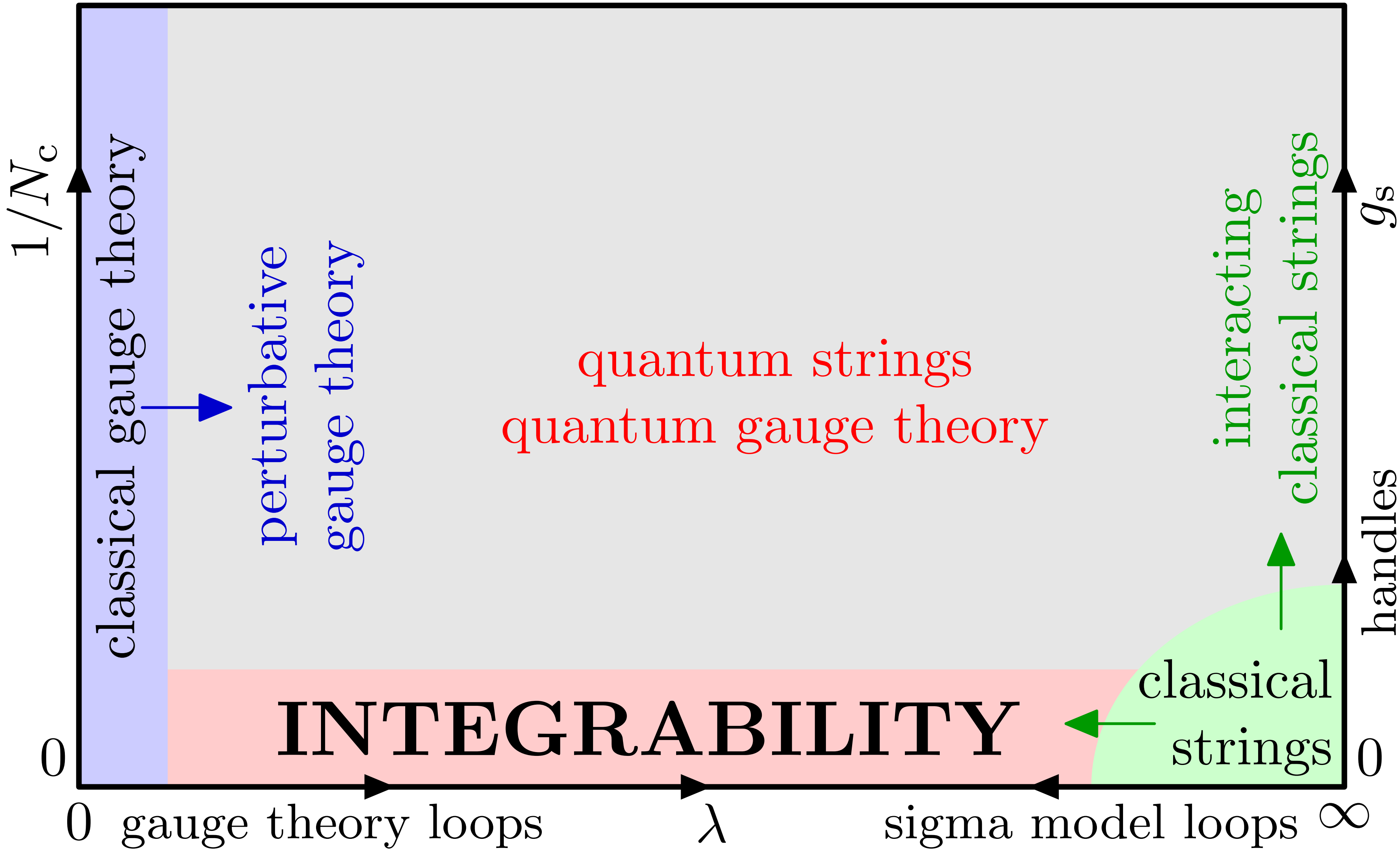


- Large-$N_c$ limit: $N_c = \infty$, $g_{YM} = 0$, 't Hooft coupling $\lambda = g_{YM}^2 N_c$ remains.
- Only planar Feynman graphs, no crossing propagators.
- Drastic combinatorial simplification.

**Planar Limit:** $g_s = 0$, $N_c = \infty$.

- Surprises [Lipatov hep-th/9311037] [Lipatov hep-th/9812336] [Minahan Zarembo] [NB Kristjansen Staudacher] [Bena Polchinski Roiban] [Anastasiou, Bern Dixon, Kosower] [Drummond, Henn Smirnov, Sokatchev] . . .
- AdS/CFT Integrability! Efficient methods for calculation.

