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Multifractality of wave functions in disordered metals

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What is the multifractality of the eigenstates?

Schrödinger equation in a random potential

$$\hat{H}\psi_\alpha(r) = \varepsilon_\alpha \psi_\alpha(r)$$

$$\hat{H} = \hat{H}_0 + V_r$$

$\hat{H}_0$ -Kinetic energy

V_r -Random potential

The simplest gaussian distribution

$$\langle V_r \rangle = 0 \qquad \langle V_r V_{r'} \rangle = \gamma \delta(r - r')$$

$t = |\psi(r)|^2$ -Local probability

$t_n = \int_0^\infty t^n f(t) dt$ -moments

$f(t)$ -Distribution function

Distribution function of the probabilities f(t)

$$f(t) = \Delta \left\langle \sum_\alpha \delta\left(t - |\Psi_\alpha(r)|^2\right) \delta(\varepsilon - \varepsilon_\alpha) \right\rangle$$

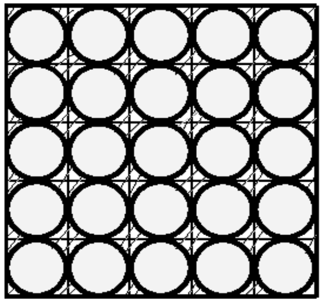
$\Delta = (\nu V)^{-1}$ -Mean level spacing, ν -Density of states, V -Volume

Trivial moments.

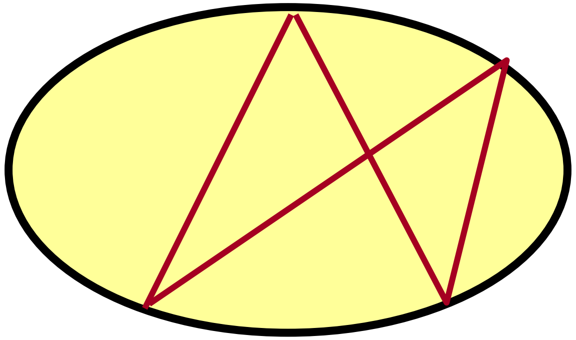
$t_0 = 1$ $\quad$ $t_1 = 1$

1. Pure metal or a ballistic chaotic regime (the wave function extends all over the sample)

$t_n \propto \frac{1}{V^n},$ $V = L^d$ d-dimension



Small metal particles



Chaotic billiards

2. Localized regime (strong disorder)

t_n are insensitive to the volume V

3. Multifractality

$Vt_n \propto L^{-\tau(n)}$ $\tau(n) = (n-1)d^*(n)$

If $d^* \neq d$ -fractality

If $d^*(n)$ is a function of n-multifractality

Examples

Porter-Thomas distribution: $f(t) = V\exp(-Vt)$ Weak disorder, $\varepsilon_0 \tau \geq 1$

⇓

$d^* = d$

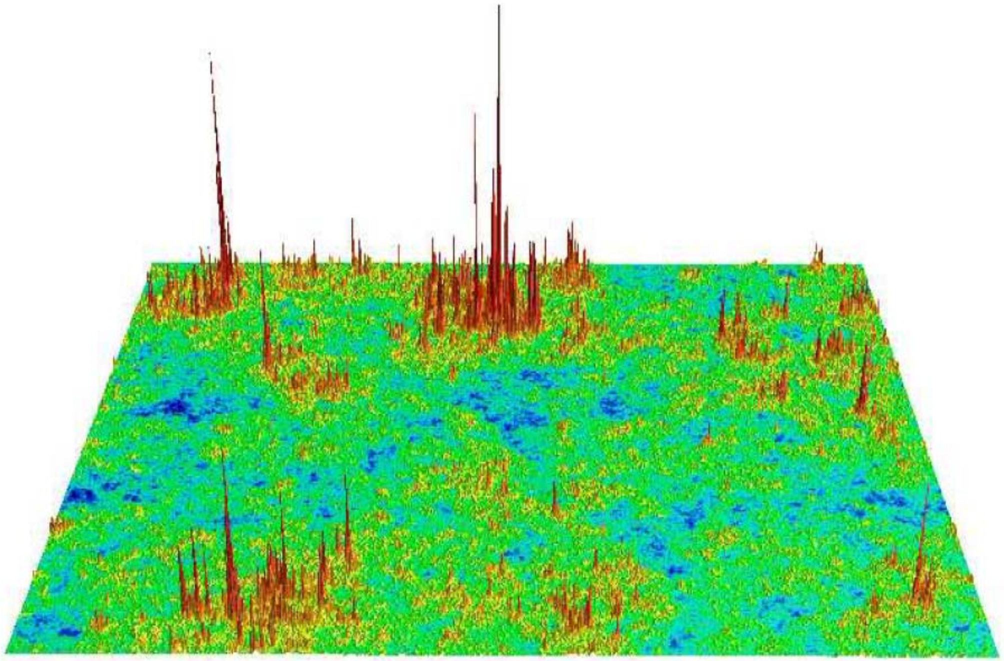
Power law decay: $\psi(r) \propto r^{-\mu}$

$\mu < d/2$ ⇒ $\int |\psi(r)|^2 dr$ is sensitive to boundary

⇓

$d^* = d - 2\mu$ -the same for all $n > \dfrac{d}{2\mu}$

If $d^*(n)$ is a function of n ⇒ A more complicated (multifractal) behavior!



