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Ergodic Properties of a Simple Model at the Crossroads between Turbulent Transport and Localization

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Ergodic properties of a simple model at the crossroads between turbulent transport and localization

Krzysztof Gawedzki[a], Trieste, August 2010

Aim of this lecture:

to teach you a useful piece[b] of the theory of stochastic differential equations (**SDE**'s) on an example well known to the **turbulence** community and related to 1D **Anderson localization**

Keywords: **inertial particles**, **SDE**'s, **hypoellipticity**, **control theory**

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[a] Based on joint work with **David P. Herzog** & **Jan Wehr**

[b] A good short review: **L. Rey-Bellet**, *"Ergodic properties of Markov processes"*, In: "Open Quantum systems II", Lect. Notes in Math. 1881, Berlin, 2006, pp. 1-78

Cross-fertilization result

Plan

1. Simple model of **inertial-particles dispersion**
2. Relation to the 1D **Anderson localization**
3. Solution of the basic **SDE**
4. **Hypoellipticity**
5. **Control theory**
6. **Invariant measure** for projective dispersion
7. Top **Lyupunov exponent**
8. Conclusions and open problems

For additional reading:

9. **Malliavin**'s approach to **hypoellipticity**

Basic model

- Our model describes the motion of a small heavy beads, called **inertial particles**, suspended in a turbulent flow (e.g. of water droplets in clouds)

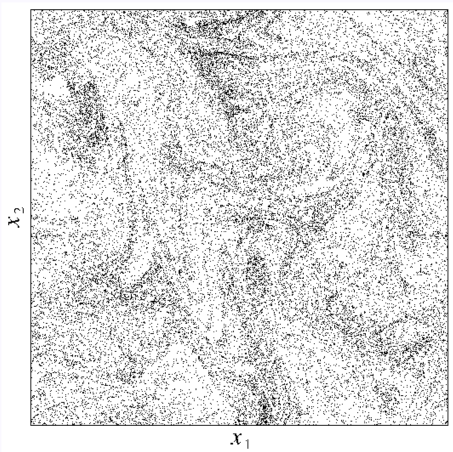
- To a good approximation, motion of inertial particles in a d-dimensional flow with velocity field $\boldsymbol{v}(t, \mathrm{r})$ is described by the equations

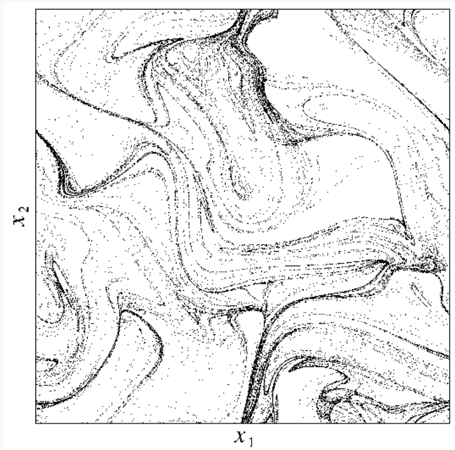
$$\ddot{\boldsymbol{R}} = -\frac{1}{\tau}\big(\dot{\boldsymbol{R}} - \boldsymbol{v}(t, \boldsymbol{R})\big)$$

friction force

Stokes time

- **Main phenomenon**: intermittent clustering of particles





from **J. Bec**, J. Fluid Mech. 528, 255-277 (2005)

- Some information about particle clustering may be extracted from the dynamics of the separation $\delta\boldsymbol{R} \equiv \boldsymbol{\rho}$ of close particles, called **pair dispersion**, that in a moderately turbulent flow obeys the linearized equation:

$$\ddot{\boldsymbol{\rho}} = -\frac{1}{\tau}\big(\dot{\boldsymbol{\rho}} - (\boldsymbol{\rho}\cdot\boldsymbol{\nabla})\boldsymbol{v}(t,\boldsymbol{R}(t))\big)$$

- Assuming the correlation time of the process $S_i^j(t) = \nabla_i v^j(t,\boldsymbol{R}(t))$ much shorter than the **Stokes time** τ, one may model $S(t)$ by a matrix-valued **white noise** with isotropic covariance

$$\Big\langle S_j^i(t)\, S_l^k(t')\Big\rangle \;=\; D_{jl}^{ik}\,\delta(t-t')$$

where

$$D_{jl}^{ik} \;=\; A\,\delta^{ik}\delta_{jl} + B(\delta_j^i\delta_l^k + \delta_l^i\delta_j^k)$$

with $A \geq |B|$, $A + (d+1)B \geq 0$, and, for incompressible flow, $A + (d+1)B = 0$ (not necessarily assumed below)