# Charted Territory

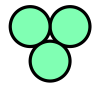
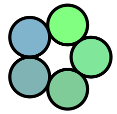


# II. Local Operators and Scaling Dimensions

# Local Operators

$\mathcal{N}=4$ SYM is a 4D conformal field theory. Principal objects of a CFT:
**Local Composite Operators** (position space picture)
Local, gauge-invariant combinations of the fields, e.g.

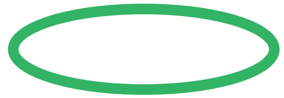
$$\mathcal{O}' = \operatorname{Tr} \varPhi_1 \varPhi_2 (\mathcal{D}_1 \mathcal{D}_2 \varPhi_2) \mathcal{F}_{23} (\mathcal{D}_1 \mathcal{F}_{24}) + \ldots .$$

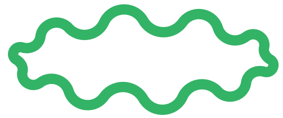
$\mathcal{O}$:  $\mathcal{O}'$:  …

Observable: scaling dimension $D_{\mathcal{O}}$, e.g. two-point function

$$\big\langle \mathcal{O}(x)\, \mathcal{O}(y) \big\rangle = \frac{C}{|x-y|^{2D_{\mathcal{O}}}} \,.$$

**String Theory:**
States: Solutions $X$ of classical equations of motion
plus quantum corrections.
Energy: Charge $E_X$ for translation along AdS-time.

$X$: 

$X'$: 

…

**AdS/CFT:** String energies and gauge dimensions match, $E(\lambda) = D(\lambda)$?!

# From Gauge Theory to Spin Chains

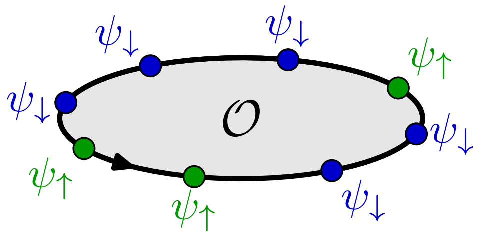
Planar limit: Use spin chain terminology to describe local operators/states.
Only single-trace operators relevant (fields $\mathcal{W}$ are $N_\mathrm{c} \times \bar{N}_\mathrm{c}$ matrices)

$$\mathcal{O} = \mathrm{Tr}\, \mathcal{W}_1\, \mathcal{W}_2\, \mathcal{W}_3 \ldots \mathcal{W}_L, \qquad \mathcal{W}_k \in \{\mathcal{D}^n \Phi, \mathcal{D}^n \Psi, \mathcal{D}^n \mathcal{F}\}.$$

Translate single-trace operators to spin chain states:

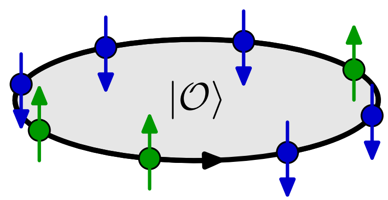
Consider, e.g. (adjoint) spin-$\frac{1}{2}$ fermions $\psi_\alpha$

$$\mathcal{O} = \mathrm{Tr}\, \psi_\downarrow \psi_\downarrow \psi_\uparrow \psi_\downarrow \psi_\downarrow \psi_\downarrow \psi_\uparrow \psi_\uparrow.$$




Identify $\psi_\uparrow \to |\uparrow\rangle$ and $\psi_\downarrow \to |\downarrow\rangle$:

$$|\mathcal{O}\rangle = |\downarrow\downarrow\uparrow\downarrow\downarrow\downarrow\uparrow\uparrow\rangle.$$

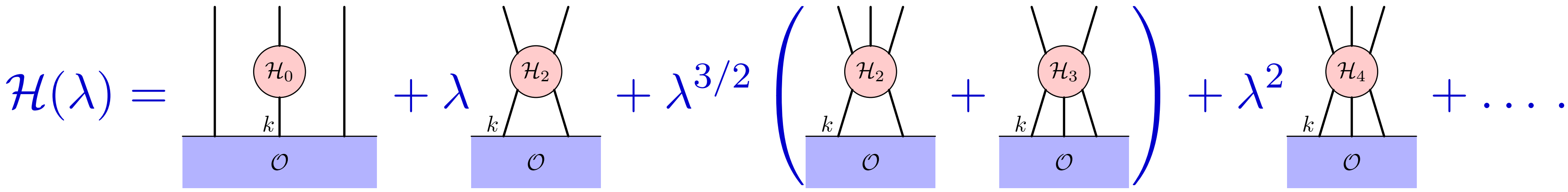



Generalised spin chain with many spin orientations: $\{\mathcal{D}^n \Phi, \mathcal{D}^n \Psi, \mathcal{D}^n \mathcal{F}\}$.

