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The Inverse Problem of Earthquake Source Mechanics Method and Applications

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Method and Applications

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Background Reading

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Forward problems are those in which we are given the cause and required to find the effect. For example, given the Earth structure, find the expected arrival times of different seismic phases. Or, given the details of the earthquake rupture process, find the seismograms at different stations.

Inverse problems are those in which we are given the effect and required to find the cause. For example, given the times of arrivals of P-waves at different stations from an earthquake, find the hypocentral location. Or, given the travel time curves, find the Earth structure.

In seismology, as in all branches of geophysics, it is inverse problems that we are most interested in.

Forward problems	*Inverse problems*
Serve to elucidate underlying physical principles	Say, given some gravity anomaly, find shape & density of anomalous body.
Solutions are UNIQUE.	Solutions are NON-UNIQUE i.e. more than one density distribution can give same anomaly.

Definition of some common terms used in inverse theory

STABILITY is defined as the property that a solution is insensitive to small random errors in the data.

NON-UNIQUENESS is the property that more than one solution (almost) equally well-fitting the data exists.

ROBUSTNESS is the property that a solution is insensitive w.r.t. a small number of large errors ("outliers") in the data.

The inverse problem of earthquake source mechanics consists of determining the spatial and temporal distribution of slip (or slip rate) over the fault area, using teleseismic as well as near-field waves.

Since the early 1980's, several attempts have been undertaken to solve this inverse problem. for the source of particular earthquakes, that is, to determine the spatial and temporal distribution of slip (or slip rate) over the fault area, using teleseismic as well as near-field waves. In addition, much work has been done on determining the source time function for specific earthquakes and the result interpreted in terms of spatial moment release using a constant rupture velocity as well as on inversion considering the source as a line source. Finally, the inverse problem for the static case, namely, the determination of the final slip distribution on the fault using geodetic data, has been studied.

The solution of all these problems are far from trivial. It is well known that this inverse problem is unstable, even in the *imaginary* case of a continuous distribution of seismic stations over the surface of Earth and its stability and uniqueness properties have been discussed extensively by *Kostrov and Das* [1988]. From the computational point of view, this instability is equivalent to nonuniqueness of the solution. The real situation is even worse because the number of stations with appropriate records is very limited (about 10-20 global stations, at present). Consequently, to obtain a definite solution of such a problem, one needs some physical constraints on the source process, in addition to the requirement of fitting the observed seismograms. In principle, these constraints should be inferred from the physics of faulting during the earthquakes, that is, from fracture and frictional mechanics. Unfortunately, our knowledge of the physics of the earthquake process is still rather limited and the only comparatively well-established constraint is the limitation on the fracture propagation speed. This is a weak constraint because the duration of the earthquake process is greater than the seismic wave propagation time across the fault whereas the limiting

fracture speed is comparable to the seismic wave velocities. To implement this constraint numerically, one requires very fine spatial gridding on the fault. A less physically founded constraint would be the requirement that the slip rate vector be directed in accordance with the average stress drop direction (inferred from the fault plane solution), that is, the projection of the slip rate vector in the stress drop direction must be positive. While, in principle, slip in the opposite direction is possible due to interference of waves on the fault, it does not seem likely and experience from three-dimensional forward modelling shows that the slip direction almost coincides with that of the stress drop [*Das*, 1981]. This suggests that with sufficient practical accuracy, one may assume the slip direction to be constant over the fault during the process and coincide with the stress drop direction and the only component of the slip rate vector to be nonnegative. In what follows, we shall refer to this constraint as the "no backslip" constraint. As we shall see later in the paper, other possible constraints may be considered. For example, one may require the solution to be in agreement with the seismic moment obtained from the centroid moment tensor solution or from geodetic measurements. Or one may require the maximum slip rate on the fault to be limited by some considerations from fracture mechanics. For example, *Hartzell and Heaton* [1983] minimized the seismic moment, which is another possible constraint on the solution.

To invert for the three-dimensional slip rate distribution (two spatial dimensions on the fault and time) is a computationally difficult task. With sufficiently fine gridding of the fault, it is almost impracticable in a regular way, even on existing supercomputers. Not surprisingly, simplified methods of solving the problem have been used. Basically, such simplifications are always a version of trial and error fitting. In any case, the method of the solution imposes some implicit constraints. Then, when one obtains a unique solution, it is not clear if it is unique due to the explicitly formulated constraints or as a result of the method used in the inversion. As an example, the generalized inverse (or pseudoinverse) leads to a unique solution which minimizes the root mean square residual even without additional constraints. In that case the implicit constraint consists of the requirement that the solution has the minimum norm. Or, in the method of *Kikuchi and Fukao* [1985] the no backslip constraint was explicitly imposed, but the method consisted of successive reduction of the root mean square residual by fitting a series of essentially instantaneous point sources. This again imposes an implicit requirement that the solution must be concentrated in as small a number of discrete patches on the fault, as possible. The interpretation of the solution obtained in this manner in terms of the asperity model is only natural, but this interpretation as being the only and unique interpretation is not supported by the data because it was implicitly imposed by the method, and the question whether there exists another, more smoothly distributed solution, equally well fitting the data for all practical purposes and satisfying the no backslip constraint, remains open.

In order for a particular slip distribution to be an acceptable solution to the inverse problem it must satisfy the following three conditions.

1. The solution must explain the data.
2. The solution must be physically reasonable (consistent with independent constraints).
3. If more than one solution fits the data equally well, additional information must be supplied to uniquely define which solution is being obtained.

