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On the limit set of Complex Klenian Groups acting on PC_2

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On the limit set of Complex Klenian Groups acting on $\mathbb{P}^2_{\mathbb{C}}$

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Let G be a subgroup of $PSL(3,\mathbb{C})$ acting on $\mathbb{P}^2_{\mathbb{C}}$.

$$L_0(G) := \overline{\{x \in \mathbb{P}^2_{\mathbb{C}} : |Stab_G(x)| = \infty\}}$$

$$L_1(G) := \overline{\bigcup_{x \in \mathbb{P}^2_{\mathbb{C}} \setminus (L_0(G))} (G \cdot x)'}$$

$$L_2(G) := \overline{\bigcup_{K \subset \mathbb{P}^2_{\mathbb{C}} \setminus (L_0(G) \cup L_1(G))} (G \cdot K)'}$$

The Limit Set of G is defined as

$$\Lambda(G) = L_0(G) \cup L_1(G) \cup L_2(G).$$

The Discontinuity Domain of G is defined as

$$\Omega(G) = \mathbb{P}^2_{\mathbb{C}} - \Lambda(G)$$

If $\Omega(G) \neq \varnothing$ we say G is Complex Kleinian.

EXAMPLE.

$$g = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$L_0(g) = \{e_1, e_2, e_3\},$$

$$L_1(g) = \{e_1, e_2, e_3\},$$

$$L_2(g) = \overleftrightarrow{e_1, e_2} \cup \overleftrightarrow{e_2, e_3}$$

then

$$\Lambda(g) = L_0(g) \cup L_1(g) \cup L_2(g) = \overleftrightarrow{e_1, e_2} \cup \overleftrightarrow{e_2, e_3}$$

The set $\Omega(g)$ is not the maximal open set where $\langle g \rangle$ acts properly and discontinuously.

If $g, g' \in PSL(3, \mathbb{C})$ are conjugated in $PSL(3, \mathbb{C})$ then there exists $h \in PSL(3, \mathbb{C})$ such that $\Lambda(g') = h\Lambda(g)$.

In order to list all possible limit sets of cyclic groups it suffices to consider the following Jordan canonical forms:

Strongly loxodromic.

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \; |\lambda_1| < |\lambda_2| < |\lambda_3|$$

$$L_0(g) = \{e_1, e_2, e_3\},$$

$$L_1(g) = \{e_1, e_2, e_3\},$$

$$L_2(g) = \overleftrightarrow{e_1, e_2} \cup \overleftrightarrow{e_2, e_3}$$

Screw

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \; |\lambda_1| = |\lambda_2| < |\lambda_3|$$

$$\frac{\lambda_2}{\lambda_1} = e^{2\pi i\theta}$$

CASE 1. θ is a *rational* number.

$$L_0(g) = L_1(g) = L_2(g) = \overleftrightarrow{e_1, e_2} \cup \{e_3\} = \Lambda(g)$$

CASE 2. θ is *not* a *rational* number

$$L_0(g) = \{e_1, e_2, e_3\},$$

$$L_1(g) = L_2(g) = \overleftrightarrow{e_1, e_2} \cup \{e_3\} = \Lambda(g)$$

Complex Homotetia

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}, \ |\lambda_1| < |\lambda_2|$$

$$L_0(g) = L_1(g) = L_2(g) = \overleftrightarrow{e_1, e_2} \cup \{e_3\} = \Lambda(g).$$

Loxoparabolic

$$\begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}, \ |\lambda_1| < |\lambda_2|$$

$$L_0(g) = L_1(g) = \{e_1, e_3\}$$

$$L_2(g) = \overleftrightarrow{e_1, e_2} \cup \overleftrightarrow{e_1, e_3} = \Lambda(g).$$