Spectrum of scaling dimensions: Eigenvalues of spin chain Hamiltonian $\mathcal{H}$.

# Integrable Spin Chain

Spin chain Hamiltonian from Feynman diagrams: [Minahan Zarembo] [NB hep-th/0307015] [NB hep-th/0310252]




Tree-level Hamiltonian $\mathcal{H}_0$ trivial, disregard.

**One-loop Hamiltonian**:

- $\mathcal{H}_2$ contains integrable Heisenberg XXX Hamiltonians. [Lipatov hep-ph/9812336] [Minahan Zarembo]
- Complete one-loop Hamiltonian $\mathcal{H}_2$ integrable! [NB, Staudacher hep-th/0307042]
- Bethe ansatz determines spectrum of integrable spin chains. [Bethe, Z. Phys. A71, 205 (1931)]

Higher-loop corrections $\mathcal{H}_{3,4,\ldots}$ describe **long-range** interactions of spins:

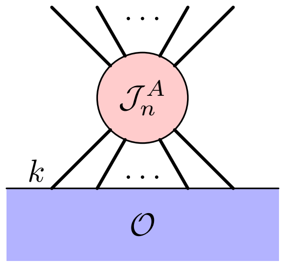
- Corrections apparently preserve integrability. [NB Kristjansen Staudacher] [NB hep-th/0310252] [NB, Dippel Staudacher]
- No established algebraic framework for higher-loop corrections.

Exact integrability in AdS/CFT is useful working hypothesis. Assumption!

# Integrability: Yangian Symmetry

Many ways to define integrability. E.g. hidden symmetry: **Yangian**!
Here: Yangian based on $\mathfrak{psu}(2,2|4)$ superconformal algebra.
Interacting superconformal representation $\mathcal{J}^A$ on local operators [NB hep-th/0310252]

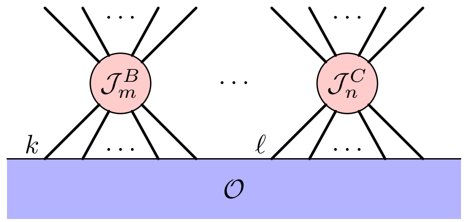
$$\mathcal{J}^A = + \ldots + \lambda^{n/2} \sum_k$$




$$+ \ldots .$$

Interacting bi-local Yangian generators $\widehat{\mathcal{J}}^A$ [Dolan Nappi Witten] [Serban Staudacher] [Agarwal Ferretti] [Zwiebel hep-th/0610283] [NB, Erkal 0711.4813]

$$\widehat{\mathcal{J}}^A = \ldots + \lambda^{(m+n)/2} F^A_{BC} \sum_{k<\ell}$$

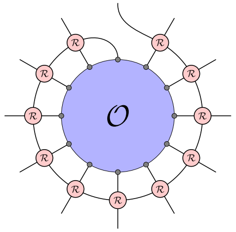



$$+ \ldots .$$

- Hamiltonian part of superconformal algebra: almost commutes.
- Hamiltonian commutes with Yangian up to boundary terms.
- Local operators form superconformal (not Yangian) multiplets.