The third condition above means that the physical constraints may be insufficient to specify an unique solution of the problem. Such constraints should be explicitly stated and not hidden in the method of inversion. It is be desiarble to use constraints which are physically based rather than non-physical ones. A commonly used example of the latter is the requirement that the solution has minimum norm.

A more reasonable approach to this difficulty would be to describe the whole set of equally acceptable solutions by obtaining some extreme representatives of this set, say.

Olson and Apsel [1982] performed one of the first full inversions of this problem. They used a form of the no-backslip constraints as well as limiting of the rupture propagation speed. Additionally, they required slipping to be confined to grids intersecting the rupture front and limited the number of times each grid could slip. Effectively, their model included two preassigned fronts, the rupture front and the healing one between which the whole slip process was confined. This last assumption permitted drastic reduction of the number of unknowns in the model. Numerical solutions of the forward problem show, however, that with an inhomogeneous strength and friction on the fault, the rupture front can become very distinct from a single line and subsequent slip behind the main rupture front is frequently obtained [*Das and Aki*, 1977; *Mikumo and Miyatake*, 1979]. Such reslipping of previously slipped regions on the fault has been found. Consequently, this assumption can be considered only as another example of the above mentioned simplification of the problem. Since this simplification decreases the size of the problem, it is often used. For a proper inversion, however, it is necessary not to confine the slippage to a region near the rupture front, nor to a priori assign the number of times each grid can slip but to determine the entire slip time history at every spatial grid on the fault.

The discretization of the problem can itself imply unwanted constraints on the solution. Namely, with a coarse grid the number of unknowns can be reduced so much as to make the corresponding numerical problem stable, that is, well conditioned, without additional physical constraints, the extreme cases being point source fitting, and fitting the source to a rectangular dislocation with constant slip and constant slip rate.

In these notes, we shall describe a method for determining the complete slip time history on a fault during an earthquake and to examine the stability of such a solution by examining not only a "best fitting" solution but also solutions close to it in data space.

Formulation of the Discrete Problem

We briefly summarize the formulation of the discrete problem under study for completeness and clarity, particularly of the notations. Using the representation theorem (e.g., equation (3.2.18) of *Kostrov and Das* [1988]) the displacement record at a station located at a point $\mathbf{x}$, on the earth's surface can be expressed in terms of the slip distribution over a fault Σ as

$$u_k(\mathbf{x}_1, t_1) = \int_0^{t_1} dt \iint_\Sigma K_{ik}(\mathbf{x}_1, \mathbf{x}, t_1, t) a_i(\mathbf{x}, t) dS \tag{1}$$

where $i, k = 1, 2, 3$, $u_k(\mathbf{x}_1, t_1)$ are the components of the displacement vector, $a_i(\mathbf{x}, t)$ are the components of the slip and $K_{ik}(\mathbf{x}_1, \mathbf{x}, t_1, t)$ are the components of the impulse response of the medium at $(\mathbf{x}_1, t_1)$, due to a dislocation point source at $(\mathbf{x}, t)$. The observed seismograms do not represent the displacement vector $\mathbf{u}$ itself but are filtered by the instrument. Convolving both sides of (1) with the instrument response for a given station and assuming the slip direction to be constant, we obtain, after some transformations:

$$S_j(t_1) = \int_0^{t_1} dt \iint_\Sigma W_j(\boldsymbol{\xi}, t_1 - t) \dot{a}(\boldsymbol{\xi}, t) dS \tag{2}$$

where j identifies the station and components of the seismogram $S(t_1)$, $W_j(\boldsymbol{\xi}, t)$ is the impulse response at $(\boldsymbol{\xi}, t)$ corresponding to a fixed slip direction and convolved with the instrument response, $\dot{a}(\boldsymbol{\xi}, t)$ is the slip rate, the two-dimensional vector $\boldsymbol{\xi}$ gives the position on the fault relative to some reference point (for example, the earthquake hypocenter) and t is the time measured from the origin time of the earthquake. With a continuous distribution of stations, equation (2) would represent an integral equation of the first kind. Such equations are known to be unstable.

We shall consider the effect of the following additional constraints:

$$\dot{a}(\boldsymbol{\xi}, t) \geq 0 \text{ for all } (\boldsymbol{\xi}, t) \tag{3a}$$

$$\dot{a}(\boldsymbol{\xi}, t) = 0 \text{ for } t < T(\boldsymbol{\xi}) \qquad (3b)$$

where, $t = T(\boldsymbol{\xi})$ gives the boundary of the area where slip is permitted (due to a causality condition) at time t,

$$\int_0^\infty dt \iint_\Sigma \mu(\boldsymbol{\xi})\dot{a}(\boldsymbol{\xi}, t)dS = M_0 \qquad (3c)$$

where M_0 is the seismic moment and μ is the modulus of rigidity of the medium, together with other constraints to be discussed later in the paper. The constraint $(3a)$ is the "no-backslip constraint," the constraint $(3b)$ is the "causality constraint" and the constraint $(3c)$ is the "seismic moment constraint." Note that $T(\boldsymbol{\xi})$ as defined above is not the usual rupture front, but in what follows we shall refer to this boundary as the "rupture front" to distinguish it from the usual definition.

The functions W_j can then be easily calculated using the ray approximation and the Thomson-Haskell technique to account for the layered structure at the source and the station when using teleseismic waves, and by other methods such as multimodal summation or ray theory for regional or strong-motion data.

