The Abdus Salam International Centre for Theoretical Physics

United Nations Educational, Scientific and Cultural Organization

IAEA

2162-34

Advanced Workshop on Anderson Localization, Nonlinearity and Turbulence: a Cross-Fertilization

23 August - 3 September, 2010

An Effective Theory of Pulse Propagation in a Nonlinear and Disordered Medium in Two Dimensions

A. FINKEL'STEIN

Texas A&M University
Dept. of Physics College Station
TX
U.S.A.

An effective theory of pulse propagation in a nonlinear *and* disordered medium in two dimensions.

Georg Schwiete & Alexander Finkel'stein

Phys. Rev. Lett.
104, 103904 (2010)

Minerva
Foundation

Georg Schwiete → Berlin

Abstract

We develop an effective theory of pulse propagation in a nonlinear *AND* disordered medium in 2d.

It is formulated in terms of a minimal equation which despite its apparent simplicity describes novel phenomena which we refer to as **''locked explosion'' and ''diffusive'' collapse**.

It can be applied to such distinct physical systems as laser beams propagating in disordered photonic crystals or Bose-Einstein condensates expanding in a disordered environment.

Experiment - visualization of the Anderson localization:

Laser beams propagating in disordered photonic crystals

Bose-Einstein condensates expanding in a disordered environment

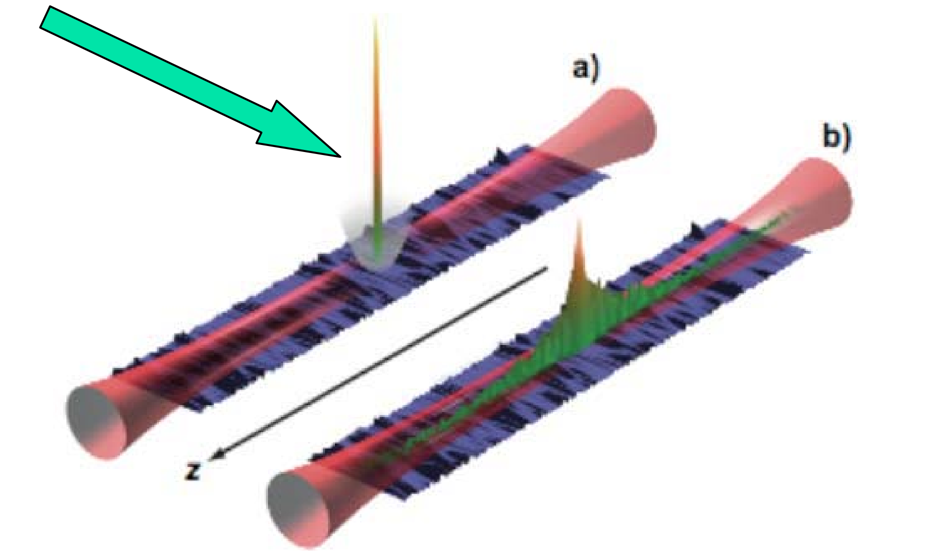
BECs with disorder

Direct observation of Anderson localization of matter-waves in a controlled disorder

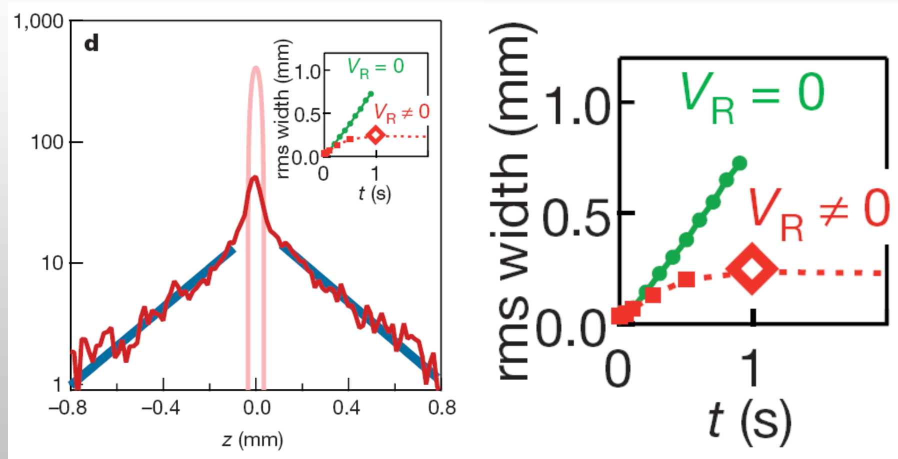
Juliette Billy[1], Vincent Josse[1], Zhanchun Zuo[1], Alain Bernard[1], Ben Hambrecht[1], Pierre Lugan[1], David Clément[1], Laurent Sanchez-Palencia[1], Philippe Bouyer[1] & Alain Aspect[1]

[1]*Laboratoire Charles Fabry de l'Institut d'Optique, CNRS and Univ. Paris-Sud, Campus Polytechnique, RD 128, F-91127 Palaiseau cedex, France*

The method presented here can be extended to localization of atomic quantum gases in higher dimensions, and with controlled interactions.



Roati et al, Nature **453**, 895 (2008)



Nature **453**, 891 (2008)

Disordered photonic 1D-lattice

PRL 100, 013906 (2008) PHYSICAL REVIEW LETTERS week ending 11 JANUARY 2008

Anderson Localization and Nonlinearity in One-Dimensional Disordered Photonic Lattices

Yoav Lahini,[1,*] Assaf Avidan,[1] Francesca Pozzi,[2] Marc Sorel,[2] Roberto Morandotti,[3]
Demetrios N. Christodoulides,[4] and Yaron Silberberg[1]
[1]*Department of Physics of Complex Systems, The Weizmann Institute of Science, Rehovot, Israel*

short time scales of δ-like wave packets in the presence of disorder. A transition from ballistic wave packet expansion to exponential (Anderson) localization is observed. We also find an intermediate regime in which the ballistic and localized components coexist while diffusive dynamics is absent. Evidence is found for a faster transition into localization under nonlinear conditions.