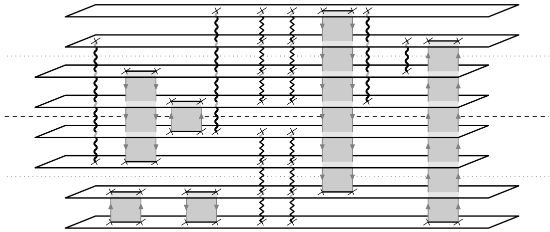
# Spectrum from Integrability

## one-loop gauge theory



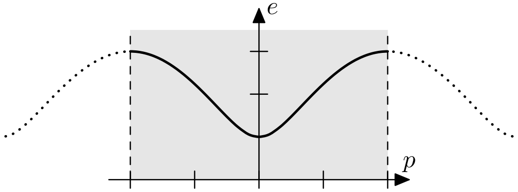

Bethe equations

## classical strings

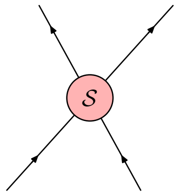


spectral curve

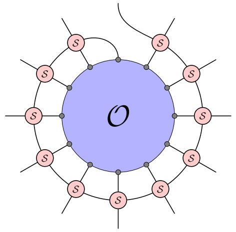
## asymptotic spectrum



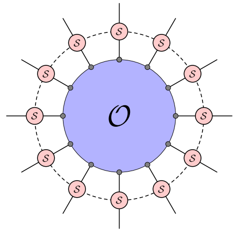

magnon
dispersion




scattering
matrix

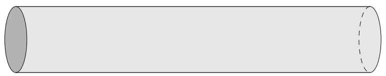



Bethe
equations

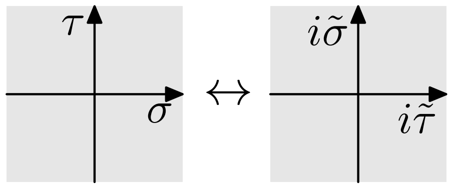



Lüscher
terms

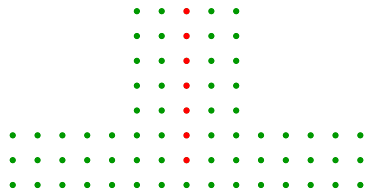
## finite-size spectrum



TBA




mirror model



Y-system

Efficient? Exact!? Proofs!

# III. Cusp Dimension and Interpolation

# Cusp Dimension from Bethe Equations

Use Bethe equations to compute some anomalous dimension.
Useful object: Twist-two operators with spin $S$ (spin-$S$ excited $q\bar{q}$ meson).

$$\mathcal{O}_S \simeq \operatorname{Tr} \mathcal{X} \overleftrightarrow{\mathcal{D}}^S \mathcal{Y}.$$

A lot is known about their anomalous dimensions $\delta D_S$:

- QCD: $\delta D_S$ responsible for scale violations in DIS.
- DGLAP evolution equations.
- BFKL evolution equations.
- Maximum transcendentality principle for $\mathcal{N} = 4$ SYM.
- Large-$S$ behaviour: cusp anomalous dimension $D_{\text{cusp}}$

$$\delta D_S = D_{\text{cusp}} \log S + \ldots .$$

# Cusp Dimension

Cusp dimension determined by AdS/CFT planar integrable system!
Compute cusp dimension using Bethe equations. Integral eq.: [Eden Staudacher]

$$\psi(x) = K(x,0) - \int_0^\infty K(x,y)\,\frac{dy\,y}{e^{2\pi y/\sqrt{\lambda}}-1}\,\psi(y).$$

Kernel $K = K_0 + K_1 + K_\mathrm{d}$ with [NB, Eden Staudacher]

$$K_0(x,y) = \frac{x\,\mathrm{J}_1(x)\,\mathrm{J}_0(y) - y\,\mathrm{J}_0(x)\,\mathrm{J}_1(y)}{x^2-y^2}\,,$$

$$K_1(x,y) = \frac{y\,\mathrm{J}_1(x)\,\mathrm{J}_0(y) - x\,\mathrm{J}_0(x)\,\mathrm{J}_1(y)}{x^2-y^2}\,,$$

$$K_\mathrm{d}(x,y) = 2\int_0^\infty K_1(x,z)\,\frac{dz\,z}{e^{2\pi z/\sqrt{\lambda}}-1}\,K_0(z,y).$$

Cusp anomalous dimension: $D_\mathrm{cusp} = (\lambda/\pi^2)\psi(0)$.

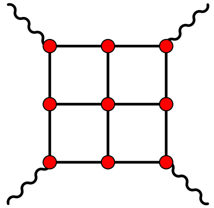
# Weak/Strong Expansion

Weak-coupling solution of integral equation [NB, Eden Staudacher]

$$D_{\text{cusp}}(\lambda) = \frac{1}{2}\frac{\lambda}{\pi^2} - \frac{1}{96}\frac{\lambda^2}{\pi^2} + \frac{11}{23040}\frac{\lambda^3}{\pi^2} - \left(\frac{73}{2580480} + \frac{\zeta(3)^2}{1024\pi^6}\right)\frac{\lambda^4}{\pi^2} \pm \ldots .$$

Confirmed by gluon scattering amplitudes [Bern Dixon Smirnov][Bern, Czakon, Dixon Kosower, Smirnov]

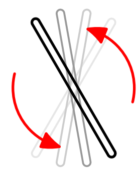
$$A(p,\lambda) \simeq A^{(0)}(p)\exp\Big(2D_{\text{cusp}}(\lambda)M^{(1)}(p)\Big).$$



Connection between integrability & scattering amplitudes? later...