For the numerical solution, this equation has to be be discretized. We divide the rectangular area on the fault plane into rectangular cells by equally spaced straight lines parallel to the strike and dip of the fault, one cell centre coincident with the hypocentre. Denote the dimensions of the cells Δ_x and Δ_h in strike and dip direction, respectively. For each cell the synthetic seismograms are computed for all stations with unit slip rate uniform within the cell and a time step Δ_τ, which is taken as the sampling interval of the slip history. Let the number of cells in strike direction be N_x, those in dip direction be N_h and the number of time steps in slip history be N_τ. The synthetics are sampled with the same sampling interval as the records used for inversion (Δt). Denote the synthetics $S(k_x, k_h, j_s, j_t.\Delta t)$, k_x, k_h being the cell numbers along strike and dip, j_t the number of the corresponding seismogram sample at the station j_s. Let $s(k_x, k_h, k_\tau)$ be the (unknown) slip rate at the cell (k_x, k_h) at the time $\Delta_\tau k_\tau$. Then the seismograms will be modelled as

$$\sum_{k_x=1}^{N_x} \sum_{k_h=1}^{N_h} \sum_{k_\tau=1}^{N_\tau} S(k_x, k_h, j_s, j_t.\Delta t - \Delta_\tau k_\tau).s(k_x, k_h, k_\tau)$$

and the problem consists of finding such $s(k_x, k_h, k_\tau)$ as to approximate with this expression the real records at stations, say $u(j_s, j_t)$. In this paper we use a weak causality condition requiring that the first signal recorded at any station must be radiated from the hypocentral cell. This condition excludes some of $s(k_x, k_h, k_\tau)$. We renumber the remaining slip rate values sequentially from 1 to n_1, $n_1 < N_x.N_h.N_\tau$. Let $G(k_x, k_h, k_\tau)$ be an integer valued array, containing zeros for the slip rate values excluded due to causality condition and the sequential numbers of remaining $s(k_x, k_h, k_\tau)$. Denote so renumbered slip rate values $x_i, i = 1, ...n_1$. Similarly, we renumber the record samples $u(j_s, j_t)$ sequentially from 1 to m_1 and denote them $u_j, j = 1, ..., m_1$. Let the correspondingly numbered values of $S(k_x, k_h, j_s, j_t\Delta t)$ be denoted a_{ji}, a_{ji} comprising an $(m_1 \times n_1)$ matrix A. Then the integral equation to be solved is discretized and leads to the approximation problem which takes the form:

$$A\mathbf{x} \approx \mathbf{u}$$

The constraints

The need for additional constraints was discussed in a previous paper (Das and Kostrov, 1990). The two constraints used in that paper are included here from the very beginning,

namely the condition (C1) of no back slip

$$x_i \geq 0$$

and the requirement (C2) that total seismic moment equals to the CMT value M_0, say:

$$\sum_{i=1}^{n_1} c_i x_i = M_0$$

where c_i is the medium rigidity at the corresponding cell multiplied by the cell area times Δ_τ. A third constraint (C3) is also included. Namely, we assume that the slip rate is zero at any cell and time step which would produce a signal before the first arrival at any station from the hypocentral cell. This is a weak causality-like constraint. No strong causility constraint, such as slip rates being zero outside some chosen rupture front, are included in this study. In Das and Kostrov (1990) and in Paper I, both constraint C3 and the strong causality constraint were used.

The linear programming problem

We follow the procedure described by Das and Kostrov (1990). Denote $\mathbf{u} - A\mathbf{x}$ by $\mathbf{r}$. We will minimize the absolute misfit

$$f = \sum_{j=1}^{m_1} |r_j|$$

If we represent $\mathbf{r}$ as $\mathbf{r}^+ - \mathbf{r}^-$, where $r_j^+ \geq 0$, $r_j^- \geq 0$, the function f will be linear

$$f = \sum_{j=1}^{m_1} (r_j^+ + r_j^-)$$

and we obtain the following linear programming (LP) problem:
Minimize

$$f = \sum_{j=1}^{m_1} (r_j^+ + r_j^-)$$

under the constraints

$$\begin{aligned} &\mathbf{u} - A\mathbf{x} - \mathbf{r}^+ + \mathbf{r}^- = 0 \\ &M_0 - \sum_{i=1}^{n_1} c_i x_i = 0 \\ &x_i \geq 0, \quad r_j^+ \geq 0, \quad r_j^- \geq 0 \end{aligned}$$

First step of solving such a problem consist of finding a feasible solution, i.e. a set of positive unknowns satisfying the constraints. For our problem it can be done analytically. First, the moment equation can be solved for one of x_i's, x_1 say, to give

$$x_1 = M_0/c_1 - \sum_{i=2}^{n_1} (c_i/c_1) x_i$$

Substitution into the first set of equations gives

$$u_j - a_{j1} M_0/c_1 - \sum_{i=2}^{n_1} (a_{ji} - a_{j1} c_i/c_1) x_i - r_j^+ + r_j^- = 0$$

Let

$$\sigma_j = \mathrm{sgn}(u_j - a_{j1}M_0/c_1)$$

and

$$r_j^{\sigma_j} = \begin{cases} r_j^+ & \sigma_j = 1 \\ r_j^- & \sigma_j = -1 \end{cases}$$

$$r_j^{-\sigma_j} = \begin{cases} r_j^- & \sigma_j = 1 \\ r_j^+ & \sigma_j = -1 \end{cases}$$

Now, the equations can be solved for $r_j^{\sigma_j}$ to obtain

$$\left.\begin{array}{l} r_j^{\sigma_j} = \sigma_j(u_j - a_{j1}M_0/c_1) - \sum_{i=2}^{n_1} \sigma_j(a_{ji} - a_{j1}c_i/c_1)x_i + r_j^{-\sigma_j} \\ x_1 = M_0/c_1 - \sum_{i=2}^{n_1}(c_i/c_1)x_i \end{array}\right\} \quad (4)$$

