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Gallavotti-Cohen symmetry for observable different from entropy production

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Miscellaneous on Large deviation theory

”Gallavotti-Cohen” symmetry for observable different from entropy production

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Mars gris, avril pluvieux, - Font l’an fertile et plantureux

Based on the work :

Entropy production and fluctuation for a KPZ interface,
A.C. Barato, R. Chetrite, H. Hinrichsen, D. Mukamel.
arXiv :1008.3463.

Large Deviation theory

"Improbable events permit themselves the luxury of occurring." C.Chan 1928

Heuristic of large deviation

- Random variable A_T which converges in probability towards $\langle a \rangle$
- Large deviation : How improbable for A_T to converge towards a which is different from the typical value $\langle a \rangle$:

$$\text{LDP} : \langle \delta(A_T - a) \rangle \asymp \exp(-TI(a))$$

- $I(a)$ is called the rate function. Large deviation theory : Prove the LDP and calculate the rate function.
- Scaled cumulant generating function : $\Lambda(s) \asymp -\frac{1}{T} \ln \langle \exp(-sTA_T) \rangle$
- Gartner-Ellis Theorem (Saddle point approximation) : If $\Lambda(s)$ is differentiable, then the LDP exists and $I(a) = \sup_{s \in \Re} (\Lambda(s) - sa)$

Gallavotti-Cohen (GC) symmetry

$$I(a) = I(-a) - Ea \Leftrightarrow \Lambda(E - s) = \Lambda(s)$$

- Where does it come from ?
- For what type of random variable ?

Mathematical set up : X_t Ergodic time homogeneous pure jump process

- Transition rates $W_{\not\lambda}(x,y)$, Intensity $\lambda(x) = \int dy W(x,y)$, Markovian generator $L = W - \lambda Id$.
- CADLAG trajectories (in French : Continue A Droite, avec Limit A Gauche) then $X_s^+ = X_s$ and the jump is $\Delta X_s = X_s - X_s^-$.
- Invariant density $\rho_{inv}(x)$

Forward trajectory and trajectorial measure

- Trajectory on $[0, T]$: number of jumps n, sequence of state x_i and jump time $t_i \Rightarrow$ [x]
- Measure of this trajectory with initial distribution ρ_0 :
  $M_{[0,T],\rho_0}[x] =$
  $\rho_0(x_0)\exp(-\lambda(x_0)t_1)W(x_0,x_1)\exp(-\lambda(x_1)(t_2-t_1))\ldots W(x_{n-1},x_n)\exp(-\lambda(x_n)(T-t_n))$

Reversed trajectory and trajectorial measure

- Reversed trajectory ...$\Rightarrow \widetilde{[x]} = \left(\overleftarrow{x}, \overleftarrow{T-t}, n\right)$
- Measure of this reversed trajectory with initial distribution ρ_0^r :
  $\widetilde{M}_{[0,T],\rho_0^r}[x] \equiv M_{T,\rho_0^r}\widetilde{[x]} =$
  $\rho_0^r(x_n)\exp(-\lambda(x_n)(T-t_n))W(x_n,x_{n-1})\ldots W(x_1,x_0)\exp(-\lambda(x_0)t_1)$

Modern definition of fluctuating total entropy production

$$\Delta S_T^{tot}[x] \equiv \ln \frac{M_{[0,T],\rho_0}[x]}{\widetilde{M}_{[0,T],\rho_T}[x]} = -\ln \rho_T(x_T) + \ln \rho_0(x_0) + \sum_{i=1}^{N_t} \ln \left(\frac{W(x_{i-1},x_i)}{W(x_i,x_{i-1})} \right)$$

Fluctuating entropy of the system

$$S_t^{sys} = -\ln \rho_t(x_t)$$

Fluctuating entropy production in the system :

$$\Delta S_T^{sys}[x] = -\ln \rho_T(x_t) + \ln \rho_0(x_0)$$

Fluctuating entropy production in the environment

$$\Delta S_T^{env}[x] = \sum_{i=1}^{N_t} \ln \left(\frac{W(x_{i-1},x_i)}{W(x_i,x_{i-1})} \right) = \sum_{0 \leq s \leq T, \Delta x_s \neq 0} \ln \left(\frac{W(X_s^-,X_s)}{W(X_s,X_s^-)} \right)$$

"In the end, a theory is accepted not because it is confirmed by conventional empirical test, but because researchers persuades one another that the theory is correct and relevant".
Fisher Black (1986)