Strong-coupling asymptotic solution of integral equation [Casteill Kristjansen][Basso Korchemsky Kotański]

$$E_{\text{cusp}}(\lambda) = \frac{\sqrt{\lambda}}{\pi} - \frac{3\log 2}{\pi} - \frac{\beta(2)}{\pi\sqrt{\lambda}} + \ldots .$$

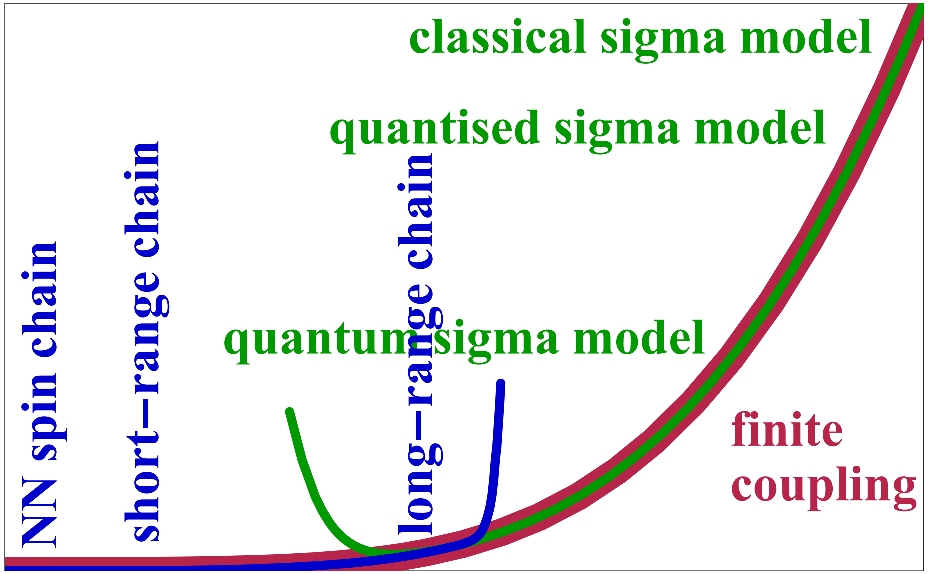


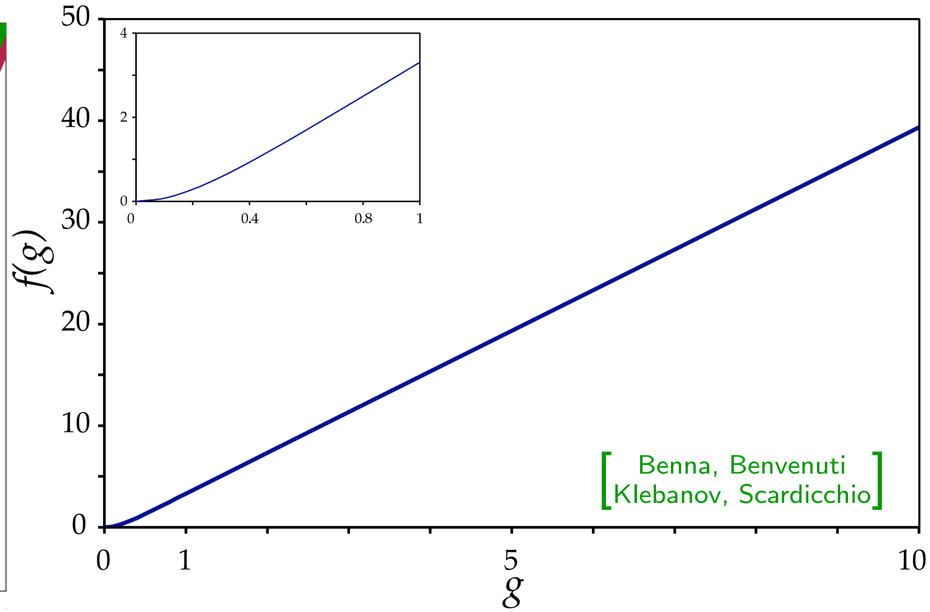
Agreement with semiclassical energy of spinning string. [Gubser Klebanov Polyakov][Frolov Tseytlin][Roiban Tirziu Tseytlin]

# Finite-Coupling Interpolation

Smooth interpolation between perturbative gauge and string theory

- in the Bethe equations (left), [NB, Eden Staudacher]
- in the cusp dimension (right). [Benna, Benvenuti Klebanov, Scardicchio]




An exact result in a (planar) 4D gauge theory at finite coupling.

