Lecture III (outline)

Cycle map. Intermediate Jacobian. Deligne cohomology

In this lecture $k = \mathbb{C}$. As before X is smooth, q.-projective irreducible variety defined over $\mathbb{C}$.
Associated with X we have now also the corresponding complex manifold X_{an}. By abuse of language we denote often this manifold also shortly by the same letter X and the topology on $X = X_{an}$ is the classical (analytic) topology.

(A) The cycle map

__Proposition__ X smooth, projective, irreducible $/\mathbb{C}$
$0 \leq p \leq d = \dim X$; put $q = d-p$. There exists a homomorphism $\gamma_{X,\mathbb{Z}}$ (shortly $\gamma_{\mathbb{Z}}$) as follows:

$$\begin{tikzcd} Z^p(X) \arrow[r, "\gamma_{\mathbb{Z}}"] \arrow[dr] & Hdg^p(X) \subset H^{2p}(X_{an}, \mathbb{Z}) \\ & CH^p(X) \arrow[u, "\gamma_{\mathbb{Z}}"] \end{tikzcd}$$

Where $Hdg^p(X) := H^{2p}(X, \mathbb{Z}) \cap j^{-1}(H^{p,p}(X))$ and with $j : H(X, \mathbb{Z}) \to H(X, \mathbb{C})$ the natural map and $H^{2p}(X, \mathbb{C}) = \bigoplus_{r+s=2p} H^{r,s}(X)$ the Hodge decomposition.
(of course we mean $H^{2p}(X_{an}, \mathbb{C})$)

__Remark__
Since $H(X_{an}, -)$ is a good topology (see lecture II) we have that the equivalence relation $Z_{hom}(X)$ is adequate and since $Z_{rat} \subseteq Z_{hom}$ the cycle map $\gamma_{\mathbb{Z}}$ factors through $CH^p(X)$ and - by abuse of language - we use the same letter $\gamma_{\mathbb{Z}}$).

Construction of $\gamma_{\mathbb{Z}}$ (outline, [V], p 252)

By linearity ~~we consider~~ it suffices to consider the case that $Z \hookrightarrow X$ is a closed subvariety of X.

Write $U = X - Z$ and use the exact sequence

$$\to H^{2p-1}(U,\mathbb{Z}) \to H^{2p}(X,U;\mathbb{Z}) \xrightarrow{\rho} H^{2p}(X,\mathbb{Z}) \to H^{2p}(U,\mathbb{Z}) \to \cdots$$

and via the theorem of Thom $T : H^{2p}(X,U,\mathbb{Z}) \xrightarrow{\sim} H^{0}(X,\mathbb{Z})$

Then put $\gamma_{\mathbb{Z}}(Z) = \rho.T(1_Z)$

Next in order to relate it to the Hodge decomposition we must use the de Rham isomorphism

$$H^i_{sing}(X,\mathbb{C}) \xrightarrow{\sim} H^i_{dR}(X,\mathbb{C}) = \bigoplus_{r+s=i} H^{r,s}(X)$$

And using this we have the

<u>Lemma</u> $j\gamma_{\mathbb{Z}}(Z) \in H^{p,p}(X) \subset H^{2p}(X,\mathbb{C})$

(to be proved in the lecture) from which we get the required result:

$$\gamma_{\mathbb{Z}}(Z) \in Hdg^p(X)$$

(B) Hodge cycles. Hodge conjecture.

So recall:

$$Hdg^p(X) := \{\eta \in H^{2p}(X,\mathbb{Z}) \,;\, j(\eta) \in H^{p,p}(X) \subset H^{2p}(X,\mathbb{C})\}$$

with $j : H^{2p}(X,\mathbb{Z}) \longrightarrow H^{2p}(X,\mathbb{C})$ the natural map.

The elements of $Hdg^p(X)$ ~~are~~ called <u>Hodge classes or Hodge cycles of type (p,p)</u>.

