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Critical Scaling at the Anderson Localization Transition in the Strong Multifractality Regime

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Critical Scaling at the Anderson Localization Transition in the Strong Multifractality Regime

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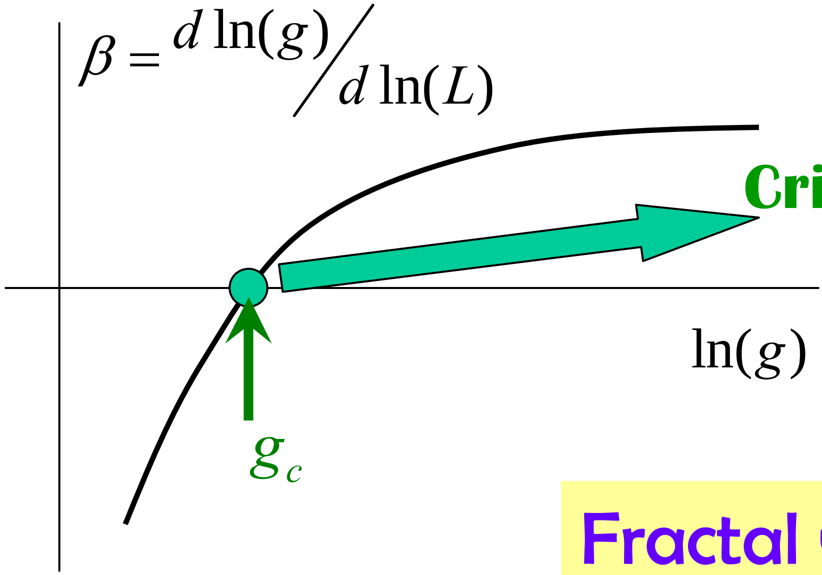
Outline of the talk

1. **Introduction:**
   - Fractal wavefunctions: weak- vs. strong- fractality
   - Critical correlations of fractal wavefunctions and the dynamical scaling hypothesis
   - Paradox of the critical correlations at strong fractality
2. **Strong multifractality regime:**
   - Model (the Critical RMT) and method (the Virial Expansion)
3. **Scaling exponents:**
   - Outline of calculations and results
4. **Conclusions**

Fractal wave-functions at the localization transition

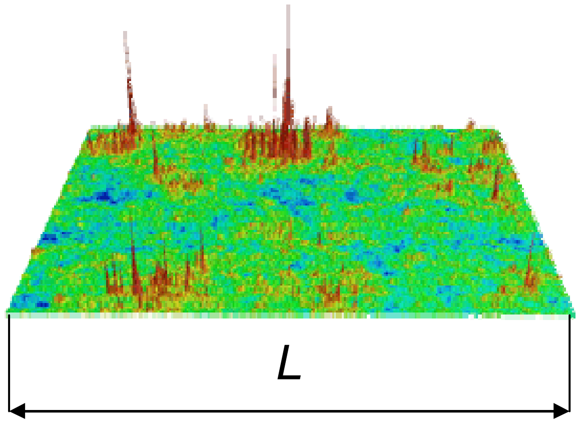


Anderson, 1958
the Gang of four, 1979

Criticality at the localization transition

Fractal wave-functions

(Wegner, 1980)



Inverse Participation Ratio

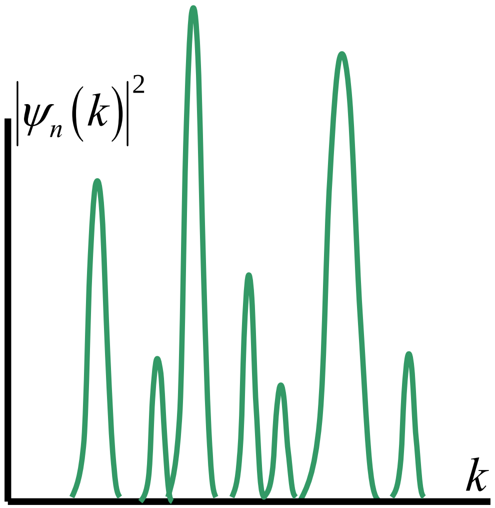
$$P_q = \nu^{-1} \left\langle \sum_{n,\vec{r}} \left|\psi_n(\vec{r})\right|^{2q} \delta(E-\varepsilon_n) \right\rangle$$