- This gives are basic **linear (!) SDE**:

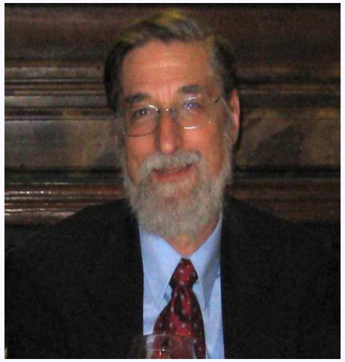
$$\boxed{\ddot{\boldsymbol{\rho}} = -\frac{1}{\tau}\big(\dot{\boldsymbol{\rho}} - S(t)\boldsymbol{\rho}\big)}$$

Relation to 1D localization

- For $\boldsymbol{\psi}(t) = \exp[\frac{t}{2\tau}]\,\boldsymbol{\rho}(t)$, one obtains:

$$-\ddot{\boldsymbol{\psi}} + \frac{1}{\tau} S(t)\,\boldsymbol{\psi} \;=\; -\frac{1}{4\tau^2}\boldsymbol{\psi}$$

- Viewing t as a spatial variable, this is the 1D **stationary Schrödinger equation** with:
  - $\boldsymbol{\psi}(t)$ a vector-valued wave function
  - $V(t) = \frac{1}{\tau} S(t)$ a random matrix-valued δ-correlated potential
  - $E = -\frac{1}{4\tau^2}$ the energy



- In $d = 1$ both $\psi(t)$ and $V(t)$ are real-valued giving the model for 1D **Anderson** localization studies already in 1967 by **Halperin**

- In $d = 2$ both $\psi(t)$ and $V(t)$ may be viewed as complex-valued giving a non-hermitian random **Schrödinger** operator not studied in the context of localization

Solution

- In the 1[st] order form with differentials, Eq. $\ddot{\boldsymbol{\rho}} = -\frac{1}{\tau}\big(\dot{\boldsymbol{\rho}} - S(t)\boldsymbol{\rho}\big)$ becomes:

$$
\begin{aligned}
d\boldsymbol{\rho} &= \frac{1}{\tau}\boldsymbol{\chi}\, dt\,, \\
d\boldsymbol{\chi} &= -\frac{1}{\tau}\boldsymbol{\chi}\, dt + dS(t)\,\boldsymbol{\rho}
\end{aligned}
$$

with, invariably, **Itô** or **Stratonovich** convention

- The solution is:

$$
\boldsymbol{p}(t) \equiv \begin{pmatrix} \boldsymbol{\rho}(t) \\ \boldsymbol{\chi}(t) \end{pmatrix} = \overleftarrow{\exp}\Big[\int_0^t d\Sigma(s)\Big]\begin{pmatrix} \boldsymbol{\rho}(0) \\ \boldsymbol{\chi}(0) \end{pmatrix}
$$

for

$$
d\Sigma(t) = \begin{pmatrix} 0 & \frac{1}{\tau}dt \\ dS(t) & -\frac{1}{\tau}1dt \end{pmatrix}
$$

- It exists for all times and is a **Markov process** with the generator

$$
\boxed{L = \frac{1}{\tau}(\boldsymbol{\chi}\cdot\boldsymbol{\nabla}_{\boldsymbol{\rho}} - \boldsymbol{\chi}\cdot\boldsymbol{\nabla}_{\boldsymbol{\chi}}) + \frac{1}{2}\sum_{i,j,k,l}\rho^j\rho^l D^{ik}_{jl}\,\nabla_{\chi^i}\nabla_{\chi^k}}
$$

Remarks

- $\boldsymbol{p}(t) = 0 \;\Leftrightarrow\; \boldsymbol{p}(0) = 0$ and the process $\boldsymbol{p}(t)$ may be restricted to $\mathbb{R}^{2d} \setminus \{0\} \equiv \mathbb{R}^{2d}_{\neq 0}$

- L is defined by the formula: $\frac{d}{dt}\Big\langle f(\boldsymbol{p}(t))\Big\rangle = \Big\langle (Lf)(\boldsymbol{p}(t))\Big\rangle$

- L is not **elliptic** (its top symbol is degenerate because it contains the second derivatives only in the directions of $\boldsymbol{\chi}$)