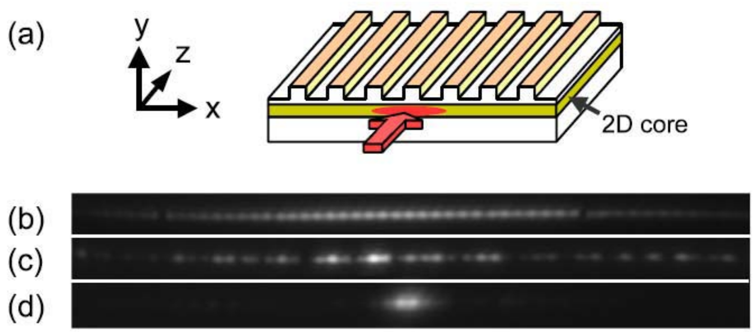
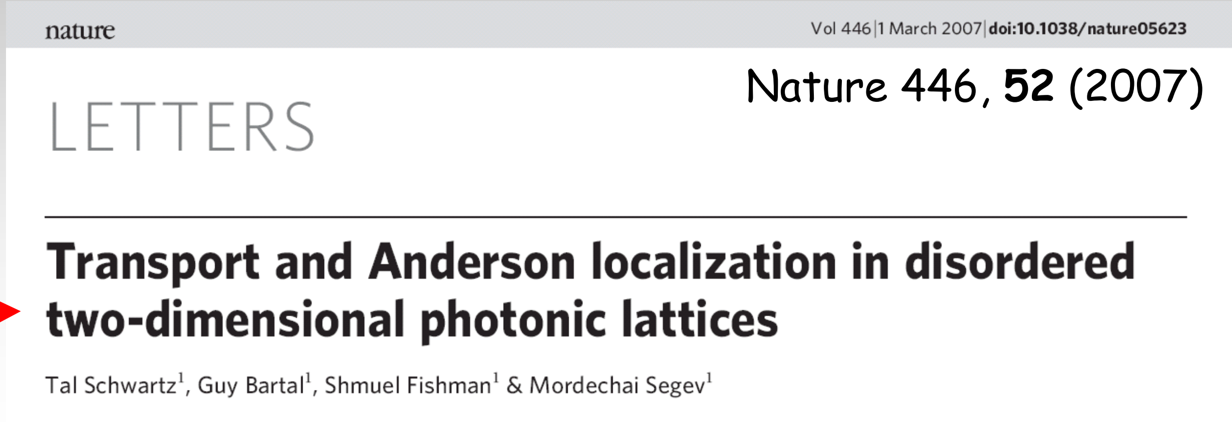


FIG. 1 (color online). (a) Schematic view of the sample used in the experiments. The red arrow indicates the input beam. (b)–(d) Images of output light distribution, when the input beam covers a few lattice sites: (b) in a periodic lattice, (c) in a disordered lattice, when the input beam is coupled to a location which exhibits a high degree of expansion, and (d) in the same disordered lattice when the beam is coupled to a location in which localization is clearly observed.

Disordered photonic 2D-lattice

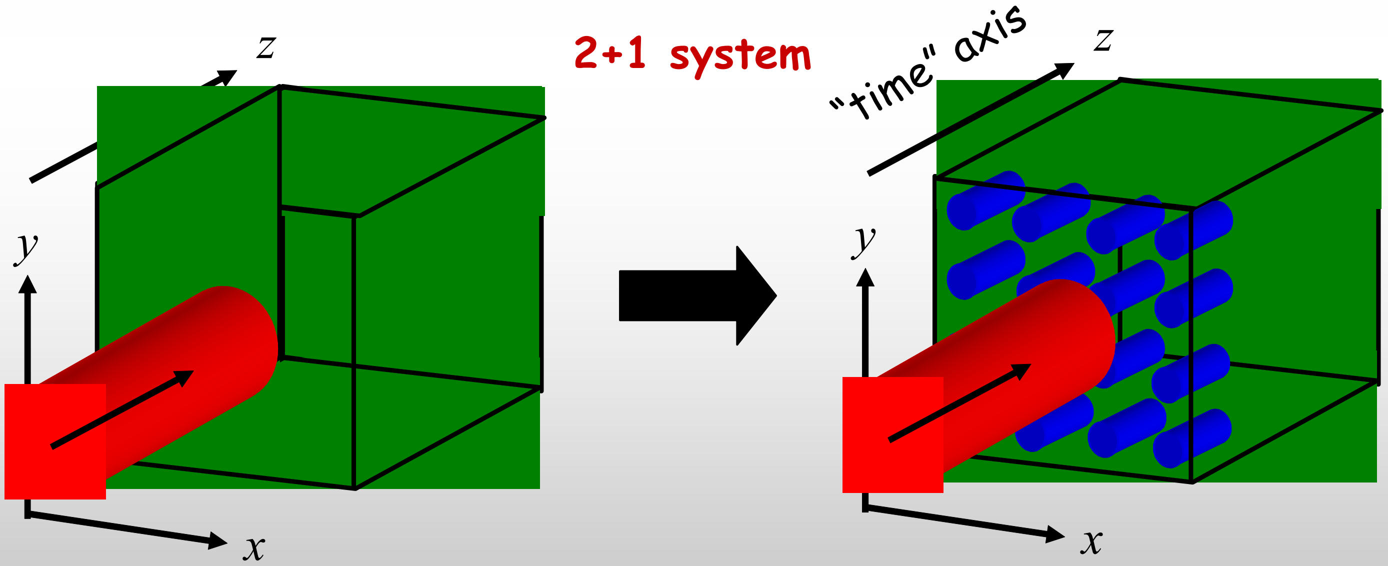

nature Vol 446 | 1 March 2007 | doi:10.1038/nature05623

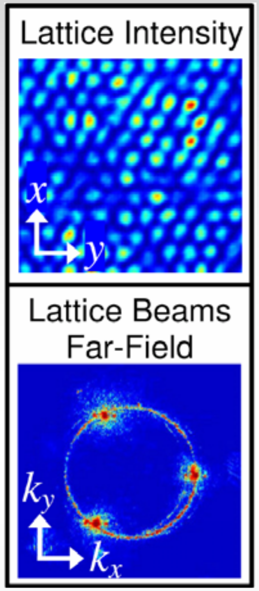
LETTERS

Transport and Anderson localization in disordered two-dimensional photonic lattices

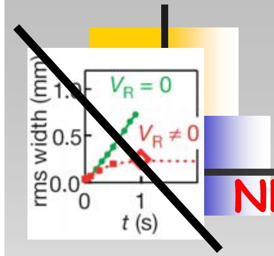
Tal Schwartz[1], Guy Bartal[1], Shmuel Fishman[1] & Mordechai Segev[1]

Nature 446, **52** (2007)