Multifractal wave functions

Multifractality in disordered metals:

nothing exotic and always happens in 2D at weak disorder (the size of the sample L is smaller than the localization length L_c).

Method of investigation: non-linear sigma model (mostly supersymmetric).

Two different approaches:

1. Using Renormalization Group equations including high gradients. Calculation of the distribution of density of states, conductances, etc.
(Wegner; Altshuler, Kravtsov, Lerner)

2. Finding non-trivial minima (instantons) of the supermatrix action. Distribution of wave functions can be found directly. (Muzykantskii, Khmel'nitskii; Falko, Efetov; Mirlin, Fyodorov)

Apparently, there is a connection between 1) and 2), but this is not understood.

Supersymmetry method makes it possible to come to RMT and go beyond (localization, mesoscopics, quantum chaos, etc.)

Literature: K.B. Efetov "Supersymmetry in Disorder and Chaos", Cambridge University Press, 1997

The main ideas

Grassmann anticommuting variables χ :

$$\{\chi_i, \chi_j\} = 0 \qquad \chi_i^2 = 0$$

Integrals (Berezin 1961):

$$\int \chi_i d\chi_i = 1 \qquad \int d\chi_i = 0$$

All other integrals are repetitions of these two.

The most important integrals (the basis of the method)

$$\int \exp(-\vec{\chi}^* A\vec{\chi})d\chi^* d\chi = \det A$$

Not $(\det A)^{-1}$ as for conventional complex numbers!

Supervector:

$$\psi = (\chi, S)$$

Supermatrix:

$$q = \begin{pmatrix} a & \sigma \\ \rho & b \end{pmatrix}$$

$$Strq = a - b$$

χ, σ, ρ - anticommuting

S, a, b -conventional

⇓

$$Str(P_1P_2) = Str(P_2P_1)$$

$$Str(P_1P_2P_3) = Str(P_3P_1P_2)$$

Supermatrix σ-model

$$F = \frac{\pi\nu}{8}\int Str[D(\nabla Q)^2 + 2i(\omega + i\delta)\Lambda Q]dr$$

Physical quantities as integrals over the supermatrices

$$\int B(Q)\exp(-F[Q])DQ$$

Adding magnetic or spin-orbit interactions one changes the symmetry of the supermatrices Q (orthogonal , unitary and symplectic).

Depending on the dimensionality (geometry of the sample) one can study different problems (localization in wires and films, Anderson metal-insulator transition, etc.)

Everything that can be written in terms of products of Green functions can be expressed in terms of an integral over the supermatrices with the σ -model.

The explicit structure of Q

$$Q = UQ_0\overline{U}$$

$$U = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$$

u,v contain all Grassmann variables

All essential structure is in Q_0

$$Q_0 = \begin{pmatrix} \cos\hat{\theta} & ie^{i\hat{\phi}}\sin\hat{\theta} \\ -ie^{-i\hat{\phi}}\sin\hat{\theta} & -\cos\hat{\theta} \end{pmatrix}$$

$$\hat{\theta} = \begin{pmatrix} \theta & 0 \\ 0 & i\theta \end{pmatrix}$$

$$\phi = \begin{pmatrix} \varphi & 0 \\ 0 & \chi \end{pmatrix}$$

(unitary ensemble)

$$\Downarrow$$

Mixture of both compact and non-compact symmetries rotations: rotations on a sphere and hyperboloid glued by the anticommuting variables.