LD of observable (Level 1) : Spectral characterization

Empirical mean of one point function

$$A_T^e \equiv \frac{1}{T}\int_0^T A(x_t)dt \to \int dx A(x)\rho^{inv}(x)$$

Feymann-Kac : $\Lambda(s) = -\inf Spectre(-L - sAId) \Rightarrow$ No generic GC symmetry

Empirical mean of two point function

$$B_T^e \equiv \frac{1}{T}\sum_{0\leq s\leq t, \Delta x_s \neq 0} B(X_s^-, X_s) \to \int dxdy B(x,y)\rho^{inv}(x)W(x,y)$$

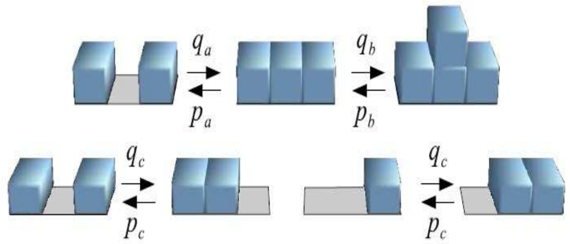
- "Feymann-Kac" formula : $E_x\left(\delta(x_T - y)\exp(-sTB_T^e)\right) = \exp(T(W\exp(-sB) - \lambda Id))(x,y)$ then

$$\Lambda(s) = -\inf Spectrum(H_s) \quad \text{with} \quad H_s = -W\exp(-sB) + \lambda Id$$

- We have $GC \equiv \left\{\inf Spectrum(H_s) = \inf Spectrum(H_{E-s})\right\} \Leftarrow CS_1 \equiv \left\{Spectrum(H_s) = Spectrum(H_{E-s})\right\} \Leftarrow$ $CS_2 \equiv \left\{H_s(x,y) = f^{-1}(x)H_{E-s}(y,x)f(y) \equiv \begin{pmatrix} W(x,y) = f^{-1}(x)W(y,x)\exp(-EB(y,x))f(y) \\ B(x,y) = -B(y,x)\end{pmatrix}\right\}$
- Then CS_2 is for an (antisymmetric) observable which is proportional to the entropy production in the large time limit : $\Delta S_T^{env}[x] = \frac{-\ln(f(x_0)) + \ln(f(x_T))}{T} + EB_T^e[x]$
- Entropy production is obtained for $B(x,y) = \ln\left(\frac{W(x,y)}{W(y,x)}\right)$, then $H_s(x,y) = -W(x,y)^{1-s}W(y,x)^s + \lambda(x)\delta(x-y) \Rightarrow CS_2$ verified : $H_s = H_{1-s}^T$
- Questions : Example of an observable for which CS_2 is not verified, but CS_1 or GC is ?

Restricted Solid On Solid model

- Deposition and evaporation with the constraint $|h_i - h_{i\pm1}| \leq 1$ and periodic BC : $h_{L+1} = h_1$



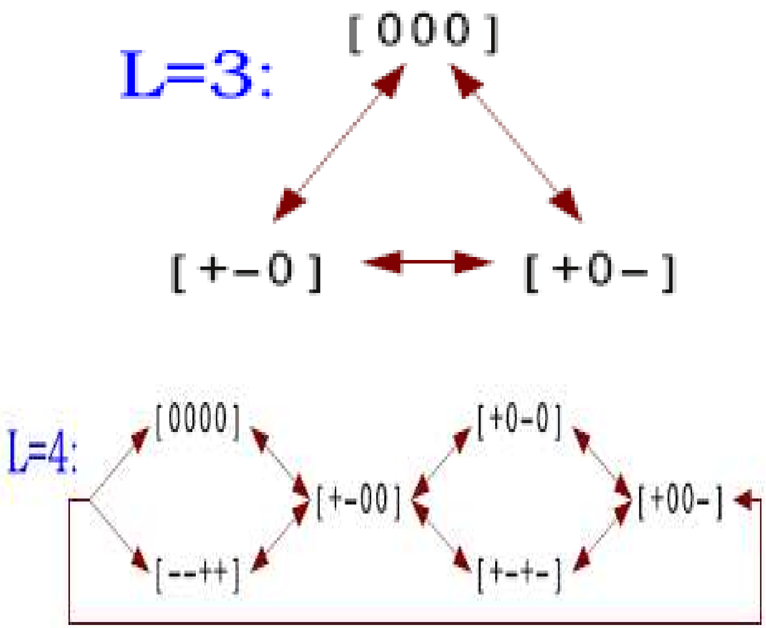
$$-+ \underset{p_a}{\overset{q_a}{\rightleftarrows}} 00 \underset{p_b}{\overset{q_b}{\rightleftarrows}} +- \qquad 0+ \underset{p_c}{\overset{q_c}{\rightleftarrows}} +0 \qquad -0 \underset{p_c}{\overset{q_c}{\rightleftarrows}} 0-$$