$$\lim_{L\to\infty}(P_q) \propto 1/L^{(q-1)d_q}$$

fractal dimension: $0 < d_q < d$

Wavefunction occupies a fraction of space

Weak- vs. strong- fractality regimes



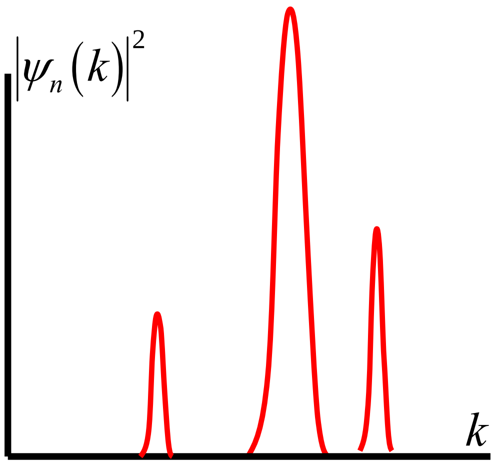
Weak fractality $d_q \simeq d$

Fractal WF are close to extended states

Examples (talk by Konstantin Efetov):
- 2d disordered systems, weak disorder
- turbulence

Method: σ-model

Localization transition in the Anderson model, d=3: $d_2 \simeq d/2, \;\; d_4 \ll d$



Strong fractality $d_q \ll d$

Fractal WF are close to localized states

Example:
- localization transition in the high-dimensional Anderson model

Method: locator expansion (talk by Boris Altshuler)

Correlations of the fractal wave-functions

Two point correlation function:

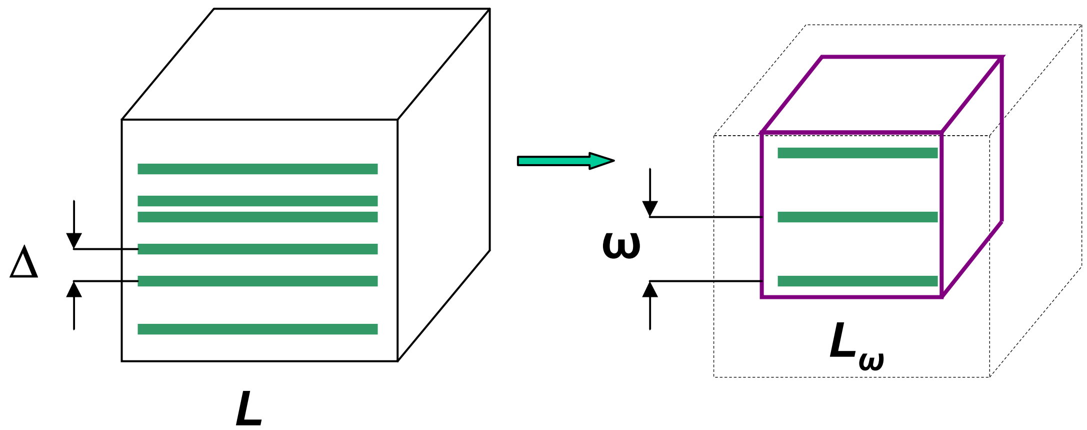
$$C_2(\omega,\mathbf{R}) \equiv \nu^{-1}\langle \sum_{\mathbf{p}} \sum_{m,n} \delta(\omega/2-\xi_n)\delta(\omega/2+\xi_m)|\psi_{\xi_n}(\mathbf{p})|^2|\psi_{\xi_m}(\mathbf{p}+\mathbf{R})|^2\rangle$$

- For a disordered system at the critical point (fractal wavefunctions)

$$C_2(\omega=0,\mathbf{R}) \propto (L/|\mathbf{R}|)^{d-d_2}, \quad |\mathbf{R}| \leq L$$

(Wegner, 1985)

- If $\omega > \Delta$ then $L_\omega = L(\Delta/\omega)^{1/d}$ must play a role of L:



Correlations of the fractal wavefunctions

Two point correlation function:

$$C_2(\omega, \mathbf{R}) \equiv \nu^{-1} \langle \sum_{\mathbf{p}} \sum_{m,n} \delta(\omega/2 - \xi_n) \delta(\omega/2 + \xi_m) |\psi_{\xi_n}(\mathbf{p})|^2 |\psi_{\xi_m}(\mathbf{p} + \mathbf{R})|^2 \rangle$$

For a disordered system at the critical point (fractal wavefunctions)

$$C_2(\omega = 0, \mathbf{R}) \propto (L/|\mathbf{R}|)^{d-d_2}, \quad |\mathbf{R}| \leq L$$