- The **transition probability densities** exist in the sense of distributions

$$P_t(\boldsymbol{p}_0, \boldsymbol{p}) \;=\; \Big\langle \delta\Big(\boldsymbol{p} - \overleftarrow{\exp}\,\Big[\int_0^t d\Sigma(s)\Big]\boldsymbol{p}_0\Big)\Big\rangle$$

and satisfy the differential equations:

$$\big(\partial_t - L \otimes 1\big)\, P_t \;=\; 0 \;=\; \big(\partial_t - 1 \otimes L^\dagger\big)\, P_t$$

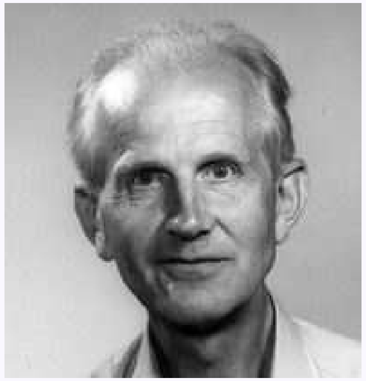
so that

$$\big(2\partial_t - L \otimes 1 - 1 \otimes L^\dagger\big)\, P_t \;=\; 0$$

Hypoellipticity

Definition. A differential operator D on a domain Ω is called **hypoelliptic** if for all distributional solutions of the equation $Df = g$ with smooth right hand side g, f is also smooth

- **Hörmander**'s **criterion**:



Suppose that

$$D = \varphi + X_0 + \frac{1}{2}\sum_{n=1}^{N} X_n^2\,,$$

where φ is a smooth function and $X_0, X_1, \ldots X_n$ are smooth vector fields on Ω and that for each $x \in \Omega$

$$X_{n_1}(x), \quad [X_{n_2}, X_{n_1}](x), \quad [X_{n_3}, [X_{n_2}, X_{n_1}]](x), \quad \ldots\ldots$$

with $n_l = 0, 1, \ldots, N$ span the tangent space at x.

Then D is **hypoelliptic**

Proposition. The generator $L = \frac{1}{\tau}\big(\boldsymbol{\chi}\cdot\boldsymbol{\nabla}_{\boldsymbol{\rho}} - \boldsymbol{\chi}\cdot\boldsymbol{\nabla}_{\boldsymbol{\chi}}\big) + \frac{1}{2}\sum \rho^j \rho^l\, D^{ik}_{jl}\, \nabla_{\chi^i}\nabla_{\chi^k}$ is **hypoelliptic** on $\mathbb{R}^{2d}_{\neq 0}$

Idea of the proof: establish that L satisfies **Hörmander**'s criterion

Lemma. One may decompose:

$$D^{ik}_{jl} = \sum_{m,n=1}^{d} \big(E\delta^i_j\delta^m_n + F\delta^{im}\delta_{jn} + G\delta^i_n\delta^m_j\big)\big(E\delta^k_l\delta^m_n + F\delta^{km}\delta_{ln} + G\delta^k_n\delta^m_l\big)$$

for

$$E = \frac{1}{d}\big(-(A+B)^{\frac{1}{2}} + (A+(d+1)B)^{\frac{1}{2}}\big),$$

$$F = \frac{1}{2}((A+B)^{\frac{1}{2}} + (A-B)^{\frac{1}{2}}),$$

$$G = \frac{1}{2}((A+B)^{\frac{1}{2}} - (A-B)^{\frac{1}{2}})$$

Proof of Lemma: a straightforward check □

Now set:

$$\boldsymbol{X}_0 = \frac{1}{\tau}\big(\boldsymbol{\chi}\cdot\boldsymbol{\nabla}_{\boldsymbol{\rho}} - \boldsymbol{\chi}\cdot\boldsymbol{\nabla}_{\boldsymbol{\chi}}\big),$$

$$\boldsymbol{X}^m_n = \sum_{i,j} \rho^j\big(E\delta^i_j\delta^m_n + F\delta^{im}\delta_{jn} + G\delta^i_n\delta^m_j\big)\nabla_{\chi^i}$$

Then

$$\boxed{L \;=\; \boldsymbol{X}_0 \,+\, \tfrac{1}{2} \sum_{m,n=1}^{d} (\boldsymbol{X}_n^m)^2}$$