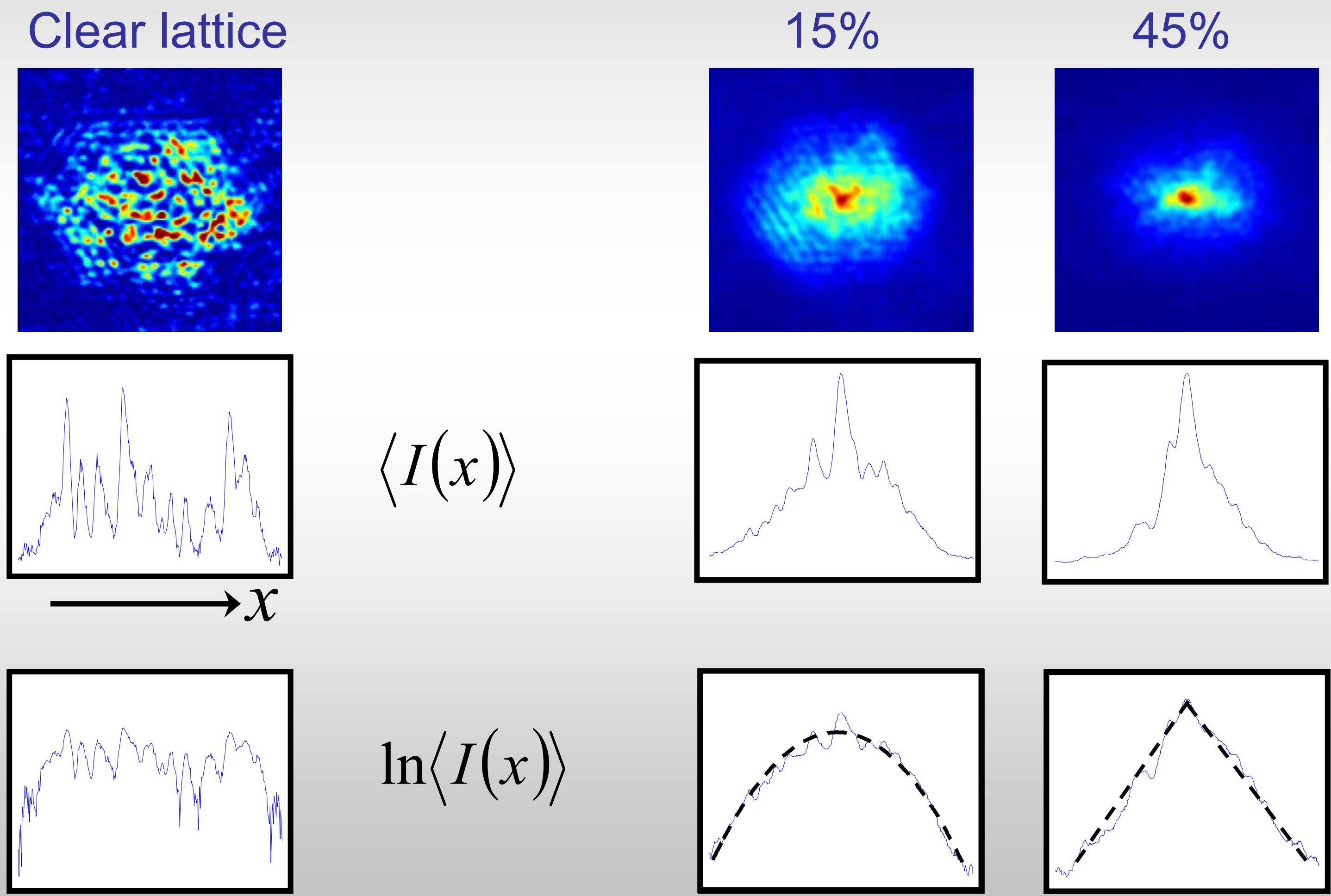


$$E(r,z,t)=\mathrm{Re}[\Psi(r,z)\exp(i(kz-\omega t))] \qquad r=(x,y)$$



Ballistics, diffusion, Anderson localization, in the linear case

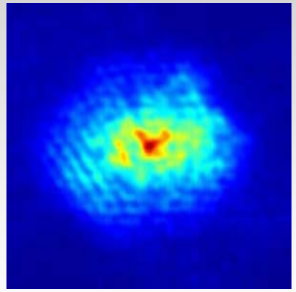
NB: "time" of observation is fixed and limited by the length of the photonic crystal.

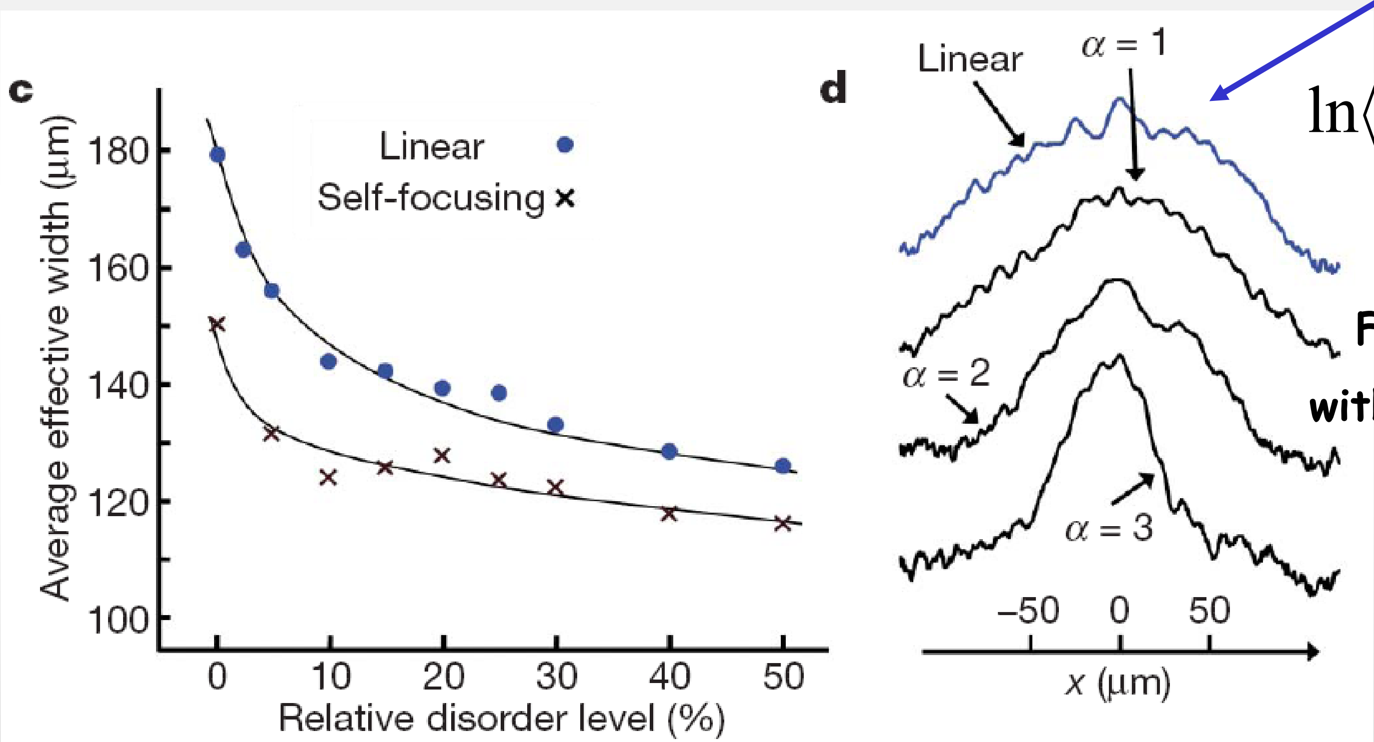


Influence of nonlinearity on localization

beam's peak. The mean free path and the localization length evaluated from these simulations are approximately $l^* \approx 5$ µm and $\xi \approx 29$ µm, respectively (see calculation details in Supplementary Information). 30% disorder.

15%





$$\ln\langle I(x)\rangle$$

15% disorder

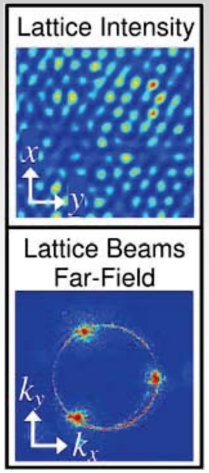
Focusing nonlinearity with increasing strength

(α=0,1,2,3)

Main conclusion:

Self-focusing nonlinearity **<u>promotes</u>** localization!