All constant terms in the right-hand side are positive, so making all unknowns in the right hand side equal to zero we obtain a feasible solution of the set of constraints. Moreover, it is a basic feasible solution, and the problem is represented in the restricted normal form (Press et al, 1986). Substituting $r_j^{\sigma_j}$ into the expression of f, we obtain

$$f = \sum_{j=1}^{m_1} \sigma_j(u_j - a_{j1}M_0/c_1) - \sum_{i=2}^{n_1}(\sum_{j=1}^{m_1} \sigma_j(a_{ji} - a_{j1}c_i/c_1))x_i + \sum_{j=1}^{m_1} 2r_j^{-\sigma_j} \quad (5)$$

The simplex tableau

At this point it is worth changing notations. First, let us expand the vector $\mathbf{x}$ to include all unknown quantities, i.e. λ_i, $r_j^{-\sigma_j}$, $r_j^{\sigma_j}$, denoting

$$\begin{array}{l} x_i = \lambda_{i-n_1} \quad \text{for} \quad i = n_1+1, ..., n_1+N_s \\ x_i = r_{i-n_1-N_s}^{-\sigma_{i-n_1-N_s}} \quad \text{for} \quad i = n_1+N_s+1, ..., n_1+N_s+m_1 \\ x_i = r_{i-n_1-N_s-m_1}^{\sigma_{i-n_1-N_s-m_1}} \quad \text{for} \quad i = n_1+N_s+m_1+1, ..., n_1+N_s+2m_1 \end{array} \quad (6)$$

Let B be the list of indexes of unknowns x_i appearing in the left-hand side of equations (4), namely,

$$B = \{n_1+N_s+m_1+1, ..., n_1+N_s+2m_1, 1\} \quad (7)$$

and N be the list of indexes of unknowns in the right-hand side, namely,

$$N = \{2, ..., n_1+N_s+m_1\} \quad (8)$$

Denote the size of B by m_2 and that of N by n_2 i.e. set $m_2 = m_1+1$, $n_2 = n_1+N_s+m_1-1$. Next, let

$$\begin{array}{l} t_{j0} = \sigma_{B(j)}(u_{B(j)} - a_{B(j),1}M_0/c_1), \quad j = 1, ..., m_1 \\ t_{m_2 0} = M_0/c_1 \\ t_{ji} = -\sigma_{B(j)}(a_{B(j),N(i)} - a_{B(j),1}c_{N(j)}/c_1) \quad \text{for} \quad j = 1, .., m_1;\ i = 1, .., n_1-1 \\ t_{ji} = \alpha_{N(i)} v_{B(j),N(i)} \sigma_{B(j)} \quad \text{for} \quad j = 1, ..., m_1;\ \ i = n_1, .., n_1+N_s-1 \\ t_{ji} = \delta_{n_1+N_s+j,i} \quad \text{for} \quad j = 1, .., m_1;\ \ i = n_1+N_s, ..., n_2, \\ \quad \text{where } \delta \text{ is the Kronecker delta} \\ t_{m_2 i} = -c_{N(i)}/c_1 \quad \text{for} \quad i = 1, .., n_1-1 \\ t_{m_2 i} = 0 \quad \text{for} \quad i = n_1, .., n_2 \end{array} \quad (9)$$

and

$$f_0 = \sum_{j=1}^{m_1} t_{j0}, \qquad f_i = \sum_{j=1}^{m_1} t_{ji}, \quad \text{for} \quad i = 1, ..., n_1 + N_s - 1 \tag{10}$$

$$f_i = 2 \quad \text{for} \quad i = n_1 + N_s, ..., n_2 \tag{11}$$

Then the equations (4) and (5) take the form

$$\left.\begin{aligned} x_{B(j)} &= t_{j0} + \sum_{i=1}^{n_2} t_{ji} x_{N(i)} \\ f &= f_0 + \sum_{i=1}^{n_2} f_i x_{N(i)}, \end{aligned}\right\} \qquad j = 1, .., m_2$$

This system of equations has the form of the tableau of the simplex method in restricted normal form and can be represented graphically as shown in Tableau 1.

Our starting solution is

$$\begin{aligned} x_{B(j)} &= t_{j0}, \qquad j = 1, .., m_2 \\ x_{N(i)} &= 0, \qquad i = 1, ..n_2 \end{aligned}$$

It is feasible, because $t_{j0} \geq 0$ and corresponds to the misfit value of f equal to f_0. These x_i with $i \in B$ are called basic variables, whereas those with $i \in N$ are called non-basic ones. Now the simplex algorithm can be applied to the tableau to minimise f. In our case the minimum exist, since f is the sum of non-negative values $\mathbf{r}^+$ and $\mathbf{r}^-$ and cannot be minimised below zero.

The accumulation of rounding errors during optimization using the simplex method may lead to a completely wrong solution. When the number of constraints and the number of variables were of the order of 10^3 and the number of optimization steps was up to several tens of thousands this misbehaviour of the algorithm was observed when using the unprotected subroutines from Numerical Recipes (Press et al., 1986). The details are discussed in the Appendix and the protective measures needed to overcome the problem are developed. It was found that with these modifications the simplex algorithm was quite stable and the problem of accumulation of rounding errors was eliminated.

Secondary optimization : most uniform moment rate distribution

The kernel S of the integral equation is smoothed due to attenuation effects. Obviously, the smoother the kernel, the rougher a solution satisfying the equations is possible. The scarceness of the data also has the same effect. The inherent instability of the problem is enhanced by this. If we consider the extreme limit of the very low frequaency solution (say, the CMT solution), or we have data from only one station, then a point source solution satisfies the equations. If we want to search for solutions which are more uniformly distributed or smoothed, then this produces additional constraints and we now proceed to formulate these constraints explicitly.