And we have

$$\gamma_{\mathbb{Z}} : Z^p(X) \longrightarrow \cancel{H^{2p}}\ Hdg^p(X)$$

(always X smooth, irreducible and proper over $\mathbb{C}$, so ~~X~~ X_{an} compact manifold and by abuse of notation we write $X_{an} = X$)

In the original form Hodge conjectured that the map $\gamma_{\mathbb{Z}}$ is surjective, ~~however~~ (integral Hodge conjecture), however Atiyah-Hirzebruch showed in 1962 that for $p>1$ this integral form is not true (later more counterexamples by Kollar and Totaro). Therefore the conjecture has to be modified to:

Hodge conjecture

$$\text{(HC)} \quad Z^p(X)\otimes\mathbb{Q} \xrightarrow{\gamma_{\mathbb{Q}}} Hdg^p(X)\otimes\mathbb{Q} \text{ onto ?}$$

$$\qquad\qquad\qquad\qquad\qquad\quad \| $$

$$H^{2p}(X,\mathbb{Q})\cap H^{p,p}(X)$$

For divisors there is the famous

Lefschetz (1,1) - theorem

$$\gamma_{\mathbb{Z}} : \ Div(X) \longrightarrow Hdg^1(X) \quad \text{is onto.}$$

(in the lecture we hope to outline the modern proof of Kodaira - Spencer & using sheaf-theory and the exponential sequence).

For cycles of codimension $p>1$ $(p\neq d-1)$ the Hodge conjecture is wide open and one of the most famous conjectures in mathematics ((for $p=d-1$ the Hodge conjecture with $\mathbb{Q}$-coefficients is true: it follows from the case $p=1$ and the so-called strong Lefschetz theorem))

(C) Intermediate Jacobians

Always X smooth, proper, irreducible over $\mathbb{C}$, $X = X_{an}$ denotes also the corresponding complex (compact) manifold.

Recall the Hodge decomposition:

$$H^i(X, \mathbb{C}) = \bigoplus_{r+s=i} H^{r,s}(X) \qquad (\text{and } H^{s,r} = \overline{H^{r,s}})$$

and the corresponding Hodge filtration.

$$F^j H^i(X, \mathbb{C}) = \bigoplus_{r \geq j} H^{r, i-r}(X) = H^{i,0} + H^{i-1,1} + \dots + H^{j,i-j}$$

So we have a descending filtration:

$$F^0 = H^i(X, \mathbb{C}) \supseteq F^1 \supseteq \dots \supseteq F^i \supseteq F^{i+1} = 0$$

Definition

The p-th intermediate Jacobian (of Griffiths) is

$$J^p(X) = H^{2p-1}(X, \mathbb{C}) / F^p H^{2p-1} + H^{2p-1}(X, \mathbb{Z})$$
$$= (H^{p-1,p} + \dots + H^{0,2p-1}) / H^{2p-1}(X, \mathbb{Z})$$

Lemma

$J^p(X)$ is a complex torus.

In general it is not an abelian variety.

Remarks

For $p=1$ $J^1(X) = H^1(X, \mathbb{C}) / H^{1,0} + H^1(X, \mathbb{Z})$ Picard variety of X.

For $p=d$ $\cancel{J^{2d}}$ $J^d(X) = H^{2d-1}(X, \mathbb{C}) / H^{d,d-1} + H^{2d-1}(X, \mathbb{Z})$ Albanese variety of X.

For $X = C$ curve (i.e. $d=1$) we get $J^1(X) = J^d(X) =: J(X)$ the Jacobian ~~of X = C~~ variety of C

The Picard and Albanese variety are abelian varieties

(D) Abel - Jacobi map

For $Z \in Z^p(X)$ we have the cycle map $\gamma_{\mathbb{Z}}(Z) \in H^{2p}(X, \mathbb{Z})$ which is a topological invariant of Z; when this $\gamma_{\mathbb{Z}}(Z) = 0$ we have an analytic invariant, namely

Theorem.

$\exists$ a homomorphism $AJ^p : Z^p_{hom}(X) \longrightarrow J^p(X)$

and AJ^p factors in fact through $CH^p_{hom}(X)$.

(AJ is called the Abel-Jacobi map)

In the lecture we shall discuss the construction.
For $X = C$ a curve we get the classical Abel-Jacobi map $Div_{(0)}(C) \to J(C)$, where $Div_{(0)}$ are the divisors of degree 0, and factorizing through linear equivalence we get the theorem of Abel-Jacobi $CH^1_{alg}(C) \xrightarrow{\sim} \cancel{J(C)}\ J(C)$; for $X = X_d$ we get for $p = 1$ the Picard map and for $p = d$ the Albanese map.

(E) Deligne cohomology. Deligne cycle map

In this section $X = X_{an}$ is the complex analytic manifold corresponding to the smooth, projective variety X (irreducible defined over $\mathbb{C}$) and, as before, the topology is the classical topology on $X = X_{an}$.