Gross-Pitaevskii vs Nonlinear Schrödinger



- **Gross-Pitaevskii Equation: Matter wave**

$$i\partial_t \Psi(\mathbf{r},t) = -\frac{1}{2m}\nabla^2\Psi(\mathbf{r},t) + u(\mathbf{r})\Psi(\mathbf{r},t) + \lambda|\Psi(\mathbf{r},t)|^2\Psi(\mathbf{r},t)$$

- **Nonlinear Schrödinger equation: Envelope of E-field**

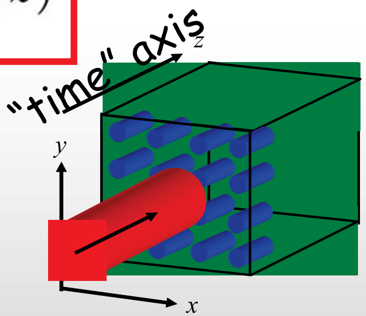
$$i\partial_z \Psi(\mathbf{r},z) = -\frac{1}{2k}\nabla^2\Psi(\mathbf{r},z) + u(\mathbf{r})\Psi(\mathbf{r},z) + \lambda|\Psi(\mathbf{r},z)|^2\Psi(\mathbf{r},z)$$

- **Disorder Potential**

$$\langle u(\mathbf{r})u(\mathbf{r}')\rangle = \frac{1}{m\tau}\delta(\mathbf{r}-\mathbf{r}')$$

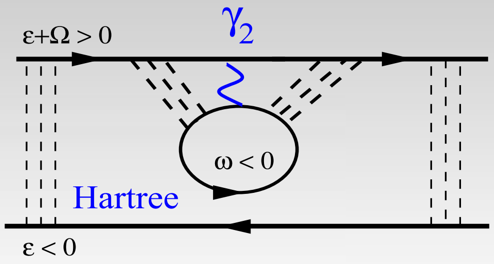
Wave vector

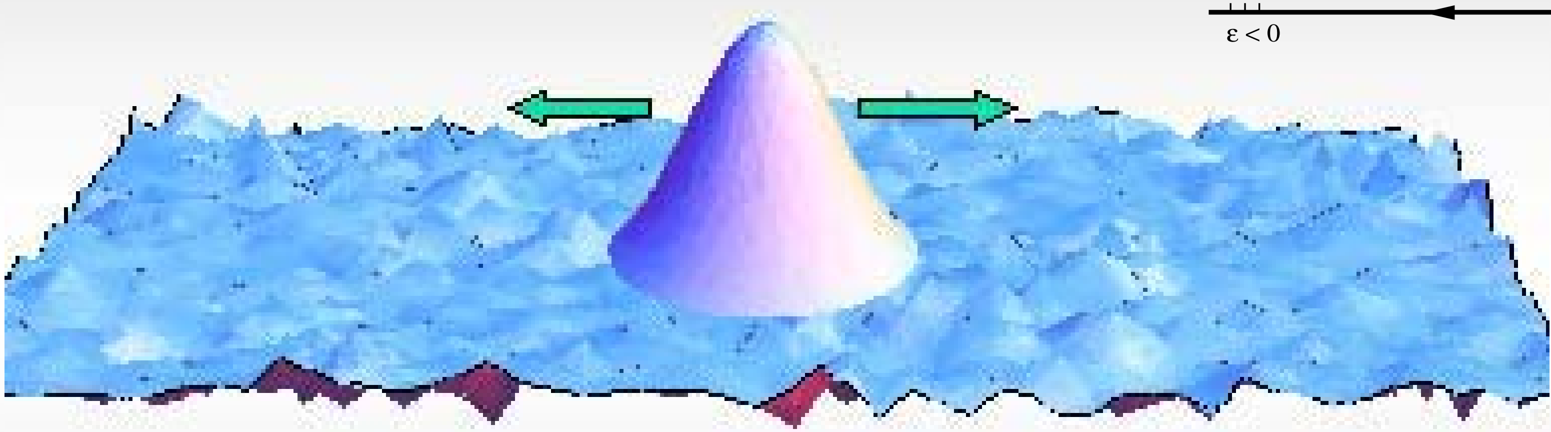
$$k = \frac{\omega}{c}$$



- **Sign of nonlinearity:**
  - **λ>0 de-focusing, repulsive**
  - **λ<0 focusing, attractive**

Theory: injection problem





Problem:

the time evolution of the injected pulse in 2d non-linear medium, averaged over impurity configurations.

in 2d there is an exponentially long diffusive regime preceding localization

Specifics of the problem:
diffusion with a broad distribution of energies

Free case: $\lambda=0$

Decoupled linear equations for different ε:

$$\partial_t n(\mathbf{r},t,\varepsilon) - D_\varepsilon \nabla^2 n(\mathbf{r},\varepsilon,t) = \delta(t) F(\varepsilon,\mathbf{r})$$

Formal solution:

$$n(\mathbf{r},\varepsilon,t) = \frac{\Theta(t)}{4\pi D_\varepsilon t} \int d^2 r_1 \, \mathrm{e}^{-(\mathbf{r}-\mathbf{r}_1)^2/(4D_\varepsilon t)} F(\varepsilon,\mathbf{r}_1)$$

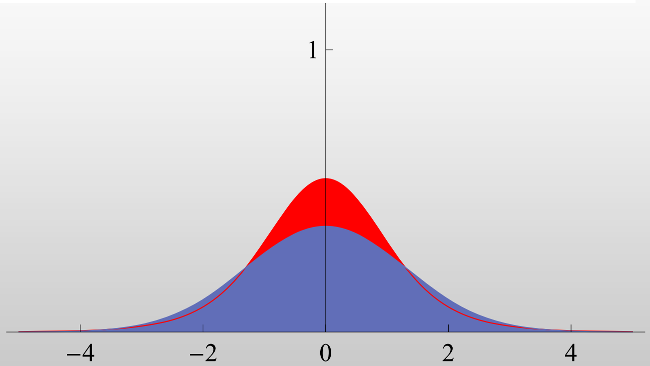
Example (B. Shapiro 2007)

$$F(\varepsilon,\mathbf{r}) = \delta(\mathbf{r})\Theta(\varepsilon)\, \mathrm{e}^{-\varepsilon/\varepsilon_0}\, 2\pi N/\varepsilon_0$$

a broad distribution of energies, i.e.,

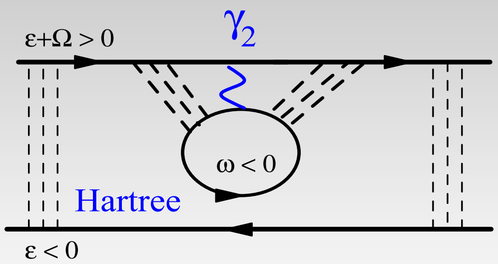
distribution of diffusion coefficients D_ε :