An expression for the distribution function in terms of the σ-model.

$$f(t)=\lim_{\gamma\to 0}\int DQ\int \frac{dr}{4V}Str\left(\pi_b^{(1)}Q(r)\right)\delta\left(t-\frac{\pi\nu\gamma}{4}Str\left(\pi_b^{(2)}Q(r_o)\right)\right)\exp(-F[Q])$$

$$F=\frac{\pi\nu}{8}\int Str\left[D(\nabla Q(r))^2-\gamma\Lambda Q\right]dr$$

$$\pi_b^{(1)}=\begin{pmatrix}\pi_b & 0\\ 0 & 0\end{pmatrix}$$

$$\pi_b^{(2)}=\begin{pmatrix}0 & 0\\ 0 & \pi_b\end{pmatrix}$$

$$\pi_b=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$$

Integration over the zero mode !

$$Q(r) = V(r_o)\tilde{Q}(r)\bar{V}(r_o)$$

$$\tilde{Q}(r_o) = \Lambda$$

In the limit $\gamma \to 0$ one comes to a reduced σ -model

$$f(t) = \frac{1}{V}\frac{d^2\Phi}{dt^2}$$

$$\Phi(t) = \left\{ \int_{\tilde{Q}(r_o)=\Lambda} \exp\left(-\tilde{F}\left[\tilde{Q}, t \right] \right) D\tilde{Q}(r) \right\}$$

The reduced σ -model

$$\tilde{F}[Q,t] = \frac{1}{8}\int Str\left[\pi\nu D_0 \left(\nabla \tilde{Q} \right)^2 - 2t\Lambda\Pi\tilde{Q} \right] dr$$

$$\Pi = \begin{pmatrix} \pi_b & \pi_b \\ \pi_b & \pi_b \end{pmatrix}$$

Only the non-compact sector is important.

The final result

$$f(t)=V\begin{cases}\exp\left(-Vt\left[1-T/2+\ldots\right]\right), & T\le 1\\ \exp\left(-\dfrac{\pi^2\nu D_0}{\ln(L/l)}\ln^2 T\right), & T\ge 1\end{cases}$$

$$T=\frac{tV\ln(L/l)}{2\pi^2\nu D_0}$$

T<1: Porter-Thomas distribution

T>1: anomalously localized states, multifractality

Coincidence (agreement(!,?)) with Altshuler, Kravtsov, Lerner

Multifractality

$$Vt_n \propto L^{-\tau(n)}$$

$$\tau(n)=(n-1)d^*(n)$$

$$d^*(n)=2-n\left(4\pi^2\nu D_0\right)^{-1}$$ -fractal dimension

The multifractality resembles the intermittency in the theory of turbulence!

1. Scaling hypothesis for developed turbulence (Kolmogorov 1941)

2. Conjecture about log-normal distribution (Kolmogorov 1962)

However, there are deviations in exponents due to rare strong fluctuations (known since 60's)!

↓

The biggest challenge for theory: no chance to derive from Navier-Stokes equations!

Simplified stochastic models: Passive scalar model

$$\left(\frac{\partial}{\partial t} - \kappa \frac{\partial^2}{\partial r^2} + \vec{u}\vec{\nabla}\right)\theta = \phi(t, r)$$

$$\vec{\nabla}\vec{u} = 0$$

ϕ is pumping

$$\overline{\phi(t_1)\phi(t_2)} = \delta(t_1 - t_2)\chi(r_1 - r_2)$$

$$\chi(r) = P_2 \begin{cases} 1, & r < L \\ 0, & r > L \end{cases}$$

$$\left\langle u^\alpha(t_1, r_1) u^\beta(t_2, r_2) \right\rangle = \delta(t_1 - t_2) V^{\alpha\beta}$$

$$V^{\alpha\beta} = V_0 \delta^{\alpha\beta} - K^{\alpha\beta}(r_1 - r_2)$$

$$K^{\alpha\beta} = \frac{D}{r^\gamma}\left(r^2 \delta^{\alpha\beta} - r^\alpha r^\beta\right) + \frac{D(d-1)}{2-\gamma}\delta^{\alpha\beta} r^{2-\gamma}$$

$$0 < \gamma < 2$$

What should one calculate?

Again:

$$f(r) = \langle \theta(r_1)\theta(r_2) \rangle$$

$$F(r_1, r_2, r_3, r_4) = \langle \theta(r_1)\theta(r_2)\theta(r_3)\theta(r_4) \rangle$$

$$S_n = \left\langle \left(\theta\left(\frac{r}{2}\right) - \theta\left(-\frac{r}{2}\right) \right)^n \right\rangle$$

Known results:

$$f(r) \propto r^{\gamma}$$

$$F(r_1, r_2, r_3, r_4) \propto r^{2\gamma} \left(\frac{L}{r}\right)^{\frac{4(2-\gamma)}{d}}$$

Intermittency!
Comes from a zero mode.

Chertkov, Falkovich, Kolokolov, Lebedev (1995) (d>>1)
Gavedskii, Kupiainen (1995) $(2-\gamma << 1)$

$$S_n \propto [S_2(r)]^{\frac{n}{2}} \left(\frac{L}{r}\right)^{\frac{2-\gamma}{2(d+2)} n(n-1)}$$

Intermittency!
Comes from instantons.

Balkovsky, Lebedev (1998)