As the first example of searching for other solutions with desired properties, we consider the following problem: obtain a solution with the most uniform distribution of moment rate in space and time corresponding to a misfit not greater than $(1 + \theta)$ times the minimum misfit f_0. For this purpose consider the moment equation

$$\sum_{i=1}^{n_1} c_i x_i = M_0$$

Each term of the sum is equal to the moment rate within corresponding cell and time interval times the source time step $\Delta\tau$. Let $M_i = c_i x_i$. Then

$$\sum_{i=1}^{n_1} M_i = M_0$$

All terms here are positive and their sum is fixed. Obviously, minimization of the greatest of M_i's with no other constraint will make all terms equal, i.e. give an uniform distribution of the moment rate. With the other constraints present we will obtain the most uniform distribution compatible with these constraints. Let the maximum of M_i be denoted M_{max}. Then

$$M_{max} \geq c_i x_i, \quad i = 1, ..., n_1$$

Introducing n_1 new variables $z_i \geq 0$ we rewrite these inequalities in the form of equations:

$$M_{max} = c_i x_i + z_i, \quad i = 1, .., n_1$$

We will start with the solution obtained by the misfit minimization. The accuracy constraint is

$$f = f_0 + \sum_{i=1}^{n_2} f_i x_{N(i)} \quad \leq \quad (1+\theta) f_0$$

or

$$\delta = \theta f_0 - \sum_{i=1}^{n_2} f_i x_{N(i)}, \quad \delta \geq 0 \tag{12}$$

Now we expand again the vector of unknowns $\mathbf{x}$, putting

$$\begin{aligned} x_{m_2+n_2+1} &= \delta \\ x_{m_2+n_2+2} &= M_{max} \\ x_{m_2+n_2+i+2} &= z_i, \quad i = 1, .., n_1 \end{aligned}$$

We expand the list B of basic variables to include all additional variables except $x_{m_2+n_2+i_{max}+2}$ and the list N of non-basic variables to include $x_{m_2+n_2+i_{max}+2}$. Denote the size of the new B as m_3 and that of new N as n_3,

$$m_3 = m_2 + n_1 + 1, \quad n_3 = n_2 + 1$$

Next, we expand the tableau (obtained at the end of first minimization) as follows:
For $j = m_2 + 1$ we let

$$\begin{aligned} t_{m_2+1,0} &= \theta f_0 \\ t_{m_2+1,i} &= -f_i, \ i = 1, .., n_2 \\ t_{m_2+1,n_3} &= 0 \end{aligned}$$

The (m_2+2)th row we obtain from M_{max} equation written for i_{max}:

$$M_{max} = c_{i_{max}} x_{i_{max}} + z_{i_{max}}$$

Let j_{max} be the row number corresponding to $x_{i_{max}}$, i.e. $B(j_{max}) = i_{max}$. We have

$$x_{i_{max}} = t_{j_{max}0} + \sum_{i=1}^{n_2} t_{j_{max}i} x_{N(i)}$$

and

$$M_{max} = x_{B(m_2+2)} = c_{i_{max}} t_{j_{max}0} + \sum_{i=1}^{n_2} c_{i_{max}} t_{j_{max}i} x_{N(i)} + x_{N(n_3)}$$

Now we let

$$t_{m_2+2,0} = c_{i_{max}} t_{j_{max},0}$$
$$t_{m_2+2,i} = c_{i_{max}} t_{j_{max},i},\ i = 1,..,n_2$$
$$t_{m_2+2,n_3} = 1$$

The rest of M_{max} equations we rewrite as follows:

$$z_k = M_{max} - c_k x_k, \quad 1 \le k \le n_1, \quad k \ne i_{max}$$

What follows depends on whether x_i is a basic or non-basic variable. For a non-basic one, let $i_k$be its corresponding column number, i.e. $N(i_k) = k$. Then

$$z_k = c_{i_{max}} t_{j_{max},0} + \sum_{i=1}^{n_2} \left(c_{i_{max}} t_{j_{max},i} - c_k \delta_{i_k i} \right) x_{N(i)} + x_{N(n_3)}$$

and we let

$$t_{j0} = c_{i_{max}} t_{j_{max},0}$$
$$t_{ji} = c_{i_{max}} t_{j_{max},i} + c_k \delta_{i_k i},\ i = 1,..,n_2$$
$$t_{jn_3} = 1$$

For x_k in the basis, let j_k be the corresponding row number. Then

$$z_k = (c_{i_{max}} t_{j_{max},0} - c_k t_{j_k 0}) + \sum_{i=1}^{n_2} (c_{i_{max}} t_{j_{max} i} - c_k t_{j_k i}) x_{N(i)} + x_{m_2+n_2+i_{max}+1}$$

and, consequently,

$$t_{j_0} = c_{i_{max}} t_{j_{max},0} - c_k t_{j_k 0}$$
$$t_{ji} = c_{i_{max}} t_{j_{max} i} - c_k t_{j_k i}, \qquad i = 1,...,n_2$$
$$t_{jn_3} = 1$$

where $j = m_2 + 2 + k$ for $k < i_{max}$ and $j = m_2 + 1 + k$ for $k > i_{max}$, i.e. $B(j) = m_2 + 1 + k$. Because our objective is to minimize M_{max}, the criterion row will coincide with $(m_2 + 2)$th row. The new tableau is shown in Tableau 2.

Application of the simplex method to this linear programming problem gives the solution in which all subsources (i.e. cells in single time step) have as equal intensities to each other as is compatible with the misfit tolerance. We shall refer to this as a "smeared-out" solution.