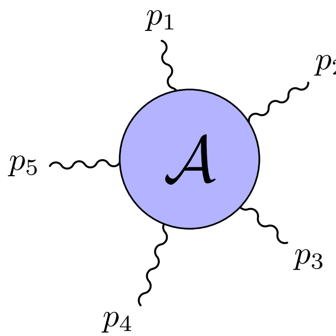
# IV. Scattering Amplitudes

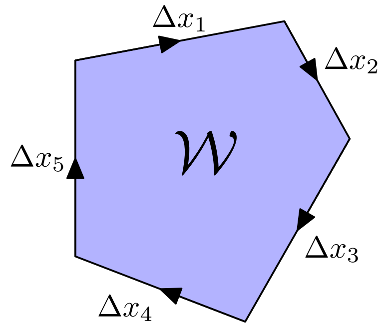
# Local Operators vs. Scattering Amplitudes

Cusp dimension $D_{\text{cusp}}$ well known from:

- IR behaviour of scattering amplitudes,
- UV behaviour of Wilson loop cusp.

MHV amplitude vs. light-like polygonal Wilson loop: [Alday Maldacena][Drummond Korchemsky Sokatchev][Brandhuber Heslop Travaglini]

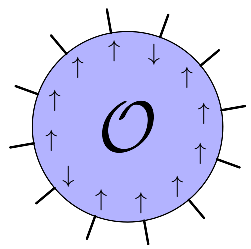


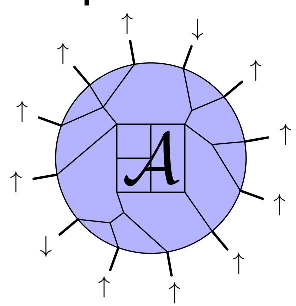




- light-like momenta $p_k^2 = 0$
- momentum conservation $\sum_k p_k = 0$
- light-like separations $\Delta x_k^2 = 0$
- closure $\sum_k \Delta x_k = 0$

Set $p_k = \Delta x_k$ and match Wilson loop expectation value with amplitude.

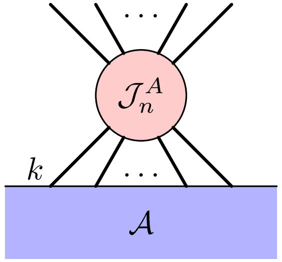
Single-trace local operator vs.
colour-ordered scattering amplitude:



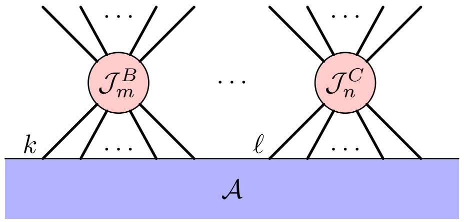
# Integrability for Scattering Amplitudes

Superconformal rep. $\mathcal{J}^A$ on amplitudes [Witten hep-th/0312171] [Bargheer, NB, Galleas Loebbert, McLoughlin] [NB, Henn McLoughlin, Plefka]

$$\mathcal{J}^A = \ldots + \lambda^{n/2} \sum_k$$ 

$$+ \ldots .$$

Yangian representation $\widehat{\mathcal{J}}^A$ (dual conformal) [Drummond, Henn Korchemsky Sokatchev] [Drummond Henn Plefka] [NB, Henn McLoughlin, Plefka]

$$\widehat{\mathcal{J}}^A = \ldots + \lambda^{(m+n)/2} F^A_{BC} \sum_{k<\ell}$$ 

$$+ \ldots .$$

- Amplitude invariant under superconformal algebra.
- Amplitude invariant under Yangian.

Algebraic determination of S-matrix?! Graßmannian? Y-system? [Cachazo]

# V. Conclusions

# Conclusions

**Integrability in Planar $\mathcal{N} = 4$ SYM**:

- Hidden (algebraic) property.
- Methods for efficient determination of observables.

Integrability observed for and applied to:

- local operators & scaling dimensions,
- scattering amplitudes,
- Wilson loops.

**Outlook: genus expansion?**