$$n(\mathbf{r},t) \propto \exp(-r/x_0) \qquad x_0^2 = D_{\varepsilon_0} t$$



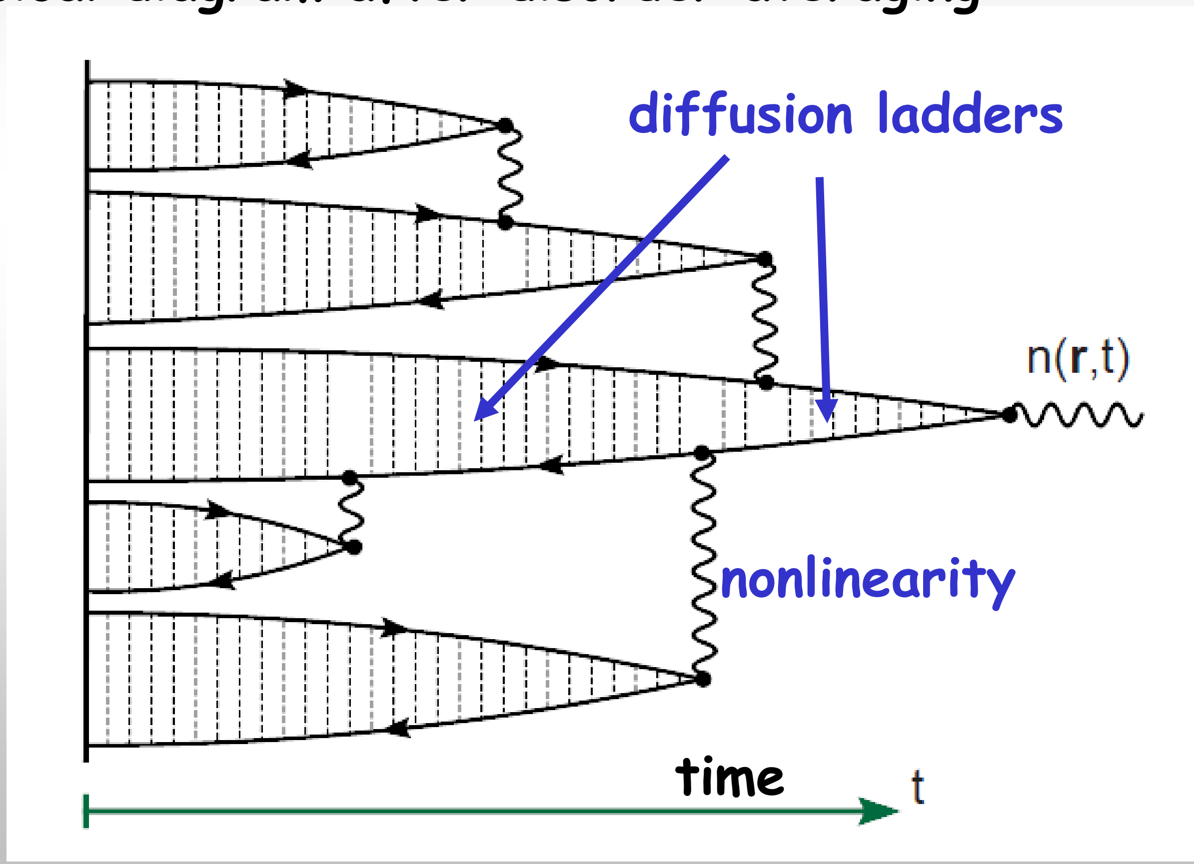
Diffusion with D(ε)

Specifics of the problem: no loops!