Recall the holomorphic de Rham complex

$$\Omega^{\cdot}_X : = \quad 0 \to \Omega^0_X \xrightarrow{d} \Omega^1_X \xrightarrow{d} \Omega^2_X \to \cdots\cdots$$

where Ω^i_X are the holomorphic differential forms of degree i ($\Omega^0_X = \mathcal{O}_{X_{an}}$) which is by the holomorphic Poincaré lemma ([G-H], p448) a resolution for $\mathbb{C}$, written

$$0 \to \mathbb{C} \to \Omega^{\cdot}_X .$$

Now let $A \subset \mathbb{C}$ be a subring (ex. $A = \mathbb{Z}, \mathbb{Q}$ or $\mathbb{R}$), ~~write~~ put $A(n) = (2\pi i)^n A \subset \mathbb{C}$ as subgroup.

Deligne (and Beilinson) introduced the complex (now called Deligne complex)

$$A(n)_{\mathcal{D}} : = \quad 0 \to \underset{0}{A(n)} \to \underset{1}{\mathcal{O}_X} \xrightarrow{d} \underset{2}{\Omega^1_X} \xrightarrow{d} \cdots \to \underset{n}{\Omega^{n-1}_X} \to 0 \to 0 \cdots$$

in degrees

~~which~~ and considered the hypercohomology ([G-H], p445) of this complex:

Def.: $$H^i_{\mathcal{D}}(X, A(n)) : = \mathbb{H}^i(X, A(n)_{\mathcal{D}})$$

the so-called Deligne cohomology (with coefficients in $A(n)$).

More generally if $Y \hookrightarrow X$ is a closed immersion of analytic manifolds one can consider the Deligne-Beilinson cohomology with support on Y:

Def.: $H^i_{Y,\mathcal{D}}(X, A(n)) := \mathbb{H}^i_Y(X, A(n)_{\mathcal{D}})$.

Ex. Take $A = \mathbb{Z}$

Ex. 1. $n = 0$ $\quad H^i_{\mathcal{D}}(X, \mathbb{Z}(0)) = H^i(X, \mathbb{Z})$.

Ex. 2 $n = 1$

The complex $\mathbb{Z}(1)_{\mathcal{D}}$ is quasi-isomorphic ([G-H] p.446) to the complex $\mathcal{O}_X^{*\,\bullet} := 1 \to \mathcal{O}_X^* \to 1$ shifted by -1, i.e. to $\mathcal{O}_X^{*\,\bullet}[-1]$ via the exponential map $\exp(z) = e^{2\pi i z}$. Explicit:

$$\begin{array}{lccccccc}
\mathbb{Z}(1)_{\mathcal{D}}: & 0 & \to & 2\pi i\mathbb{Z} & \longrightarrow & \mathcal{O}_X & \to & 0 \\
& & & \downarrow{\scriptstyle \exp} & & \downarrow{\scriptstyle \exp} & & \\
\mathcal{O}_X^{*\,\bullet}[-1]: & 1 & \to & 1 & \longrightarrow & \mathcal{O}_X^* & \to & 1
\end{array}$$

From this we get:

$$H^2_{\mathcal{D}}(X, \mathbb{Z}(1)) \cong H^1(X_{an}, \mathcal{O}^*_{X_{an}}) = \mathrm{Pic}(X_{an}) \underset{\uparrow \text{ GAGA}}{=} \mathrm{Pic}(X)$$

Now "recall" (lecture 2) the exact sequence

$$(*) \qquad 0 \to \underset{\substack{\| \\ J^1(X)}}{\mathrm{Pic}^0(X)} \longrightarrow \mathrm{Pic}(X) \longrightarrow \underset{\substack{\| \\ \mathrm{Hdg}^1(X)}}{NS(X)} \longrightarrow 0$$

Ex. 3 . $n = p$ (general case! $1 \leq p \leq d = \dim X$)

The following theorem shows that Deligne cohomology gives a beautiful generalization of (*) :

<u>Theorem</u> (Deligne) There is an exact sequence :

$$0 \to J^p(X) \to H^{2p}_{\mathcal{D}}(X, \mathbb{Z}(p)) \to Hdg^p(X) \to 0$$

Proof : to be discussed in lecture.