Another possibility is making the solution uniform only in space letting the subsource intensities vary with the time step number. It was shown above that the "source time function" i.e. total moment release at a single time step vs. time step number is quite stable. Dividing the source time function value at a time step obtained during misfit optimization by the number of cells permitted to be active at this time step by condition (C3), we obtain the average moment release per a cell at this time step. To each $x_i, \quad i = 1,..,n_1$ corresponds such an average moment value $\bar{M}_i$, say. To obtain the required most-uniform-in-space solution we minimize the maximum of $w_i = c_i x_i / \bar{M}_i$. Construction of the tableau for this case is exactly the same as that for the previous case except that instead of c_i we must use $c_i / \bar{M}_i$.

Secondary optimization : smoothest solution

The most uniform solution obtained above still looks like a collection of independent concentrated subsources. Basically, the reason for this, as was mentioned earlier, is that the kernel of the underlying integral equation (for example, Das and Kostrov, 1990; equation (2)) is smooth and permits approximation of the integral with a relatively small number of the integrated samples, this number being smaller the higher the noise in the records.

From the mechanical point of view, the slip rate can not be discontinuous except at the rupture front and at some wave fronts. Elsewhere it must be a smooth function of time and space variables. If the rupture process is not perfectly brittle but involves some critical slip or slip-rate weakening, the slip rate will be continuous everywhere on the fault (Andrews, 1976a,b, 1985; Burridge, Conn and Freund, 1979; Kostrov and Das, 1988). Consequently, it is of importance to find the smoothest solution compatible with other constraints within essentially the same level of misfit. We consider the problem of finding the solution which is smoothest in strike direction. For this purpose we minimize the sum of moduli of second difference of slip rate,

$$\Delta^2_{k_x} = s(k_x - 1, k_h, k_t) + s(k_x + 1, k_h, k_t) - 2s(k_x, k_h, k_t)$$

for those k_x, k_h, k_t for which none of the terms in this expression is rejected by condition (C3). Let

$$\begin{aligned} i &= G(k_x, k_h, k_t) \\ i^- &= G(k_x - 1, k_h, k_t) \\ i^+ &= G(k_x + 1, k_h, k_t) \end{aligned}$$

and I be the set of indexes i for which i, i^+ and i^- are not equal to zero. We define

$$d_i = x_{i^-} + x_{i^+} - 2x_i, \quad i \in I$$

We substitute the expressions for those unknowns in the right-hand side which are in the basis at the end of misfit minimization to obtain

$$d_i = d_i^0 + \sum_{k=1}^{n_2} D_{ik} x_{N(k)},$$

denoting by D_{ik} the resulting coefficient matrix. Analogously to the misfit vector, the d_i's are split into positive and negative parts:

$$d_i = d_i^+ - d_i^-$$

Let σ_i^d be the sign function of d_i^0

$$d'_i = \begin{cases} d_i^+ & \text{for} \quad \sigma_i^d = 1 \\ d_i^- & \text{for} \quad \sigma_i^d = -1 \end{cases}$$

$$d''_i = \begin{cases} d_i^- & \text{for} \quad \sigma_i^d = 1 \\ d_i^+ & \text{for} \quad \sigma_i^d = -1 \end{cases}$$

Then

$$d'_i = |d_i^0| + \sum_{k=1}^{n_2} \sigma_i^d D_{ik} x_{N(k)} + d''_i, \qquad i \in I$$

As in the previous case we add the accuracy constraint as $(m_2 + 1)$th row and n_I rows corresponding to d'_i, n_I being the number of indexes in I. The number of columns is increased by n_I as well to contain zeros everywhere except at intersections of that additional rows and columns. The lists B and N are expanded to contain indexes of the new unknowns, i.e. d'_i,d''_i , renumbered sequentially so that $\mathbf{d}''$ corresponds to components of expanded $\mathbf{x}$ from $(m_2 + n_2 + 2)$ to $(m_2 + n_2 + n_I)$, and $\mathbf{d}'$ corresponds to those from $(m_2 + n_2 + 2 + n_I)$ to $(m_2 + n_2 + 2 + 2n_I)$. The new criterion row is obtained by summing columnwise the last n_I columns of the expanded tableau.

Third level of optimization: Concentrating the moment release

The solution obtained by misfit optimization with (or without) total moment constraint tends to spread along the fault beyond the area delimited by aftershocks. To decide whether this is a real feature or just an artefact we performed yet another set of optimizations minimizing either the moment released outside a given domain in time and space, or maximizing the moment released within such a domain. The optimization has been performed at the third level after obtaining the most uniform or the smoothest solution. The construction of the tableau in this case is very simple. We again introduce a tolerance for the value of previous object function and construct the required additional row of the tableau from the previous criterion row similarly to equation (10). The new object function is constructed as the sum of moment rates $c_i x_i$ for those variables which correspond to the domain. The criterion row is obtained by substituting the expressions for basic variables involved, into the sum.

Multiobjective optimization

The second and third level optimizations produced solutions which are optimal in two or three respects, i.e. best fitting and smoothest, say. Broadly implemented approach to such 'multiobjective optimization' consist in combining the two or three object function into one by assigning them with some positive weights (Franklin, 1980, p. 104-111). The weights determine the relative importance of the aims, to say, how much of the misfit is traded for a unit decrease of the second derivative. We prefer the successive optimization as described above, where it is decided beforehand, how much of the previous optimal value (misfit measure, say) can be sacrificed to obtain a more desirable solution.