Typical diagram after disorder averaging

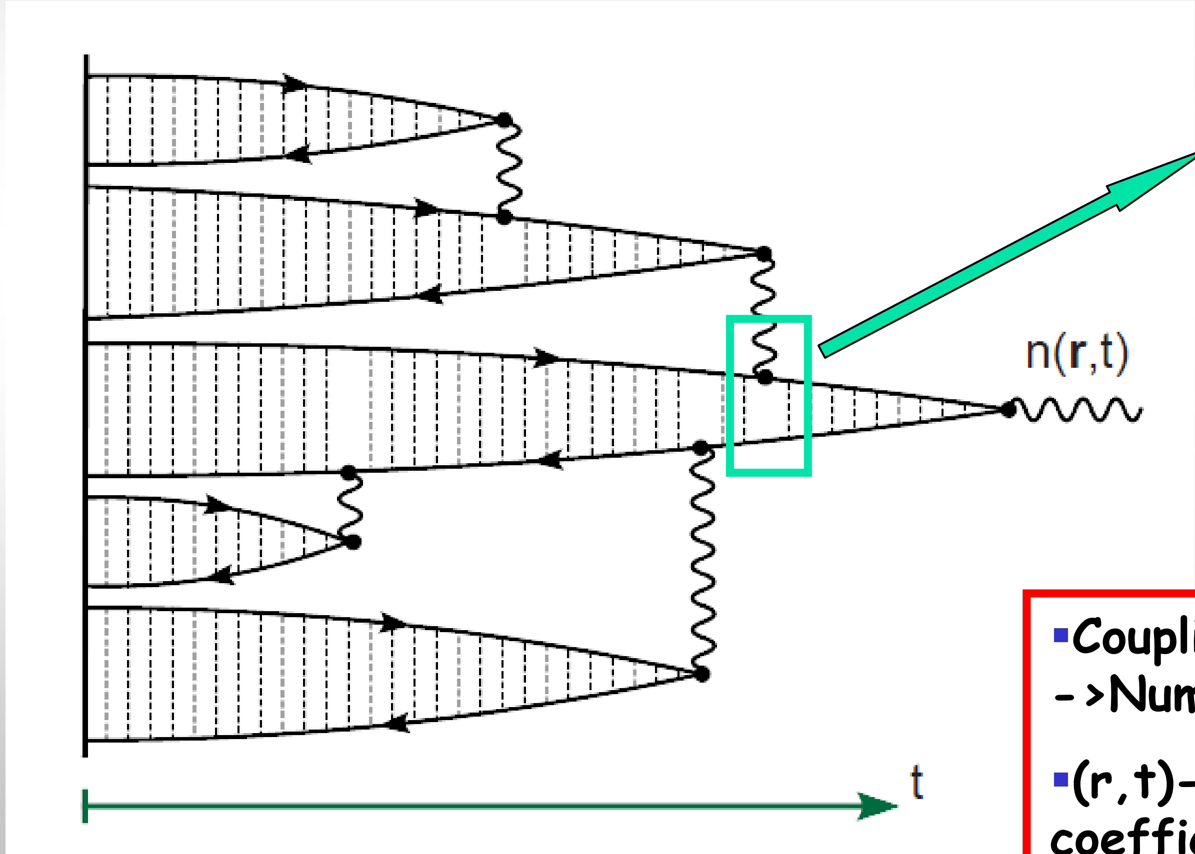
Injection

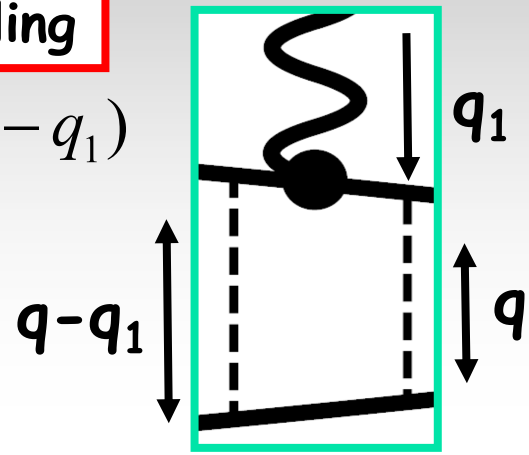


Typical diagram after disorder averaging

coupling

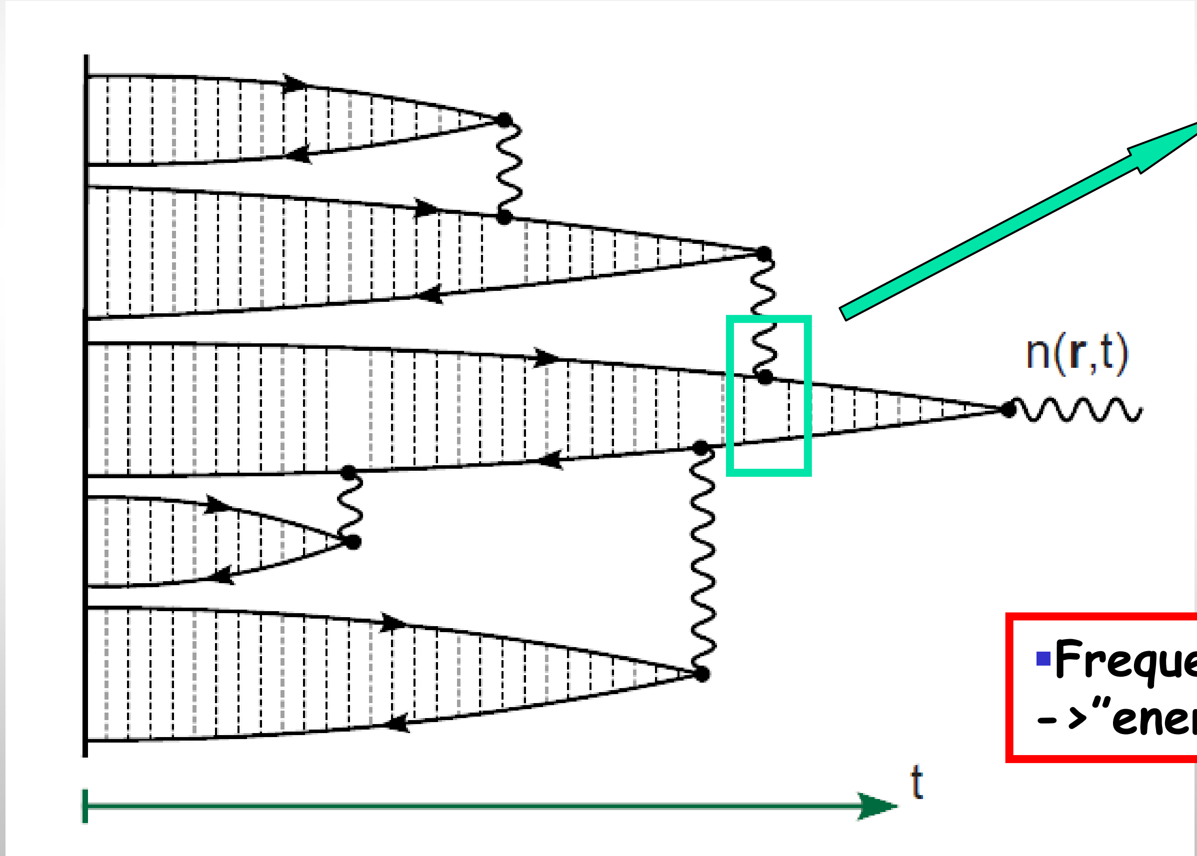
$\propto q(q-q_1)$

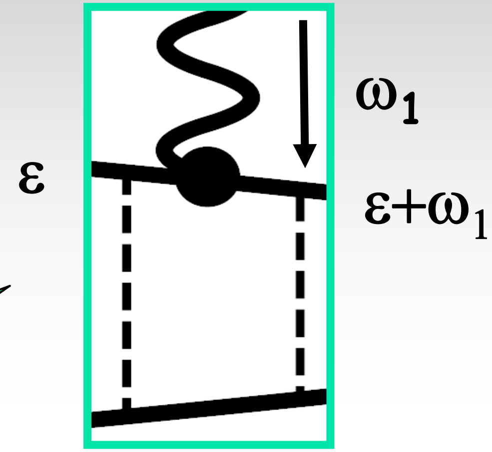




- Coupling vanishes for q->0 ->Number conservation
- (r,t)-dependent Diffusion-coefficient D(r,t)

Typical diagram after disorder averaging





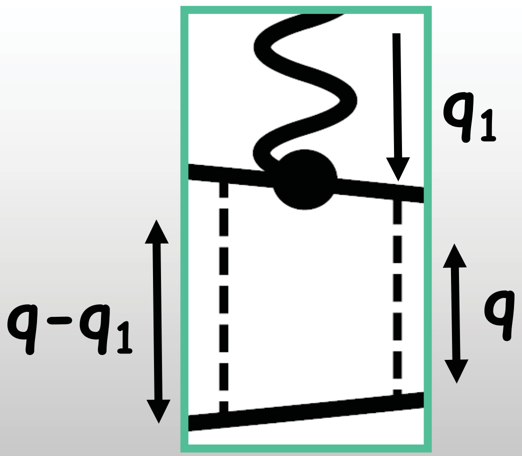
- Frequency transfer
->"energy gradually changes"

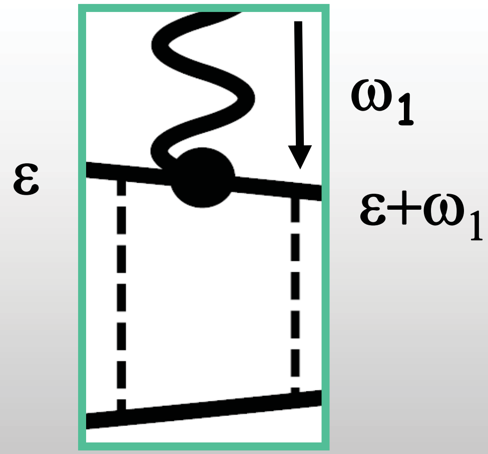
Kinetic equation

$$\vartheta(\mathbf{r},t) = 2\lambda n(\mathbf{r},t)$$

- **The kinetic equation:**

$$\partial_t \tilde{n}(\mathbf{r},t,\varepsilon) - \nabla(D_{\varepsilon-\vartheta}\nabla\tilde{n}(\mathbf{r},t,\varepsilon)) + \partial_t\vartheta(\mathbf{r},t)\partial_\varepsilon\tilde{n}(\mathbf{r},t,\varepsilon) = \delta(t)\, F(\varepsilon - \vartheta(\mathbf{r},0),\mathbf{r})$$