Moreover Deligne constructed also a cycle map

$$\gamma_{\mathcal{D}} : Z^p(X) \longrightarrow H^{2p}_{\mathcal{D}}(X, \mathbb{Z}(p))$$

which "catches" both the classical cycle map and the Abel - Jacobi map

<u>Theorem</u> There is a commutative diagram, with exact rows,

$$\begin{tikzcd}
0 \arrow[r] & Z^p_{hom}(X) \arrow[r] \arrow[d, "AJ"] & Z^p(X) \arrow[r] \arrow[d, "\gamma_{\mathcal{D}}"] & Z^p(X)/Z^p_{hom}(X) \arrow[r] \arrow[d, "\gamma_{\mathbb{Z}}"] & 0 \\
0 \arrow[r] & J^p(X) \arrow[r] & H^{2p}_{\mathcal{D}}(X, \mathbb{Z}(p)) \arrow[r] & Hdg^p(X) \arrow[r] & 0
\end{tikzcd}$$

(see for instance [Green et al], lect. 2 of Green and lectures ~~[MT], chap~~ *ibid* Murre, chap. IV).

References for lecture III.

For the basics of complex algebraic geometry see the book of Griffiths - Harris [Gr-H.]

The topics discussed in this lecture III are all thoroughly treated in the book of Voisin [V]. For some of the ~~lecture~~ topics one can also look to some of the lectures ~~in~~ from the CIME lectures in Torino 1993 (~~Springer~~ [Green et al], Springer LNM 1594). For the Deligne - Beilinson cohomology one can look also to [E.V].

[G-H] Griffiths - Harris, "Principles of algebraic geometry", Wiley & Sons

[E.V] Esnault - Viehweg, Deligne - Beilinson cohomology, in: "Beilinson's conjectures on Special Values of L-Functions", Rapoport, etc. editors, Perspectives in Math., Vol 4, Academic Press

[Green et al], Green, etc.. "Algebraic Cycles and Hodge Theory", Springer Lect. in Math., vol 1594

[V] Voisin, "Hodge Theory and Complex Algebraic Geometry". Vol I, II Cambridge ~~D~~ Studies in advanced mathematics, Vol. 76 and 77.

Lecture IV (outline)

Algebraic versus homological equivalence. Griffiths group.

In this lecture we discuss the important discovery of Griffiths (1969) that ~~in codim~~ for cycles of codimension > 1 algebraic and homological equivalence are different. This shows that there is a striking difference between the theory of divisors and the theory of algebraic cycles of codimension > 1.

(A) The starting point is:

Theorem 1 (Griffiths 1969)

Let Y be a smooth, projective, irreducible variety defined over $\mathbb{C}$ of even dimension $d = 2m$. Let $\{X_t\}$ be a Lefschetz pencil of hyperplane sections of Y. Assume that $H^{2m-1}(Y) = 0$ and that

$$H^{2m-1}(X_t, \mathbb{C}) \neq H^{m,m-1}(X_t) + H^{m-1,m}(X_t)$$

Let $Z \in \mathcal{Z}^m(Y)$ be such that $Z \cap X_t$ is algebraically equivalent to zero for "very general" $t \in \mathbb{P}^1$ (the parameter space of the pencil), then Z is homologically equivalent to zero on Y.

Remarks.

1. "Very general" means here that $\exists\, S \subset \mathbb{P}^1$ a countable set of points and that we must take $t \notin S$.
2. The proof (to be discussed) uses the theory of the intermediate Jacobians and normal functions on the one hand and also the "classical" Picard–Lefschetz theory of the Lefschetz pencil. A key role is played by the irreducibility of the monodromy ~~represent~~ representation
$$\rho : \pi_1(U) \longrightarrow \mathrm{Aut}\, H^{2m-1}(X_{t_0}, \mathbb{Q})$$
where $U = \{t \in \mathbb{P}^1 \,;\, X_t \text{ is smooth}\}$

(B) Griffiths group.

$$\mathrm{Griff}^p(X) := Z^p_{hom}(X) \Big/ Z^p_{alg}(X) \cong CH^p_{hom}(X) \Big/ CH^p_{alg}(X)$$

Due to the existence of the so-called "Chow varieties" the $\mathrm{Griff}^p(X)$ is a countable group.