- Equivalent model of charge = "Reduction" of the number of state : $\sigma_i = h_{i+1} - h_i$, BC $\Rightarrow \sum_{i=1}^{L} \sigma_i = 0$
- Height $H_T = \sum_{0 \leq s \leq t, \Delta X_s \neq 0} B(X_s^-, X_s)$ with $B(X_s^-, X_s) = 1$ for a deposition and $B(X_s^-, X_s) = -1$ for an evaporation
- If $\frac{q_a}{p_a} = \frac{q_b}{p_b} = \frac{q_c}{pc} \equiv r$ then $\Delta S_T^{env} = (\ln r)\, H_T$ and then we have GC symmetry for the large deviation of $\frac{H_T}{T}$: $I(h) = I(-h) - (\ln r)\, h$
- Now we treat the famous case : $q_a = q_b = q_c = q$, $p_b = p_c = 1$ and $p_a = p$.

$L=3$: GC with $E = \ln q - \frac{\ln p}{3}$

- First view : we can check that CS2 is verified by diagonalizing the deformed operator
- Physical view : the 7 states can be grouped by symmetry into 3 super-state : $[000], [+-0] = \{(+-0), (0+-), (-0+)\}, [-+0] = \{(-+0), (0-+), (+0-)\}$ with the network of transition forming a unique cycle

"Les preuves fatiguent la vérité"
Georges Braque. Le jour et la nuit



L=4 : GC with $E = \frac{1}{4}\ln\left(\frac{3q^4}{2p+p^2}\right)$ and CS2 non verified

First view : check that CS1 is verified by diagonalizing the deformed operator

Physical view : trajectorial analysis

For the entropy production : property of individual path

- $\exp(\Delta S_T^{env}) = \frac{M_{[0,T]}[x]}{M_{[0,T]}[\widetilde{x}]}$
- $\frac{\left\langle \delta\left(\frac{\Delta S_T^{env}}{T} - j\right)\right\rangle}{\left\langle \delta\left(\frac{\Delta S_T^{env}}{T} + j\right)\right\rangle} = \frac{\int M_{[0,T]}[x]\delta\left(\frac{\Delta S_T^{env}[x]}{T} - j\right)}{\int M_{[0,T]}[x]\delta\left(\frac{\Delta S_T^{env}[x]}{T} + j\right)} = \frac{\int M_{[0,T]}[\widetilde{x}]\exp(\Delta S_T^{env}[x])\delta\left(\frac{\Delta S_T^{env}[x]}{T} - j\right)}{\int M_{[0,T]}[\widetilde{x}]\delta\left(\frac{-\Delta S_T^{env}[x]}{T} + j\right)} = \exp(Tj)$

For the height : property of group of path

- $\exp(EH_T) \neq \frac{M_{[0,T]}[x]}{M_{[0,T]}[\widetilde{x}]}$
- A cycle analysis of the embedding Markov chains gives : $\frac{\left\langle \delta\left(\frac{H_T}{T} - h\right)\middle| N_T = n\right\rangle}{\left\langle \delta\left(\frac{H_T}{T} + h\right)\middle| N_T = n\right\rangle} = \exp(TEh)$

and then $\frac{\left\langle \delta\left(\frac{H_T}{T} - h\right)\right\rangle}{\left\langle \delta\left(\frac{H_T}{T} + h\right)\right\rangle} = \frac{\sum_n \left\langle \delta\left(\frac{H_T}{T} - h\right)\middle| N_T = n\right\rangle P(N_T = n)}{\sum_n \left\langle \delta\left(\frac{H_T}{T} + h\right)\middle| N_T = n\right\rangle P(N_T = n)} = \exp(TEh)$

Open issue : GC in disordered system

Large deviations for a random walk in a random environment, Ann. Prob. 22 (1994),
A.Greven, F.Den Hollander.

- Random environment w_k, $k \in Z$ iid $(0,1)$ random variable with distribution α
- Discrete random walk X_n in Z with
$$P_w(X_{n+1} = x \pm 1 | X_{n+1} = x) = \begin{cases} w_x \\ 1 - w_x \end{cases}$$
- Large deviation for the speed of the w-conditioned random walk :
$$P_w\left(\frac{X_n}{n} = v\right) \asymp \exp(-nI(v))$$
  - $I(v)$ is deterministic
  - $I(v)$ verify the "GC" symmetry
$$I(v) = I(-v) + v\left\langle \ln\left(\frac{1-w}{w}\right)\right\rangle$$