Remark. The original **SDE** for $\boldsymbol{p}(t)$ is equivalent to

$$d\boldsymbol{p} \;=\; \boldsymbol{X}_0(\boldsymbol{p})\,dt \,+\, \sum_{m,n=1}^{d} \boldsymbol{X}_n^m(\boldsymbol{p})\,d\beta_n^m(t)$$

with independent **Brownian motions** $\beta_n^m(t)$

Lemma. If $A+2B>0$ in $d=1$ or $A>0$ in $d\geq 2$ then

$$\boldsymbol{X}_n^m(\boldsymbol{p})\,, \quad [\boldsymbol{X}_0, \boldsymbol{X}_{n_1}^{m_1}](\boldsymbol{p})\,, \quad [\boldsymbol{X}_0, [\boldsymbol{X}_0, \boldsymbol{X}_{n_2}^{m_2}]](\boldsymbol{p})$$

span $\mathbb{R}^{2d}$ if $\boldsymbol{p}\neq 0$

Proof of Lemma: a direct check with different arguments when $\boldsymbol{\rho}=0$ and when $\boldsymbol{\rho}\neq 0$

□

The **hypoellipticity** of L on $\mathbb{R}^{2d}_{\neq 0}$ follows from **Hörmander**'s criterion

■

- The same way one proves that:

$$L^\dagger \quad \text{on} \quad \mathbb{R}^{2d}_{\neq 0}\,,$$

$$\partial_t - L, \quad \partial_t - L^\dagger \quad \text{on} \quad (0,\infty)\times\mathbb{R}^{2d}_{\neq 0}\,,$$

$$2\partial_t - L\otimes 1 - 1\otimes L^\dagger \quad \text{on} \quad (0,\infty)\times\mathbb{R}^{2d}_{\neq 0}\times\mathbb{R}^{2d}_{\neq 0}$$

are **hypoelliptic**.

Remark. For the last three ones, one uses the fact that $\boldsymbol{X}_0(\boldsymbol{p})$ was not needed to generate $\mathbb{R}^{2d}$ and that the vector field ∂_t generates the time direction

Corollary. The transition probability densities $P_t(\boldsymbol{p}_0, \boldsymbol{p})$ are smooth on $(0,\infty)\times\mathbb{R}^{2d}_{\neq 0}\times\mathbb{R}^{2d}_{\neq 0}$ (as they are annihilated by $2\partial_t - L\otimes 1 - 1\otimes L^\dagger$)

Control theory

Proposition. $P_t(\boldsymbol{p}_0, \boldsymbol{p})$ are strictly positive for $\boldsymbol{p}_0 \neq 0 \neq \boldsymbol{p}$

Proof. This follows via the **control theory** developed by **Stroock-Varadhan** from the following:

Lemma. Given $T > 0$, $\boldsymbol{p}_0 \neq 0$ and $\boldsymbol{p}_1 \neq 0$ then for there exists control functions $[0,T] \ni t \mapsto u_m^n(t) \in \mathbb{R}$ s. t. the solution of the **ODE**

$$\dot{\boldsymbol{p}} \;=\; \boldsymbol{X}_0(\boldsymbol{p}) + \sum_{m,n=1}^{d} u_m^n(t)\, \boldsymbol{X}_n^m(\boldsymbol{p})$$

with the initial condition $\boldsymbol{p}(0) = \boldsymbol{p}_0$ satisfies $\boldsymbol{p}(t) = \boldsymbol{p}_1$

Proof of Lemma: not very difficult □

■

Remark. This proves that the dispersion process is **irreducible** on $\mathbb{R}^{2d}_{\neq 0}$ (it connects with positive probability arbitrary two points)

Projective dispersion

- The pair-dispersion process $\boldsymbol{p}(t) = (\boldsymbol{\rho}(t), \boldsymbol{\chi}(t))$ on $\mathbb{R}^{2d}_{\neq 0}$ does not have an **invariant probability measure** (it diffuses to ∞ **!**)

- To study ergodic properties of the process $\boldsymbol{p}(t)$ we shall project it on

$$\mathbb{R}^{2d}_{\neq 0}/\mathbb{R}_{+} \;\cong\; S^{2d-1}$$

  where the action of $\mathbb{R}_{+}$ is by the multiplication $\boldsymbol{p} \xrightarrow{\Theta_\sigma} \sigma \boldsymbol{p}$

- $[\Theta_\sigma, L] = 0$ implies that the projected process $\boldsymbol{\pi}(t)$ is still **Markov**

- Generator Λ of $\boldsymbol{\pi}(t)$ is L acting on homogeneous functions of degree 0

- Since $[\Theta_\sigma, \boldsymbol{X}_0] = 0 = [\Theta_\sigma, \boldsymbol{X}_n^m]$, all the hypoelliptic properties still hold