$\mathrm{Griff}^1(X) = 0$ and $\mathrm{Griff}^d(X_d) = 0$. However using the above theorem 1 Griffiths proved in 1969:

Theorem Consider the quintic hypersurface in $\mathbb{P}^4$.
For a "very general" such quintic hypersurface $X_t \subset \mathbb{P}^4$ we have $\mathrm{Griff}^2(X_t) \otimes \mathbb{Q} \neq 0$

~~V~~ Remarks

1. "Very general" means $t \notin B$ where B is a union of countably many algebraic subsets of the parameter space $V = H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(5))$ of such quintic hypersurfaces.
2. The proof of Griffiths uses $J^2(X_t)$ and in fact he shows that $\exists\, Z \in Z^2_{hom}(X_t)$ s.t. $AJ^2(Z)$ is not a torsion point on $J^2(X_t)$ for such very general X_t.
3. We mention that Clemens proved later (1983) that for such X_t we ~~get~~ moreover get $\dim_{\mathbb{Q}} \mathrm{Griff}(X_t) \otimes \mathbb{Q} = \infty$.
4. In the above results a crucial role is played by the fact that ~~$\mathbb{P}(X)$~~ $J^2(X) \neq 0$. However in 1993 Nori proved that there exists (very general) varieties ~~such that~~ and cycles $Z \in Z^p(X)$ for $p \geq 3$ such that $AJ^p(Z) = 0$ but such that the image of Z in $\mathrm{Griff}^p(X) \otimes \mathbb{Q}$ is not zero!
(See the nice lectures of J. Nagel [N]).

5. In the above examples the varieties are always "very general". However there exists also varieties X defined over a number field, or even over $\mathbb{Q}$ itself, for which $\mathrm{Griff}^p(X) \otimes \mathbb{Q} \neq 0$. The first such example (to my knowledge) is due to Bruno Harris.

References to lecture IV

For the results of Griffiths and related results see Voisin's book [V] (Vol II, chap 2, section 8) or also her lecture 7 in [Green et al].
See also J. Nagel's lectures [N]

[Green et al] M. Green, – "Algebraic Cycles and Hodge Theory", Springer LNM, vol 1594

[N] J. Nagel, "Lectures on Nori's connectivity theorem" in Transcendental Aspects of Algebraic Cycles (Proc. Grenoble Summer School 2001) London Math. Soc. Lecture Note Series 313, Cambridge Univ. Press

[V] C. Voisin "Hodge Theory and Complex Algebraic Geometry", Vol II. Cambridge Studies in Adv. Math, Vol. 77

Lecture V (outline)

The Albanese kernel

Results of Mumford, Bloch and Bloch-Srinivas

In this lecture we return to the case of an algebraically closed field k (but otherwise arbitrary).

In the following we always assume (unless explicitly stated otherwise) that the varieties are smooth, projective, irreducible and defined over k.

Let $X = X_d$ be such a variety.

Recall: $CH^1_{alg}(X) \xrightarrow{\sim} Pic^0(X)_{red}$, the <u>Picard variety</u>, an abelian variety (if $k = \mathbb{C}$ the $J^1(X)$)

For zero-cycles we have a homomorphism

$$\alpha_X : CH^d_{alg}(X) \twoheadrightarrow Alb(X),$$

the <u>Albanese variety</u> again an abelian variety and α_X is the albanese map (if $k = \mathbb{C}$, $Alb(X) = J^d(X)$ and $\alpha_X = AJ^d$ the Abel-Jacobi map), however in general α_X is not injective (on the contrary!)

Put $T(X) := CH^d_{alb}(X) := Ker(\alpha_X)$ the <u>albanese kernel</u>

Ⓐ <u>Theorem of Mumford</u> (1969)

Let S be an algebraic surface $/ \mathbb{C}$. Let $p_g(S) := \dim H^{2,0}(S) = \dim H^0(S, \Omega^2_S)$ (geometric genus of S).

If $p_g(S) \neq 0$ then $T(S)$ is "infinite dimensional".

In the lecture we shall make this precise, but it implies that $T(S)$ can not be represented ~~by~~ (parametrized) by an algebraic variety, i.e. $T(S)$ is "very large" for such a surface.

(B) Generalization and reformulation by Bloch.

The notion "infinite dimensional" is closely related to the concept "weakly representable" introduced and used by Bloch.

Let $\Omega \supset k$ be a so-called "universal domain", i.e. Ω is an algebraically closed field of infinite transcendence degree over k, hence every $L \supset k$ of finite transcendence degree over k can be embedded in Ω, i.e. $k \subset L \subset \Omega$ (ex. $k = \overline{\mathbb{Q}}$, $\Omega = \mathbb{C}$).

Definition. X/k
A subgroup $A \subset CH^d(X)$ (for instance $CH^d_{alg}(X)$) is weakly representable if there exist a curve C smooth, but not necessarily irreducible, and a cycle class $T \in CH^d(C \times X)$ such that the corresponding morphism $T_* : CH_0(C_L) \longrightarrow A_L$ is surjective for all $k \subset L \subset \Omega$ (To be more specific: say $A = CH^d_{alg}(X)$; assume now that we have choosen on each component of C a point $e \in C(k)$ and normalized such that $T(e) = 0$ then we require that $T_* : CH_{0,alg}(C_L) \to CH^d_{alg}(X_L)$ is surjective*) for all fields L with $k \subset L \subset \Omega$ and transc. deg L/k finite).