Appendix: Computational considerations

The accumulation of rounding errors during optimization using the simplex method may lead to a completely wrong solution. Let m denote the number of rows and n the number of columns of the matrix A i.e. the number of basic and non-basic variables, respectively. In our case, when m and n were of the order of 10^3 and the number of optimization steps was up to several tens of thousands this misbehaviour of the algorithm was observed when using the unprotected subroutines from Numerical Recipes (Press et al., 1986). To explain the causes of the algorithm misbehaviour and the protective measures to be taken, let us describe in short the simplex method. The problem in restricted normal form is as follows:

$$F = \mathbf{f}^T \mathbf{x}_N + f_0$$

$$\mathbf{x}_B = \mathbf{t}_0 + T\mathbf{x}_N$$

where $\mathbf{x}_B$ consists of m components of the unknown vector $\mathbf{x}$, $\mathbf{x}_N$ consists of the remaining n components and the object function F expressed in terms of components $\mathbf{x}_N$ with the coefficient vector $\mathbf{f}^T$. The step of the simplex algorithm consist of selection of most negative component of $\mathbf{f}$ which determines the pivot column of the matrix T. Then within this column the element is selected which permits maximum increase of the pivot unknown keeping $\mathbf{x}_B \geq 0$. Let i_P be the pivot column number. Then the pivot row number is determined by

$$j_P = \min_j \{-t_{j0}/t_{ji_P}; \quad t_{ji_P} < 0\}$$

Next the unknowns $x_{N(j_P)}$ and $x_{B(i_P)}$ are swapped and the system is solved for new $\mathbf{x}_B$.

Let us consider first the pivoting of rows. If due to rounding errors some t_{ji_P} becomes negative while actually being zero it will be selected as the pivot element with catastrophic

consequences. Remember that in all our cases the object function is non-negative by definition and cannot be reduced below zero. It follows then that if any $t_{ji_P} = 0$, the corresponding f_{i_P} must be zero, for otherwise the object function would be unlimited from below. It means then that the pivot column was selected mistakenly f_{i_P} being negative due to rounding errors. To protect from this situation we select some $\epsilon > 0$ of the order of uncertainty in the elements of the matrix A (i.e. of the synthetic seismograms). Let us assume that this uncertainty is about one percent of the maximum amplitude, say. The columns of matrix A are scaled by the maximum absolute value of elements of its columns. The row selection rule is replaced by

$$j_P = \min_j \{-t_{j0}/t_{ji_P}; \quad t_{ji_P} < -\epsilon\}$$

If all $t_{ji_P} \geq -\epsilon$, the column pivoting is discarded and i_Pth column is marked not to be selected at subsequent steps. When the starting tableau is constructed the coefficients f_i are obtained by summation of misfit rows. We select the maximum by absolute value of f_i, f_{max}, say, and consider ϵf_{max} as the uncertainty measure for f_i's in all subsequent steps of the algorithm. Then the column selection rule is replaced by

$$i_P = \min_i \{f_i; \quad f_i < -\epsilon f_{max}\}$$

If all $f_i \geq -\epsilon f_{max}$ we consider the solution to be the optimal one. Strictly speaking, this is not yet optimum for the problems under consideration, but it is the optimum for some other problem with the matrix A^* different from A within the uncertainty ϵ. The role of ϵ in the row pivoting and that in the column pivoting is different. When pivoting a row the large enough ϵ protects from division by small numbers and consequently, from inverting a badly conditioned matrix, but with column pivoting the large ϵ leads only to a premature stopping of optimization. The two ϵ are chosen different, to obtain a solution more close to the exact optimal one. We denote this latter ϵ by ϵ_1 in Table I and in the main text.

Finally, some components of $\mathbf{t}_0$ may be made negative due to rounding errors. If unprotected, these components grow to large negative values and the object function increases instead of decreasing. To protect from this, the $\mathbf{t}_0$ is inspected at the end of each step and all negative components are zeroed out.

It was found that with these modifications the simplex algorithm is quite stable.

These modifications do not eliminate the problem of accumulation of rounding errors. Because of large number of steps and the fact that in the simplex algorithm pivoting selects not the best element of tableau matrix with respect to rounding error, the problem gets distorted in the course of optimization. To reduce this error accumulation, we refresh the tableau after every few thousand steps of the simplex algorithm. The refreshing is done as follows. Let B_0 and N_0 be lists of basic and non-basic variables, respectively, at the beginning of optimization, whereas B and N are these lists when refreshing. We construct the list ENTER containing the variables indexes present in B and not present in B_0 and the list LEAVE for those present in B_0 but not in B. First we construct the original tableau as at the beginning of optimization. Then we use the same algorithm of swapping variables from ENTER and LEAVE as in the simplex method but with different pivoting. Namely, for each variable to leave the basis, j_P-th, say, we select the index from ENTER, i_P, such that

$$i_P = \max_i(|t_{j_P i}|, \quad i \in \text{ENTER})$$

This is equivalent to matrix inversion with partial row-wise pivoting. The number of swapping steps is equal to the size of ENTER list and is much less than the number of steps involved in the optimization. Both better pivoting and reducing the number of loops give

rise to lesser error accumulation. To check the accuracy of the solution we calculated the sum of absolute values of misfits by direct substitution of $\mathbf{x}$ into $\mathbf{r} = \mathbf{u} - \mathbf{A}\mathbf{x}$ and compared this value with the final value of the object function obtained by the misfit (primary) optimization. The value coincided to more than 10 decimal digits.