- The transition probabilities of $\boldsymbol{\pi}(t)$, have smooth densities $P_t(\boldsymbol{\pi}_0, \boldsymbol{\pi}) > 0$ w. r. t. $SO(2d)$-invariant measure $\nu(\boldsymbol{\pi})$ on $S^{2d-1} \cong \mathbb{R}^{2d}_{\neq 0}/\mathbb{R}_{+}$

Invariant measure for $\boldsymbol{\pi}(t)$

- The gain from the projection is that the quotient space is **compact**
- Any **Markov** process on a compact space has invariant probability measures, e.g. obtained as weak limits subsequences of **Cesaro** means:

$$\frac{1}{T}\int_0^T P_t(\boldsymbol{\pi}_0, d\boldsymbol{\pi})\, dt$$

(probability measures on a compact space form a weakly compact set)

Theorem.

1. The projectivized pair dispersion process $\boldsymbol{\pi}(t)$ has a unique invariant probability measure $\mu(d\boldsymbol{\pi})$
2. $\mu(d\boldsymbol{\pi})$ is the weak limit of the above **Cesaro** means
3. $\mu(d\boldsymbol{\pi}) = n(\boldsymbol{\pi})\,\nu(d\boldsymbol{\pi})$ where $\nu(d\boldsymbol{\pi})$ is the normalized $S(2d)$-invariant measure on S^{2d-1}
4. $n(\boldsymbol{\pi})$ is smooth and strictly positive

Idea of the proof:

- The **hypoellipticity** of $\Lambda^\dagger$ implies that every invariant measure $\mu(d\boldsymbol{\pi})$ has a smooth density $n(\boldsymbol{\pi})$ (annihilated by $\Lambda^\dagger$) w.r.t. $\nu(\boldsymbol{\pi})$

- Smoothness and positivity of $P_t(\boldsymbol{\pi}_0, \boldsymbol{\pi})$ together with the invariance relation

$$\int n(\boldsymbol{\pi}_0)\; P_t(\boldsymbol{\pi}_0, \boldsymbol{\pi})\; d\nu(d\boldsymbol{\pi}_0) \;=\; n(\boldsymbol{\pi})$$

  imply that $n(\boldsymbol{\pi}) > 0$ everywhere

- The uniqueness of $\mu(d\boldsymbol{\pi})$ follows since two different ergodic invariant smooth measures necessarily have disjoint supports

$\blacksquare$

One also has:

Theorem. Time-correlations of $\boldsymbol{\pi}(t)$ decay exponentially (exp. **mixing**)

More symmetries

- Consider the action of $SO(d)$ on $\mathbb{R}^{2d}$ given by

$$(\boldsymbol{\rho}, \boldsymbol{\chi}) \stackrel{\Theta_O}{\longmapsto} (O\boldsymbol{\rho}, O\boldsymbol{\chi})$$

  - It induces an action of $SO(d)$ on S^{2d-1} that preserves $\nu(d\boldsymbol{\pi})$
  - It commutes with L and Λ (due to the assumed isotropy of the flow) and hence preserves the measure $\mu(\boldsymbol{\pi})$

- **Corollary**. Density $n(\boldsymbol{\pi})$ is $SO(d)$-invariant and must depend only on the $SO(d)$-invariants

- The (dimensionless) $SO(d)$-invariants on $\mathbb{R}^{2d}_{\neq 0}/\mathbb{R}_+ \cong S^{2d-1}$ are:
  - in $d = 1$: $x = \frac{\chi}{\rho}$,
  - in $d = 2$: $x = \frac{\boldsymbol{\rho}\cdot\boldsymbol{\chi}}{\boldsymbol{\rho}^2}$ and $y = \frac{\rho^1\chi^2 - \rho^2\chi^1}{\boldsymbol{\rho}^2}$,
  - in $d \geq 3$: $x = \frac{\boldsymbol{\rho}\cdot\boldsymbol{\chi}}{\boldsymbol{\rho}^2}$ and $y = \frac{\sqrt{\boldsymbol{\rho}^2\boldsymbol{\chi}^2 - (\boldsymbol{\rho}\cdot\boldsymbol{\chi})^2}}{\boldsymbol{\rho}^2}$

$d = 1$ **case**

- Here the normalized rotation-invariant measure on S^1 is:

$$\nu(d\boldsymbol{\pi}) \;=\; \frac{dx}{\pi(1+x^2)} \qquad \text{and} \qquad \mu(d\boldsymbol{\pi}) \;=\; n(\boldsymbol{\pi})\,\nu(d\boldsymbol{\pi}) \;=\; \eta(x)\,dx$$