(Wegner, 1985)

Dynamical scaling hypothesis: $L^{d-d_2} \rightarrow (L_\omega)^{d-d_2}$

$$\Rightarrow C_2(\omega > \Delta, \mathbf{R}) \propto (L_\omega/|\mathbf{R}|)^{d-d_2}, \; l \leq |\mathbf{R}| \leq L_\omega < L$$

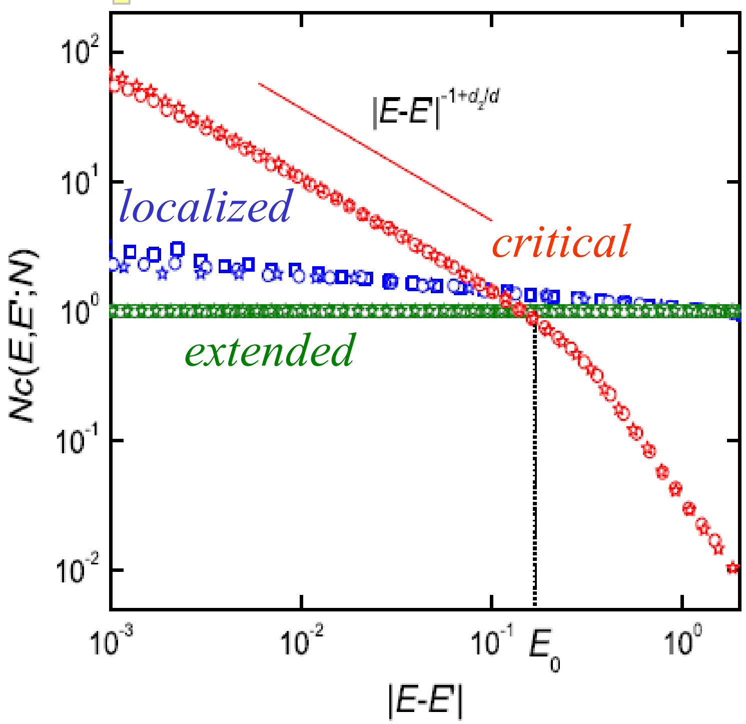
(Chalker, Daniel, 1988; Chalker, 1990)

d –space dimension, Δ - mean level spacing, l – mean free path, <…> - disorder averaging

Fractal enhancement of correlations

Dynamical scaling: $$C_2(\Delta < \omega < E_0, |\mathbf{R}| \ll l) \propto \left(\frac{E_0}{\omega}\right)^{1-d_2/d}$$

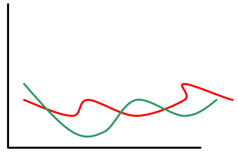
$E_0/\omega > 1,\ \ 1 - d_2/d > 0$ **- Enhancement of correlations**



(Cuevas , Kravtsov, 2007)

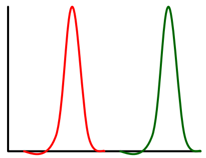
(The Anderson model: tight binding Hamiltonian)

Extended WF:



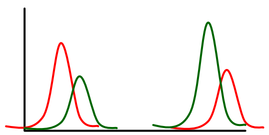
small amplitude
substantial overlap in space

Localized WF:



high amplitude
small overlap in space

Fractal WF:



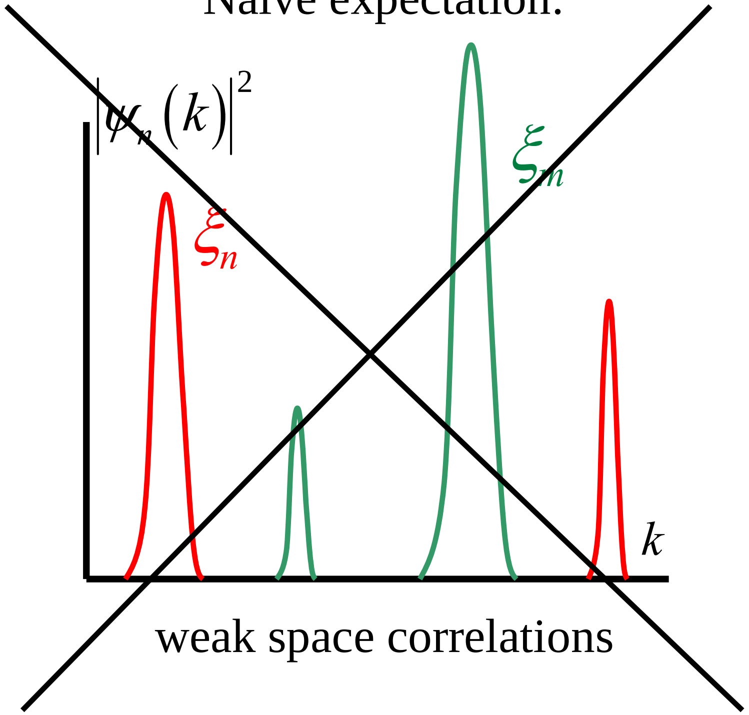
relatively high amplitude and

the fractal wavefunctions strongly overlap in space

Strong fractality regime: do WFs really overlap in space?

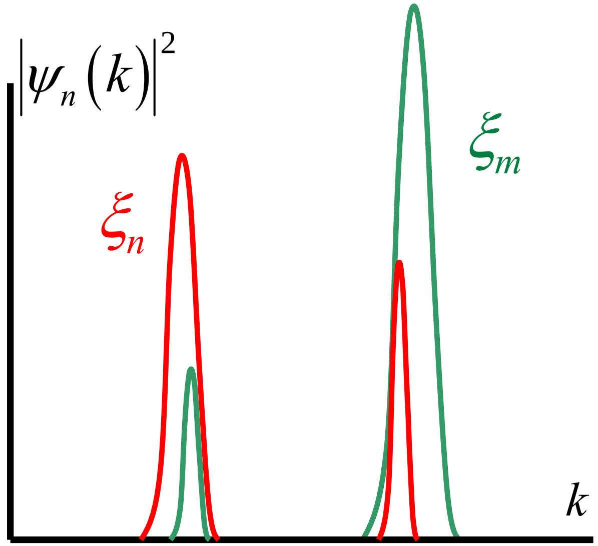
$0 < d_2 \ll d$ ***- sparse fractals,*** $\Delta \ll |\xi_m - \xi_n| \ll E_0$

Naïve expectation:



weak space correlations

The consequence of the dynamical scaling:



strong space correlations

So far, no analytical check of the dynamical scaling; just a numerical evidence

Does the dynamical scaling hypothesis hold true in the strong fractality regime?

IQH WF: Chalker, Daniel (1988),
Huckestein, Schweitzer (1994), Prack, Janssen, Freche (1996)

Anderson transition in 3d: Brandes, Huckestein, Schweitzer (1996)

WF of critical RMTs: Cuevas, Kravtsov (2007)

Coinciding space point: scaling in energy-domain

$\mathbf{R} = 0,$ energy representation

$$C_2(\omega, 0) \equiv \nu^{-1} \langle \sum_{\mathbf{p}} \sum_{m,n} \delta\left(\frac{\omega}{2} - \xi_n\right) \delta\left(\frac{\omega}{2} + \xi_m\right) |\psi_{\xi_n}(\mathbf{p})|^2 |\psi_{\xi_m}(\mathbf{p})|^2 \rangle = C_{2,\mathrm{diag}} + C_{2,\mathrm{off-diag}}$$

Diagonal part

$$C_{2,\mathrm{diag}} = \delta(\omega) \underbrace{\left\langle \nu^{-1} \sum_{\mathbf{p}} \sum_{m} \delta(\xi_m) |\psi_{\xi_m}(\mathbf{p})|^4 \right\rangle}_{\mathrm{IPR},\ \mathcal{P}_2} \propto \delta(\omega) V^{-d_2/d}, \qquad V = L^d$$

(space scaling)

Off-diagonal part

$$C_{2,\mathrm{off-diag}}|_{\Delta \ll \omega \ll E_0} \propto \omega^{-\mu}$$

(dynamical scaling)