For the numerical solution, the integrals in (2) must be discretized. This can be done in several ways. The commonest way is to divide the fault area Σ into a number of rectangular cells and to approximate $\dot{a}$ within each cell by linear functions in time and along strike and by a constant along dip. W_j is then integrated over each cell analytically, and the integrals over the fault are replaced by sums. The time at the source is discretized by taking a fixed time step, Δt say, and assuming that the slip rate $\dot{a}$ during the time step varies linearly with time. Since we use only long-period body wave records, we may use comparatively large Δt, this being desirable to reduce the number of unknowns. Strictly speaking, both the fault area and the total source duration should be determined as part of the inversion process. Due to the non-uniqueness of the solution, however, we shall assume a finite fault size, obtained from aome independent means such as the aftershock area, and a finite source duration which cannot be longer than the longest record used and cannot be much shorter than that obtained from the centroid moment tensor solution. The seismograms S_j are sampled with a time step Δt_1, say. For long period data, usually $Deltat_1 = 1$s. This value might seem redundant, but it is worth keeping to obtain better control of noise, if of course the computer power allows it. Let us renumber the observations in a one-dimensional way, ordering them by component, station number and time. Denote the value of $S_j(t_1)$ by b_k, say, k being the index in the ordering adopted. Similarly, let us renumber the values of $\dot{a}$ by cell number and time, denoting the vector by x_i, say. In addition, to allow for the possibility of weighting the different stations and components differently in the inversion, let us include this weighting into equation (2). Then equation (2) takes the form

$$A\mathbf{x} \approx \mathbf{b} \tag{13}$$

where A is the matrix obtained by integration of W_j, each column of A being a set of synthetic seismograms for all stations corresponding to different cells and time instants of the source duration, ordered in the same way as the observed seismograms and A and $\mathbf{b}$ are appropriately weighted. The condition $(3a)$ then becomes:

$$\mathbf{x} \geq 0 \tag{5a}$$

the inequality meaning that every component of $\mathbf{x}$ is nonnegative. The condition (3b) can be replaced by

$$x_i = 0 \tag{5b}$$

for those i corresponding to cells and time samples outside the "rupture front" and the condition $(3c)$ becomes:

$$\Sigma c_i x_i \approx M_0 \tag{5c}$$

where c_i is time-independent and for each cell is equal to the product of the average rigidity times the area of the cell. So, the inverse problem has been reduced to the solution of the linear system (4) under one or more of the constraints (5). In the system (4), the number of equations m is equal to the total number of samples taken from all the records involved and the number of unknowns n is equal to the number of cells times the number of time steps at the source. We shall take m greater than n to reduce the influence of the noise contained in the observations $\mathbf{b}$ on the solution. Then, the system (4) is overdetermined

and we can only obtain a solution $\mathbf{x}$ which provides a best fit to the observations, under constraints (5).

It is well known that the matrix A is ill conditioned for the problem under consideration, which implies that the system (4) admits more than one solution, equally well fitting the observations. The constraints (5) are introduced just for the purpose of reducing the set of permissible (feasible) solutions. It is to be investigated if these conditions are sufficient to make the solution unique. If not, then we have to obtain some description of the set of permissible solutions of the problem. In any case, even if an unique solution does exist, there may be many other solutions that almost satisfy the equations. Since the data used in geophysical applications often contains experimental noise and the models used are themselves approximations to reality, solutions almost satisfying the data are also of great interest.

The system of equations (4) together with the constraints (5) do not yet comprise a complete mathematical problem. It remains to formulate in exact form what the "best fit" to observations means. In the next section, we only include constraint (5a) in the mathematical formulation for the sake of simplicity, the inclusion (5b) being trivial and the inclusion of (5c) will be discussed later. We have to minimize the vector of residuals:

$$\mathbf{r} = \mathbf{b} - A\mathbf{x} \tag{6}$$

For this purpose, some norm of the vector $\mathbf{r}$ must be adopted. Usually in such problems one may choose to minimize the l_1, the l_2 or the l_∞ norm [*Noble and Daniel*, 1977; *Tarantola*, 1987], all three being equivalent in the sense that they tend to zero simultaneously. In this paper we shall use the linear programming method to solve the system (4) and minimize the l_1 norm subject to the condition (5a), but we shall also evaluate the other two norms of the solution to investigate how they behave.

Linear Minimization Problem

Investigation of Positiveness Constraint on Slip Rate

To express the l_1 minimization problem in the standard form of linear programming, we represent the residual vector $\mathbf{r}$ as the difference of two vectors with nonnegative components:

$$\mathbf{r} = \mathbf{y}^+ - \mathbf{y}^-$$

Then

$$\Sigma|\mathbf{r}| \leq \Sigma(\mathbf{y}^+ + \mathbf{y}^-)$$

Obviously

$$\min\Sigma|\mathbf{r}| = \min\Sigma(\mathbf{y}^+ + \mathbf{y}^-)$$

Introducing additional unknowns as

$$\left.\begin{array}{l} x_{n+i} = y_i^+ \\ x_{n+m+i} = y_i^- \end{array}\right\} \text{ for } i = 1, 2, ..., m$$

the problem can be rewritten as follows:

$$\text{minimize } f = (1/m) \sum_{i=n+1}^{n+2m} x_i \tag{7}$$

subject to the constraints:

$$\sum_{j=1}^{n} a_{ij}x_j - x_{n+i} + x_{n+m+i} = b_i, \quad i = 1, 2, ..., m \tag{8}$$

$$x_i \geq 0, \quad i = 1, 2, ..., n + 2m \tag{9}$$

In equation (7), f is the mean absolute residual.

The constraints (8) and (9) define a convex polytope in $(n+2m)$ dimensions, each point of which represents a feasible solution. In general, the feasible set is a continuum containing an infinite number of feasible solutions. It is well known [*Franklin*, 1980] that the feasible solution which gives the minimum (7) corresponds to a vertex of the polytope. We use the simplex method of solving the linear programming problem [*Press et al.*, 1986].