- Smoothness and strict positivity of $n(\boldsymbol{\pi})$ on S^1 imply that

$$\eta(x) \;=\; \mathcal{O}(|x|^{-2}) \qquad \text{for} \quad |x| \to \infty$$

- In fact (**Halperin** 1967)

$$\eta(x) = \; Z^{-1}\Big(\mathrm{e}^{-\frac{1}{\tau(A+2B)}\left(\frac{2}{3}x^3+x^2\right)}\int_{-\infty}^{x}\mathrm{e}^{\frac{1}{\tau(A+2B)}\left(\frac{2}{3}x'^3+x'^2\right)}dx'\Big)$$

  and the behavior at ∞ may also be extracted from the above formula

- The projected process $x(t)$ explodes in finite time to $-\infty$ but re-enters immediately from $+\infty$

$d = 2$ **case**

- Here

$$\nu(d\boldsymbol{\pi}) \;=\; \frac{dx\,dy\,d\arg(\boldsymbol{\rho})}{\pi(1+x^2+y^2)^2} \qquad \text{and} \qquad \mu(d\boldsymbol{\pi}) \;=\; \eta(x,y)\,dx\,dy\,d\arg(\boldsymbol{\rho})$$

- Smoothness and positivity of the ratio of the two measures on S^3 imply that for $z = x + iy$

$$\eta(x,y) \;=\; \mathcal{O}(|z|^{-4}) \qquad \text{for} \quad |z| \to \infty$$

and, as conjectured by **Bec-Cencini-Hillerbrand** (2007),

$$\int_{-\infty}^{\infty} \eta(x,y)\,dy \;=\; \mathcal{O}(|x|^{-3}) \qquad \text{for} \quad |x| \to \infty$$

- In this case $S^3/SO(2) \cong \mathbb{PC}^1 \cong \mathbb{C} \cup \{\infty\}$ (the **Riemann** sphere) and the projected process $z(t)$ may be shown to stay finite at all times if $z(0)$ is finite (no explosion**!**)

$d \geq 3$ case

- Now for the measures on S^{2d-1} one has:

$$\nu(d\boldsymbol{\pi}) = \frac{2^{d-1}(d-1)y^{d-2}dx\,dy\,d[O]}{\pi(1+x^2+y^2)^d} \quad \text{and} \quad \mu(d\boldsymbol{\pi}) = \eta(x,y)\,dx\,dy\,d[O]$$

where $d[O]$ is the normalized $SO(d)$ invariant measure on $SO(d)/SO(d-2)$

- Smoothness and positivity of the ratio of the two measures on S^{2d-1} imply:

$$\begin{aligned} \eta(x,y) &= \mathcal{O}(|x|^{-2d}) && \text{for} \quad |x| \to \infty \\ \eta(x,y) &= \mathcal{O}(|y|^{d-2}) && \text{for} \quad y \searrow 0 \\ \int_0^\infty \eta(x,y)\,dy &= \mathcal{O}(|x|^{-d-1}) && \text{for} \quad |x| \to \infty \end{aligned}$$

- In $d \geq 3$ the quotient space $S^{2d-1}/SO(d)$ is not smooth (unlike for $d=2$). Lack of explosions of the projected process $(x(t), y(t))$ to $y=0$ or $|x|=\infty$ was shown

Lyapunov exponent

- For the original dispersion process $\boldsymbol{p}(t) = (\boldsymbol{\rho}(t), \boldsymbol{\chi}(t))$, one may define the **Lyapunov** exponents as

$$\lambda \;=\; \lim_{t\to\infty} \frac{1}{2t}\left\langle \ln\left(\frac{\boldsymbol{p}^2(t)}{\boldsymbol{p}^2(0)}\right)\right\rangle$$

Theorem.

1. The above limit exists and is independent of the initial point of the process $\boldsymbol{p}(t)$

2. In $d = 1$ what results is in the formula:

$$\lambda \;=\; \frac{1}{\tau}\, p.v.\!\int x\,\eta(x)\,dx \;=\; -\frac{1}{2\tau} + \frac{1}{4\tau\sqrt{c}}\,\frac{d}{dc}\,\ln\left(\mathrm{Ai}^2(c) + \mathrm{Bi}^2(c)\right)$$

   for $c = (4\tau(A+2B))^{-\frac{2}{3}}$

3. In $d \geq 2$, one gets:

$$\lambda \;=\; \frac{1}{\tau}\int x\,\eta(x,y)\,dxdy$$

Remarks

1. The main input into the **proof** is the formula $L(\boldsymbol{\rho}^2) = 2\frac{\boldsymbol{\rho}\cdot\boldsymbol{\chi}}{\boldsymbol{\rho}^2} = \frac{2x}{\tau}$ and the ergodicity of the projected dispersion $\boldsymbol{\pi}(t)$, but some work is required