$$\boxed{1 - \mu = d_2/d}$$

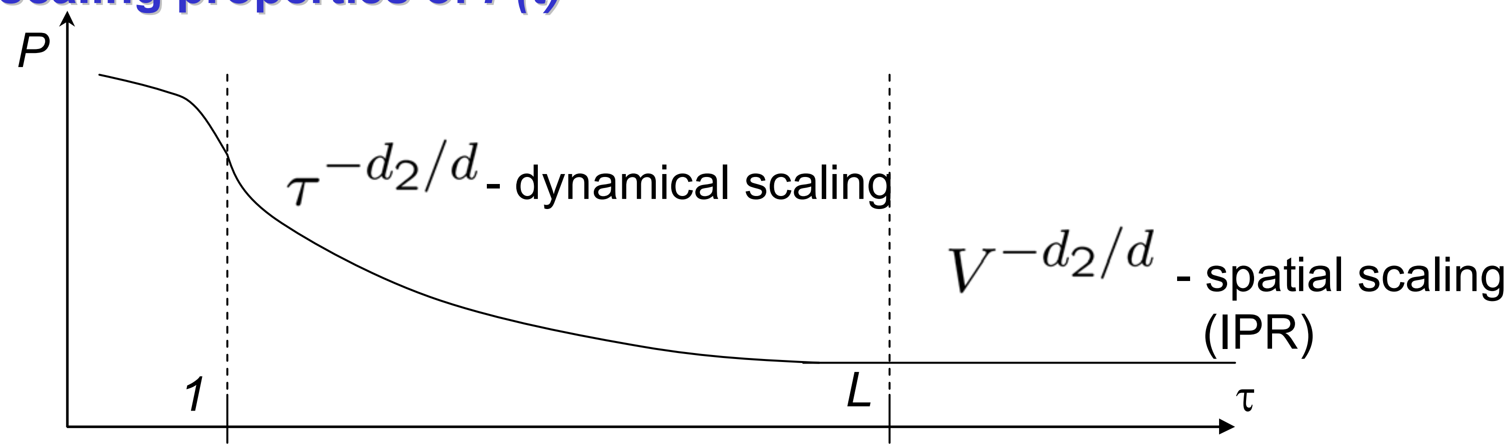
Coinciding space point: scaling in time-domain

$\mathbf{R} = 0,$ time representation

Fourier transform of $C_2(\omega,0)$: $P(t) = \int C_2(\omega,0)e^{-\imath\omega t}\frac{d\omega}{2\pi}$

– averaged return probability for a wave packet

- **Expected scaling properties of *P(t)***



(τ - scaled time)

- ***P(t)* is more convenient for the further analysis**

Scaling of the return probability

$\Upsilon = \min(\tau, L)$ - IR cutoff of the theory

Expected behaviour of *P(t) in the long time limit*

$$P(t) = A(\Upsilon)\left(\frac{B(\Upsilon)}{\Upsilon}\right)^{\kappa}$$

Universal exponent

$$\log(P) = \underline{\log(A)} + \kappa\left(\underline{\log(B)} - \log(\Upsilon)\right)$$

can be non-universal → let's eliminate them

Scaling of the return probability

$\Upsilon = \min(\tau, L)$ - IR cutoff of the theory

Expected behaviour of *P(t) in the long time limit*

$$P(t) = A(\Upsilon)\left(\frac{B(\Upsilon)}{\Upsilon}\right)^{\kappa}$$

$$\partial_{\log(\Upsilon)} \log(P) = \log(A) + \kappa \log(B) - \partial_{\log(\Upsilon)}[\kappa \log(\Upsilon)]$$

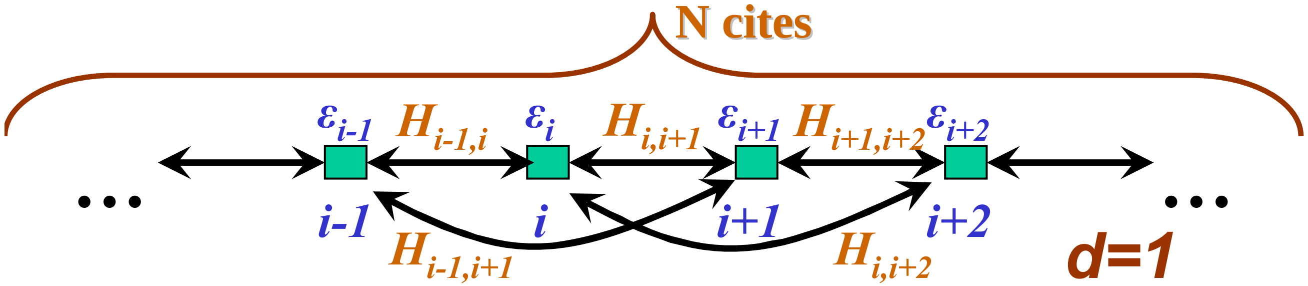
(the terms $\log(A)$ and $\log(B)$ are crossed out)