Elliptoparabolic

$$\begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}, \ |\lambda_1| = |\lambda_2|, \lambda_1 \neq \lambda_2,$$

$$\frac{\lambda_1}{\lambda_2} = e^{2\pi i\theta}$$

CASE 1. θ is a *rational* number

$$L_0(g) = \overleftrightarrow{e_1, e_3},$$

$$L_1(g) = \{e_1\} = L_2(g)$$

$$\Lambda(g) = \overleftrightarrow{e_1, e_3}$$

Elliptoparabolic

$$\begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}, \ |\lambda_1| = |\lambda_2|, \lambda_1 \neq \lambda_2,$$

$$\frac{\lambda_1}{\lambda_2} = e^{2\pi i\theta}$$

CASE 2. θ is *not* a *rational* number

$$L_0(g) = \{e_1, e_3\}$$

$$L_1(g) = \overleftrightarrow{e_1, e_3} = \Lambda(g)$$

$$L_2(g) = \{e_1\}$$

Parabolics

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$L_0(g) = \overleftrightarrow{e_1, e_3}$$

$$L_1(g) = \{e_1\} = L_2(g)$$

$$\Lambda(g) = \overleftrightarrow{e_1, e_3}$$

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

$$L_0(g) = L_1(g) = \{e_1\}$$

$$L_2(g) = \overleftrightarrow{e_1, e_2} = \Lambda(g)$$

Elliptic

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \ |\lambda_1| = |\lambda_2| = |\lambda_3|$$

CASE 1. g has finite order

$$L_0(g) = L_1(g) = L_2(g) = \varnothing$$

CASE 2. g has infinite order

$$L_0(g) = \{e_1, e_2, e_3\},$$

$$L_1(g) = \mathbb{P}^2_{\mathbb{C}}$$

$$L_2(g) = \varnothing$$

$$\Lambda(g) = \mathbb{P}^2_{\mathbb{C}}$$

What about limit sets of discrete subgroups of $PU(2,1)$ acting on $\mathbb{P}^2_{\mathbb{C}}$?

If $G \leq PU(2,1)$ is discrete then $L(G)$ the *limit set* according to Chen-Greenberg is the set of accumulation points in $\partial\mathbb{H}^2_{\mathbb{C}}$ of the G-orbit of any point $p \in \mathbb{H}^2_{\mathbb{C}}$

THEOREM. The limit set $L(G)$ is the intersection

$$L(G) = \Lambda(G) \cap \partial\mathbf{H}^2_{\mathbb{C}} ,$$

and $\Lambda(G)$ is the union of all complex projective lines l_z tangent to $\partial\mathbf{H}^2_{\mathbb{C}}$ at points in $L(G)$:

$$\Lambda(G) = \bigcup_{z \in L(G)} l_z .$$

Furthermore, if G is non-elementary then the action of G is minimal on $L(G) \subset \partial\mathbf{H}^2_{\mathbb{C}}$, i.e. all orbits are dense, while the orbit of each line l_z is dense in $\Lambda(G)$ (though the G-action on $\Lambda(G)$ is not minimal).

THEOREM. (Barrera-Navarrete)If $G \subset PU(2,1)$ is an infinite discrete group acting on $P^2_{\mathbb{C}}$ without invariant complex projective lines, then the connected component of the domain of discontinuity containing $\mathbb{H}^2_{\mathbb{C}}$ is G-invariant and complete Kobayashi hyperbolic.

THEOREM.(Barrera-Cano-Navarrete)
If $\Gamma \subset PSL(3, \mathbb{C})$ is a discrete group then,

$$Eq(\Gamma) \subset \Omega(\Gamma).$$

If U is an open Γ invariant subset with at least three lines in general position lying on its complement, then $U \subset Eq(\Gamma)$.

If $\Lambda(\Gamma)$ contains at least three lines in general position, then $\Omega(\Gamma) = Eq(\Gamma)$.

THEOREM. (Barrera-Cano-Navarrete)
Let $\Gamma \subset PSL(3, \mathbb{C})$ be a discrete group, if the number of complex lines in general position in $\Lambda(\Gamma)$ and $C(\Gamma) = \overline{\bigcup_{\gamma \in \Gamma} \Lambda(\gamma)}$ is at least three, then

$$\Lambda(\Gamma) = C(\Gamma),$$

and $\Lambda(\Gamma)$ is the union of complex lines.

Moreover, if Γ acts on $\mathbb{P}^2_{\mathbb{C}}$ without global fixed points, then $\Omega(\Gamma)$ is the largest open set where Γ acts properly and discontinuously.