2. $\lambda + \frac{1}{2\tau}$ is the **Lyapunov** exponent of the associated 1D localization problem and it is positive in $d = 1$ (permanent localization)

3. λ itself may change sign in $d = 1$ signaling a phase transition in the advection of inertial particles in one dimension (**Wilkinson-Mehlig** 2003)

4. In $d \geq 2$ no explicit formula for λ exists (except for very special cases) but numerical simulations and asymptotic analysis are available (**Wilkinson-Mehlig** *et al.* 2004-2006, **Horvai** 2005, **Bec-Cencini-Hillerbrand** 2007)

Conclusions and open problems

- The use of **hypoellipticity** and **control theory** techniques in ergodic theory of **SDE**'s was illustrated on a model for the **inertial-particle dispersion** in moderate turbulence, related to the 1D localization
- Our analysis permitted to prove certain properties of that model conjectured in the literature (**Bec** *et al.*, 2007) and to establish some new ones
- Similar treatment (with more $SO(d)$ representation theory) should apply to the **multiparticle dispersion** process $\delta_1 \boldsymbol{r}(t) \wedge \cdots \wedge \delta_k \boldsymbol{r}(t)$, leading to formulae for the other **Lyapunov** exponents λ_k
- Another open problem is the existence of the **large deviations** regime (providing more insight to the particle-clustering and its **multifractality**) for fluctuations of finite-time **Lyapunov** exponents about λ_k
- Application of the techniques discussed here to the model of inertial-particle dispersion in **fully developed turbulence** introduced by **Bec**-**Cencini**-**Hillerbrand** (2007) remains to be worked out

- An infinite-dimensional version of similar techniques was used by **Hairer-Mattingly** (2006-2008) to prove ergodicity of 2D **Navier-Stokes** **equation** with random large-scale forcing

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"Everything has its beauty but not everyone sees it"

Confucius

Malliavin's approach to hypoellipticity

- Suppose that $\boldsymbol{F}:\mathbb{R}^N\to\mathbb{R}^d$ is smooth with pollynomially bounded derivatives. Consider a **Gaussian** measure on $\mathbb{R}^N$ with covariance 1.
- $\boldsymbol{F}(\boldsymbol{\eta})$ becomes then a random vector with the distributional **PDF**

$$\int \delta(\boldsymbol{x}-\boldsymbol{F}(\boldsymbol{\eta}))\ \mathrm{e}^{-\frac{1}{2}\boldsymbol{\eta}^2}\frac{d\boldsymbol{\eta}}{(2\pi)^{N/2}}$$

- **Question**: when is the right hand side a smooth function of $\boldsymbol{x}$**?**
- A sufficient condition for that is the rank equal to d of the **Jacobi** matrix

$$\frac{\partial F^i(\boldsymbol{\eta})}{\partial\eta^n}\quad \text{or, equivalently, of}\quad M^{ij}(\boldsymbol{\eta})=\sum_{n=1}^{N}\frac{\partial F^i(\boldsymbol{\eta})}{\partial\eta^n}\,\frac{\partial F^j(\boldsymbol{\eta})}{\partial\eta^n}$$

- Under this assumption, one may change locally the integral over d of variables η^k to the one over $\boldsymbol{F}$ getting rid of the δ-function
- The resulting sum over the (discrete) set of solutions of Eq. $\boldsymbol{x}=\boldsymbol{F}(\boldsymbol{\eta})$ for fixed $\boldsymbol{x}$ is controlled by the bounds on $\boldsymbol{F}$ and its derivatives

- **Malliavin** applied the above argument in the ∞-dimensional setup where $\boldsymbol{\eta} = \frac{d\boldsymbol{\beta}}{dt}$ is a N-dimensional **white noise** on the interval $[0, T]$ and $\boldsymbol{F}(\boldsymbol{\eta}) = \boldsymbol{x}(T)$, with $\boldsymbol{x}(t)$ the solution of the **Stratonovich SDE**

$$d\boldsymbol{x} \;=\; \boldsymbol{X}_0(\boldsymbol{x})dt + \sum_{n=1}^{N} \boldsymbol{X}_n(\boldsymbol{x}) \circ d\boldsymbol{\beta}\,, \qquad \boldsymbol{x}(0) = \boldsymbol{x}_0$$