IF the dynamical scaling hypothesis holds true then

1) $\partial_{\log(\Upsilon)} \log(P) = -\kappa$ **- Equation for κ**

2) $\kappa|_{V \gg \tau} = \kappa|_{\tau \gg V} = d_2/d$ **- Universality of κ**

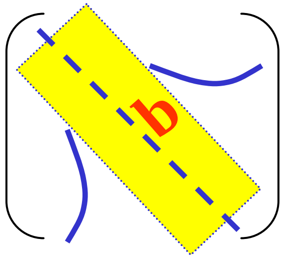
Model: MF RMT (Power-Law-Banded Random Matrices)



1d chain with random long-range hopping

$\hat{H}$ - *NxN* Hermitian BRM

b is the bandwidth



$$\langle |H_{i,j}|^2 \rangle \sim \begin{cases} 1, & |i-j| < b \\ \left(1/_{i-j} \right)^{2\alpha}, & |i-j| > b \end{cases}$$

$\alpha=1$: RMT with multifractal eignestates at any band-width

(Mirlin, Fyodorov et. al., 1996, Mirlin, Evers, 2000)

$2\pi b \gg 1$

$$d_2 \approx 1 - \mathrm{const}/_{2\pi b}$$

$1-d_2 \ll 1$ – regime of **weak multifractality**

$b<1$

$$d_2 \approx \mathrm{const}\, b$$

$d_2 \ll 1$ – regime of **strong multifractality**

Variance of matrix elements for almost diagonal MF RMT

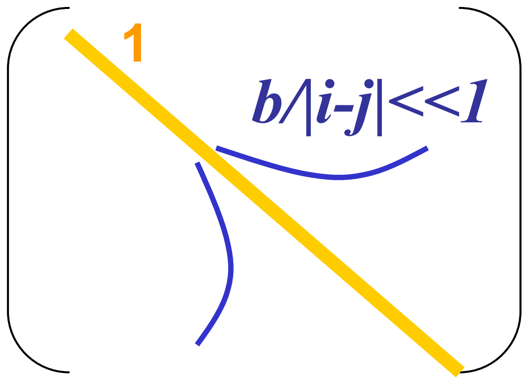
MF RMT from the GUE symmetry class

$$\left\langle \varepsilon_i^2 \right\rangle = \frac{1}{2},$$

$$\left\langle |H_{i\neq j}|^2 \right\rangle = \frac{1}{2}\frac{1}{1+(i-j)^2/b^2}\Big|_{b\ll 1} \simeq \frac{1}{2}\left(\frac{b}{i-j}\right)^2$$

$b \ll 1$ - small band width

$\rightarrow$ almost diagonal MF RMT



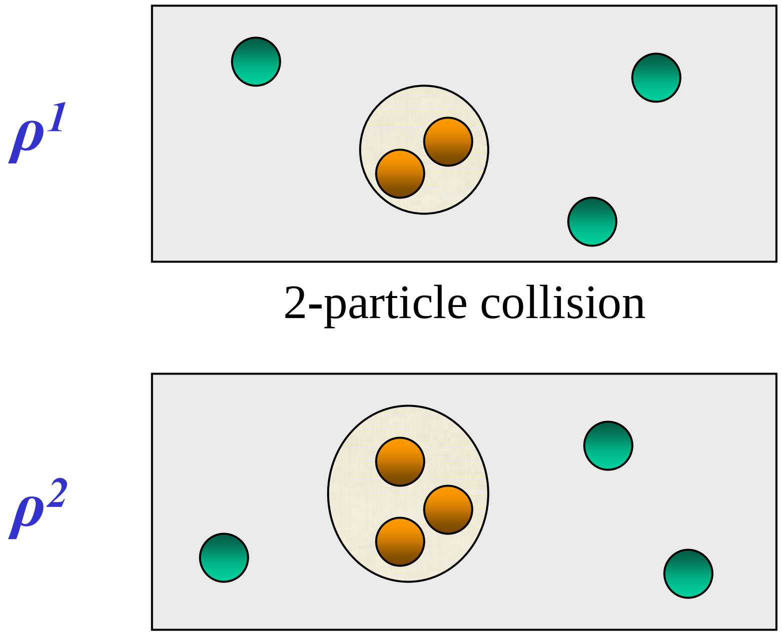
Method: The virial expansion

Note: **a field theoretical machinery of the σ–model cannot be used in the case of the strong fractality**

As an alternative to the *σ-model*, we use

- ***the virial expansion in the number of interacting energy levels***.