Kinetic equation

$$\vartheta(\mathbf{r},t) = 2\lambda n(\mathbf{r},t)$$

- **The kinetic equation:**

$$\partial_t \tilde{n}(\mathbf{r},t,\varepsilon) - \nabla(D_{\varepsilon-\vartheta}\nabla\tilde{n}(\mathbf{r},t,\varepsilon)) + \partial_t\vartheta(\mathbf{r},t)\partial_\varepsilon\tilde{n}(\mathbf{r},t,\varepsilon) = \delta(t)\; F(\varepsilon-\vartheta(\mathbf{r},0),\mathbf{r})$$

We demand $\omega\tau \ll 1$ and $lq \ll 1$ for typical frequencies ω and momenta q characterizing the average density distribution $n(\mathbf{q},\omega)$. The nonlinearity can be strong, but gradients should not be too large, $lq\vartheta(\mathbf{q},\omega)/\varepsilon_0 \ll 1$ and $\omega\tau\vartheta(\mathbf{q},\omega)/\varepsilon_0 \ll 1$, for typical kinetic energies ε_0.

that the initial wave-function sets a momentum scale p_0 characterizing the main part of the momentum distribution, so that the weak disorder condition $p_0 l \gg 1$ is fulfilled, where $l = p_0\tau/m$ is the mean free path. We further assume that the density varies smoothly on scales of l, in particular that the size of the condensate is much larger than the mean free path. Both of these conditions can be met simultaneously. The phase of Ψ, which is related to the momentum, may change rapidly, while the amplitude, which determines the density, may vary smoothly. Even if the density does not satisfy the smoothness condition initially, it is natural to expect that in the case of an expansion it will become sufficiently smooth after

The size of the cloud is much larger

than the mean free path

Formalism

- Step 1: Statistical field theory approach in real time
  - Introduce pair of fields similar to Martin-Siggia-Rose, Keldysh formalisms
  - Fix initial conditions
  - Average over disorder configurations
  - Express density via components of 2x2 matrix Green's functions
- Step 2: Method of quasi-classical Green's functions
  - Introduce momentum integrated Green's function
  - Derive analog of Usadel equation (theory of superconductivity)
  - Find kinetic equation for the density in the diffusive limit
- Assumptions:
  - The initial energy distribution has characteristic energy ε_0, so that $\varepsilon_0\tau >> 1$.
  - The initial density distribution is smooth on scales of the typical mean free path.

Kinetic equation
(set of two self-consitent equations)

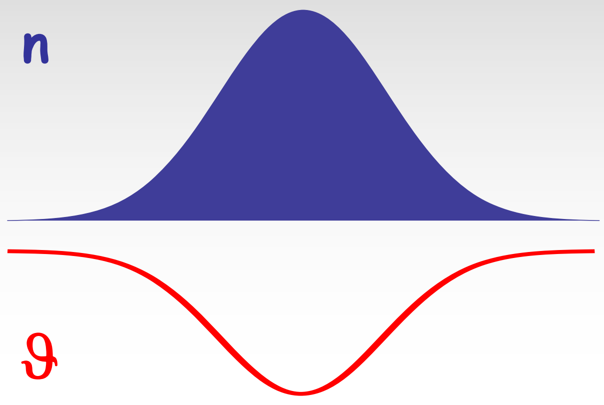
- **The kinetic equation:**

$$\partial_t \tilde{n}(\mathbf{r},t,\varepsilon) - \nabla(D_{\varepsilon-\vartheta}\nabla\tilde{n}(\mathbf{r},t,\varepsilon)) + \partial_t\vartheta(\mathbf{r},t)\partial_\varepsilon\tilde{n}(\mathbf{r},t,\varepsilon) = \delta(t)\ F(\varepsilon-\vartheta(\mathbf{r},0),\mathbf{r})$$

  - **Nonlinear integro-differential equation.**
  - **Density:** $$n(\mathbf{r},t) = \int d\varepsilon/(2\pi)\ \tilde{n}(\mathbf{r},t,\varepsilon)$$
  - **Self-consistent potential:** $$\vartheta(\mathbf{r},t) = 2\lambda n(\mathbf{r},t)$$
  - **Initial distribution:** $$F(\varepsilon,\mathbf{r}) = \int (d\mathbf{p})(d\mathbf{q})\ F(\mathbf{p},\mathbf{q})\ \exp(i\mathbf{q}\mathbf{r})\ 2\pi\delta(\varepsilon-\varepsilon_{\mathbf{p}})$$

$$F(\mathbf{p},\mathbf{q}) = \Psi_0(\mathbf{p}+\mathbf{q}/2)\Psi_0^*(\mathbf{p}-\mathbf{q}/2) \qquad \varepsilon_{\mathbf{p}} = p^2/(2m)$$

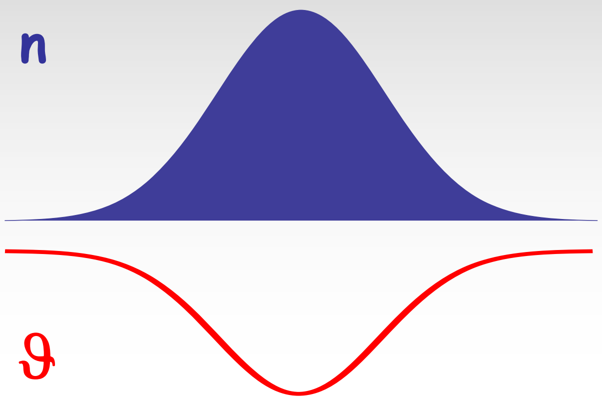
Diffusion in the presence of a non-linearity



The density distribution creates a self-consistent potential ϑ.

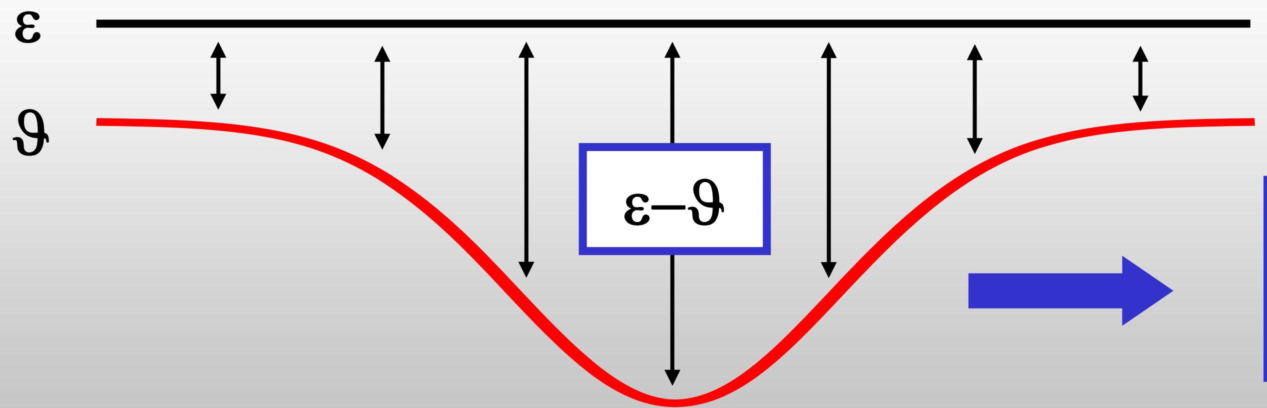
$$\vartheta(\mathbf{r},t) = 2\lambda n(\mathbf{r},t)$$