Now ~~returning to~~ again let X_d/k, assume that we have choosen a Weil cohomology theory with coefficient field $F \supset \mathbb{Q}$ (for instance $F = \mathbb{Q}$ if $X/\mathbb{C}$ or $F = \mathbb{Q}_\ell$ if $H(X) = H_{et}(X_{\bar{k}})$); consider $NS(X) \otimes_{\mathbb{Q}} F \hookrightarrow H^2(X)$, put $H^2(X)_{alg}$ for the image and $H^2(X)_{tr} := H^2(X) / H^2(X)_{alg}$

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*) i.e. the subgroup generated by the images of T_* is $CH^d_{alg}(X)$ itself

Now Bloch proved the following theorem

<u>Theorem</u> (see [B], p I-24).
Let S be an algebraic surface / $k = \bar{k}$. Assume that $H^2(S)_{tr} \neq 0$, then $CH^2_{alg}(S)$ is not weakly representable.

We get the result of Mumford if we take $k = \mathbb{C}$ and note that $p_g(S) = \dim H^2(S)_{tr}$ since in that case $H^2(S)_{tr} = H^{20}(S) = H^0(S, \Omega^2_S)$.

The proof (to be discussed in the lecture) goes via the theorem in (C) below.

(C) <u>A result on the diagonal.</u>

<u>Theorem</u> (Bloch-Srinivas)
Let $X = X_d$ be as usual. Assume that there exists a closed subset $V \subsetneqq X$ (V not nec. smooth or irreducible) such that for $U = X - V$ we have $CH_0(U_\Omega) = 0$.
Now consider the diagonal $\Delta(X) \subset X \times X$.
Then there exist an integer $N > 0$, a divisor D (or better a closed subset $D \subsetneqq X$) and two d-dimensional <u>cycles</u> Γ_1 and Γ_2 on $X \times X$ with $\sup \Gamma_1 \subset V \times X$ and $\sup \Gamma_2 \subset X \times D$ such that in $CH^d(X \times X)$ we have

$$N . \Delta = \Gamma_1 + \Gamma_2$$

<u>Remarks</u>
1. The theorem above follows from this since if $X = S$ and if $CH^2_{alg}(S)$ is <u>weakly representable</u> then we get such cycles Γ_1 and Γ_2 and both these cycles ~~opera~~ Γ_i $(i=1,2)$ operate <u>trivially</u> on $H^2(S)_{trans}$.

2. The above theorem has important consequences on the $CH^2_{alg}(X)$ (to be discussed if time permits).

(D) Bloch conjecture

Let S be again such an algebraic surface. Bloch Conjectures that, conversely, if $H^2(S)_{tr} = 0$ that $CH^2(S_{alg})$ is weakly representable.

~~Based upon~~ Using the classification of surfaces this can be proved for surfaces with "Kodaira dimension" $\kappa(S) < 2$. For surfaces of "general type" (i.e. $\kappa(S) = 2$) it has been proven for some special cases (like Godeaux surfaces), but otherwise it is wide open!

(E) Bloch-Beilinson filtration

In his book [B], ~~book~~ p. I-12 Bloch stressed the existence of a filtration on $CH^2(S)$ namely

$$F^0 = CH^2(S) \supset F^1 = CH^2_{alg}(S) = CH^2_{hom}(S) \supset F^2 = T(S) = CH^2_{alb}(S) \supset F^3 = 0$$

Bloch and Beilinson (and later many others) conjecture that more generally there is ~~some~~ a filtration of a similar nature on ~~each~~ every $CH^i(X_d) \otimes_{\mathbb{Z}} \mathbb{Q}$

If time permits we shall say something about this in the lecture (but - in order to avoid confusion - the existence of such a filtration is a conjecture, wide open)

References to Lecture 5

[B] S. Bloch, "Lectures on algebraic cycles"
Duke Univ., 1980

[B-Sr] S. Bloch – V. Srinivas, "Remarks on correspondences and algebraic cycles"
Am. J. of Math. 105 (1983)

[V] C. Voisin, "Hodge Theory and Algebraic Cycles", II
Cambridge studies in adv. math, Vol. 77
See especially Chap III.