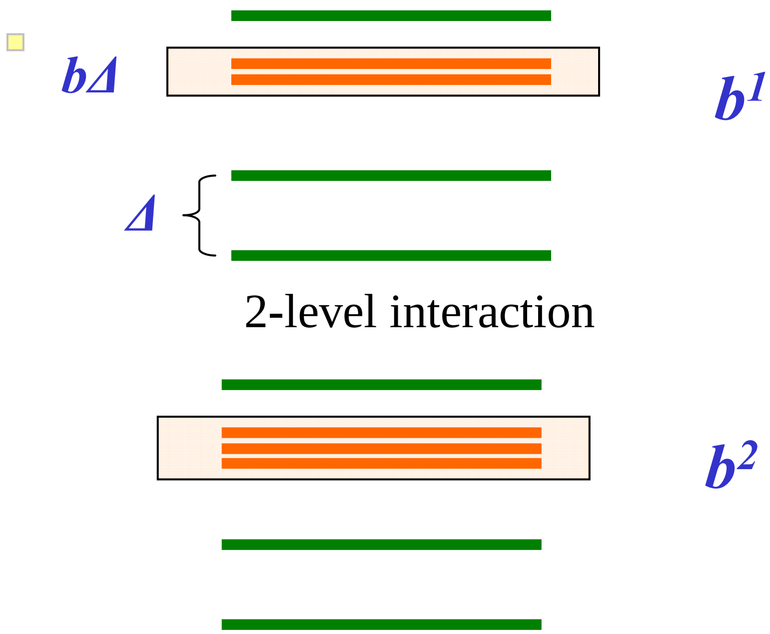
Gas of low density ρ



2-particle collision

3-particle collision

Almost diagonal RM $\Delta >> b\Delta$



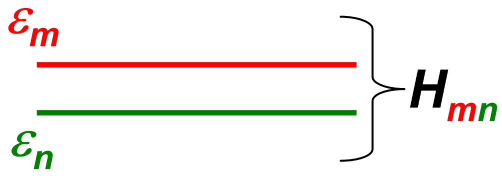
2-level interaction

3-level interaction

- **VE allows one to expand correlations functions in powers of $b<<1$**

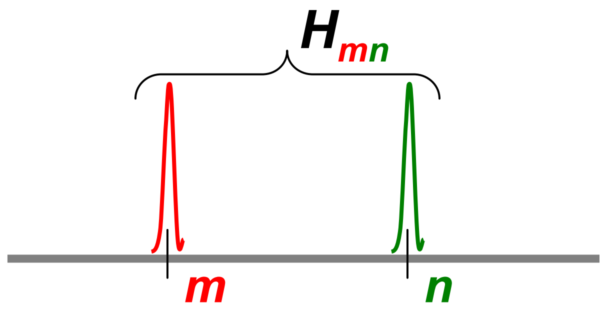
SuSy virial expansion

Interaction of energy levels

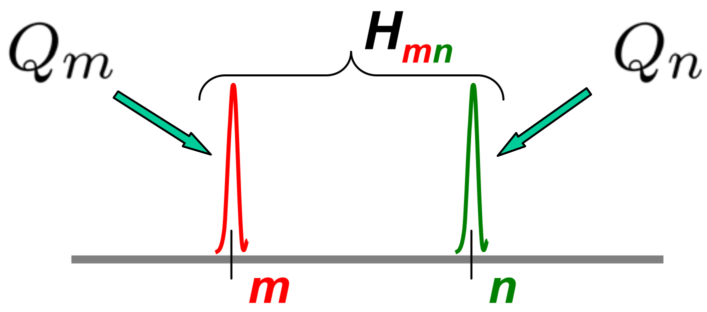


$\Longleftrightarrow$

Hybridization of localized stated



SuSy is used to average over disorder (OY, Ossipov, Kronmüller, 2007-2009)



$\longrightarrow$

Coupling of supermatrices

$$\exp\left(-\langle|H_{mn}|^2\rangle \mathrm{Str}(Q_m Q_n)\right)$$

Summation over all possible configurations

$$\sim \int \mathrm{d}x \, \overline{\exp\left(-\langle|H_{mn}|^2\rangle \mathrm{Str}(Q_m Q_n)\right) \underbrace{\mathcal{F}[Q]}_{\text{SuSy breaking factor}}}\,, \qquad x \equiv m - n$$