- Matrix $M^{ij}(\boldsymbol{\eta})$, called here the **Malliavin matrix**, is given by

$$M^{ij} \;=\; \sum_{n=1}^{N} \int_0^T \frac{\delta x^i(T)}{\delta \eta^n(t)} \, \frac{\delta x^j(T)}{\delta \eta^n(t)} \, dt$$

- An easy calculation gives:

$$\frac{\delta \boldsymbol{x}(T)}{\delta \eta^n(t)} \;=\; W(T)\, W(t)^{-1} \boldsymbol{X}_n(\boldsymbol{x}(t))$$

where $W(t)$ is the $d \times d$ matrix that solves the linearized **SDE**

$$dW \;=\; D\boldsymbol{X}_0(\boldsymbol{x}) W dt + \sum_n D\boldsymbol{X}_n(\boldsymbol{x}) W \circ d\boldsymbol{\beta}\,, \qquad W(0) = 1$$

- We infer that the matrix $M = W(T)NW(T)^{\dagger}$ is of maximal rank if and only if the matrix N is, for

$$N \ = \ \sum_{n=1}^{N} \int_0^T \left(W(t)^{-1}\boldsymbol{X}_n(\boldsymbol{x}(t))\right) \left(W(t)^{-1}\boldsymbol{X}_n(\boldsymbol{x}(t))\right)^{\dagger}$$

- Suppose now that some vector $\boldsymbol{y} \in \mathbb{R}^d$ is orthogonal to the image of N. Then $\boldsymbol{y} \cdot N\boldsymbol{y} = 0$ and one infers that

$$\boldsymbol{y} \cdot W(t)^{-1}\boldsymbol{X}_n(\boldsymbol{x}(t)) \ = \ 0 \qquad \text{for} \qquad n = 1, \ldots, N, \quad 0 \leq t \leq T$$

- At $t = 0$ this gives: $\boldsymbol{y} \cdot \boldsymbol{X}_n(\boldsymbol{x}_0) = 0$

- Differentiating over t the identity $\boldsymbol{y} \cdot W(t)^{-1}\boldsymbol{X}_n(\boldsymbol{x}(t)) = 0$ with the use of the **SDE**'s for $\boldsymbol{x}(t)$ and $W(t)$ one infers that

$$\boldsymbol{y} \cdot W(t)^{-1}[\boldsymbol{X}_{n_1}, \boldsymbol{X}_n](\boldsymbol{x}(t)) = 0$$

  for $n_1 = 0, 1, \ldots, N$, so that also $\boldsymbol{y} \cdot [\boldsymbol{X}_{n_1}, \boldsymbol{X}_n](\boldsymbol{x}_0) = 0$

- By iteration, one infers that $\boldsymbol{y}$ is orthogonal to the span of

$$\boldsymbol{X}_n(\boldsymbol{x}_0), \quad [\mathbf{X}_{n_1}, \boldsymbol{X}_n](\boldsymbol{x}_0), \quad [\boldsymbol{X}_{n_2}, [\boldsymbol{X}_{n_1}, \boldsymbol{X}_n]](\boldsymbol{x}_0), \quad \ldots\ldots$$

for $n = 1, \ldots, N$, $n_l = 0, 1, \ldots, N$, and hence vanishes if the latter vectors span $\mathbb{R}^d$. Invertibility of the **Malliavin** matrix $M(\boldsymbol{\eta})$ follows

- The space of square integrable functionals of the **white noise** $\boldsymbol{\eta}$ may be naturally identified with the symmetric **Fock space** $\Gamma(L^2([0.T], \mathbb{R}^N))$

- The regularity assumptions on $\boldsymbol{F}(\boldsymbol{\eta})$ from the finite-dimensional case become here the (satisfied) conditions that the functionals of $\boldsymbol{\eta}$:

$$\sum_{i=1}^{d} \sum_{\substack{n_l=1 \\ l=1..k}}^{N} \int_{[0,T]^k} \left| \frac{\delta^k x^i(T)}{\delta\eta^{n_1}(t_1) \cdots \delta\eta^{n_k}(t_k)} \right|^2 dt_1 \cdots dt_k \,,$$

are in the domain of all powers of the **Fock space number operator**

- A good brief review: **P. K. Friz**: *"An Introduction to Malliavin Calculus"*, http://www.statslab.cam.ac.uk/~peter/mystuff/papers.html