Application of the virial expansion

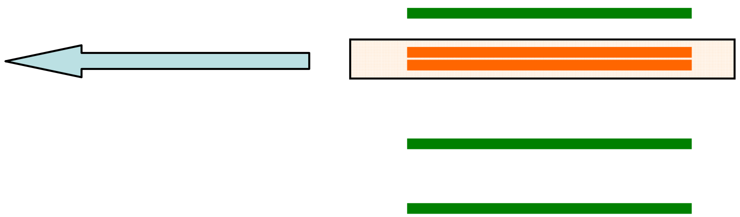
Expected behavior: $P \propto \Upsilon^{-\kappa}; \qquad \Upsilon = \mathrm{min}(\tau, N)\,, \;\; \kappa \sim b \ll 1$

VE for the return probability: $P = 1 + \sum_{j=2} b^{j-1} P_j\,, \;\; P_j \sim \log(\Upsilon)^{j-1}$

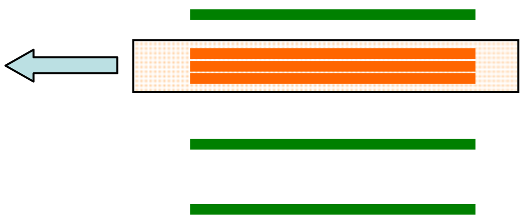
Pertubation theory for the scaling exponent

$$\kappa = -\partial_{\log(\Upsilon)} \log(1+b^1 P_2+b^2 P_3+\ldots) = \kappa_2+\kappa_3+\ldots$$

$$\kappa_2 \;\equiv\; -b^1 \partial_{\log(\Upsilon)} P_2$$



$$\kappa_3 \;\equiv\; -b^2 \partial_{\log(\Upsilon)} \left(P_3 - \frac{1}{2} P_2^2\right)$$



What do we calculate and check

1) *Dynamical scaling:*

$$P \propto \Upsilon^{\kappa}; \quad \kappa_2(\Upsilon) \equiv -b^1 \partial_{\log(\Upsilon)} P_2 , \ \kappa_3(\Upsilon) \equiv -b^2 \partial_{\log(\Upsilon)} \left(P_3 - \frac{1}{2} P_2^2\right)$$

a) *Log-behavior of P_j:*

$$P_2 \sim \log(\Upsilon) , \ P_3 \sim \log(\Upsilon)^2$$

b) *Pure power-law dependence of* $P(\Upsilon)$

❾ ***$\log^2(\Upsilon)$*** **must cancel out in *P_3- $(P_2)^2/2$***

2) Universality: *scaling* exponent is cut-off independent

$$\kappa_j|_{\Upsilon=N} = \kappa_j|_{\Upsilon=\tau} , \quad j = 2, 3$$

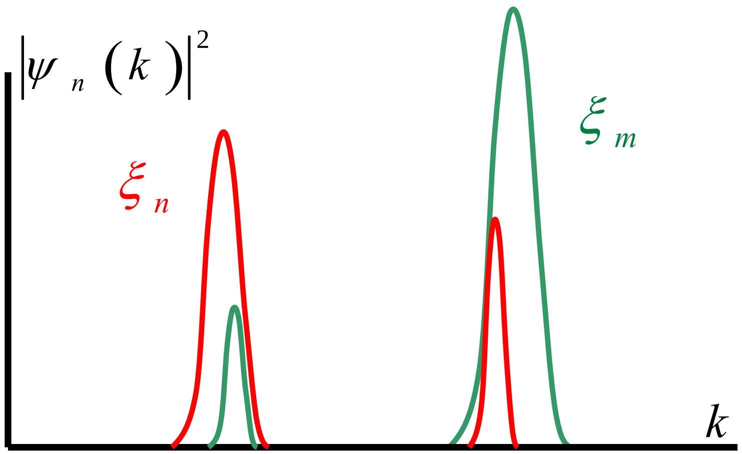
Resuls

$$1-\mu = d_2 \simeq \kappa_2 + \kappa_3 = \frac{\pi b}{\sqrt{2}} + 0.083(\pi b)^2$$

Conclusions

- Using the model of the **of the almost diagonal RMT with multifractal eigenstates in the strong fractality regime** we have shown that:

- assumptions about the dynamical scaling and the relation μ=1-d_2 hold true up to the leading and the subleading terms of the VE



- our results confirm strong correlation of the sparse fractal wave functions