Diffusion in the presence of a non-linearity



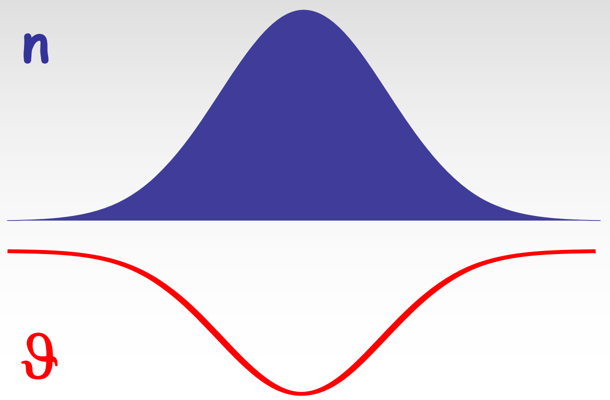
Imagine a particle with total energy ε and kinetic energy ε–ϑ diffusing in 2d under the influence of this potential. Its local diffusion coefficient is $D_{\varepsilon-\vartheta}$.

kinetic energy

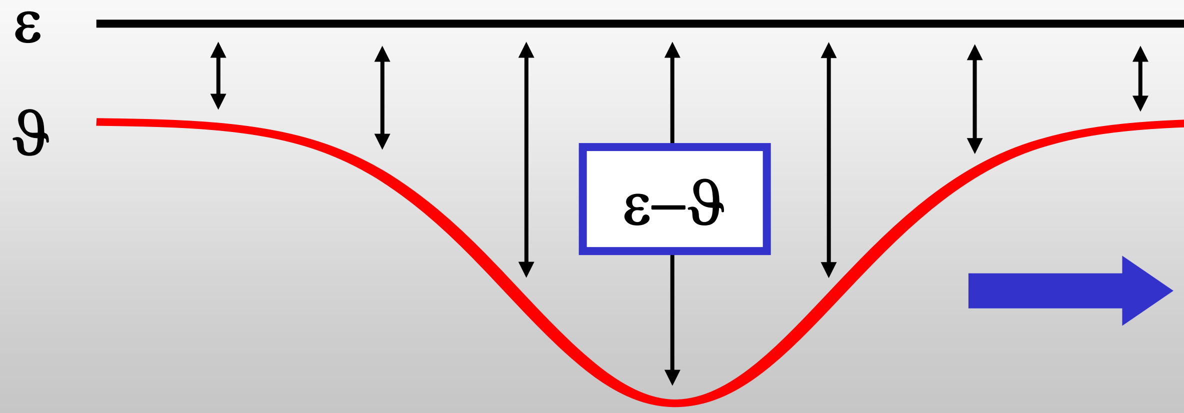


$$D_{\varepsilon-\vartheta}$$

Diffusion in the presence of a non-linearity



- The potential ϑ depends on coordinates AND time.
- A purely time dependent potential ϑ(t) (without coordinate dependence) would not lead to a physical effect.



$$D_{\varepsilon-\vartheta}$$

Kinetic equation: a more physical form

- **Let us shift ε –> ε+ϑ. Now, ε represents the kinetic energy of an interacting particle, and the kinetic equation takes the form**

$$\partial_t n(\mathbf{r},t,\varepsilon) - \Big[\nabla_{\mathbf{r}} - \nabla_{\mathbf{r}}\vartheta(\mathbf{r},t)\partial_\varepsilon\Big] D_\varepsilon \Big[\nabla_{\mathbf{r}} - \nabla_{\mathbf{r}}\vartheta(\mathbf{r},t)\partial_\varepsilon\Big] n(\mathbf{r},\varepsilon,t) = \delta(t)\; F(\varepsilon,\mathbf{r})$$

- **Upon integration in ε:**

$$\partial_t n(\mathbf{r},t) - \frac{\tau}{m}\nabla^2\left(\overline{\varepsilon}(\mathbf{r},t) + \lambda n^2(\mathbf{r},t)\right) = \delta(t)\; n(\mathbf{r},0)$$

$$\begin{aligned} \overline{\varepsilon}(\mathbf{r},t) &= \int (d\varepsilon)\; \varepsilon n(\mathbf{r},t,\varepsilon) \\ &\equiv \varepsilon_{kin}(\mathbf{r},t)\; n(\mathbf{r},t) \end{aligned}$$

- **Nonlinear diffusion**

$$\partial_t n - \nabla^2(D_{eff}n) = \delta(t)n$$

$$D_{eff}(\mathbf{r},t) = (\varepsilon_{kin} + \lambda n)\tau/m$$

Warning: simplicity of this equation could be misleading

$$\partial_t n(\mathbf{r},t) - \frac{\tau}{m}\nabla^2\left(\bar{\varepsilon}(\mathbf{r},t) + \lambda n^2(\mathbf{r},t)\right) = \delta(t)\, n(\mathbf{r},0)$$

$$\bar{\varepsilon}(\mathbf{r},t) = \int (d\varepsilon)\, \varepsilon n(\mathbf{r},t,\varepsilon) \equiv \varepsilon_{kin}(\mathbf{r},t)\, n(\mathbf{r},t)$$

- Warning:
  - This is not a closed equation for n(r,t). $\bar{\varepsilon}$ depends on the energy distribution and needs to be determined independently.
    Equation for $\bar{\varepsilon}$ depends on $\overline{\varepsilon^2}$, etc.
  - Moreover, the information about initial energy distribution enters through $\bar{\varepsilon}$.

- Our strategy:
  - Understand limiting cases
  - Make use of conservation laws to find properties that do not depend on the details of the initial energy distribution.
  - Seek for qualitative picture.

$\bar{\varepsilon}$-> 0, λ>0

$$\partial_t n(\mathbf{r},t) - \frac{T}{m}\nabla^2\left(\bar{\varepsilon}(\mathbf{r},t) + \lambda n^2(\mathbf{r},t)\right) = \delta(t)\, n(\mathbf{r},0)$$

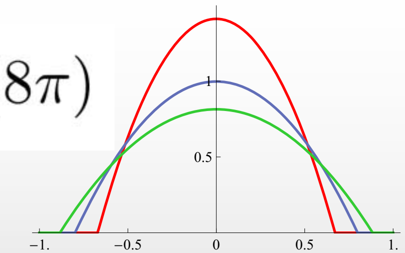
- **Porous medium equation (Zeldovich, Kompaneets, 1950)**

$$\partial_t n = \nabla^2 n^2$$

- **"Barenblatt's solution" starting from** $M\delta(r)$ **is**

$$n(\mathbf{r},t) = (C - r^2/(16t^{1/2}))_+/t^{1/2}$$

$$C^2 = M/(8\pi)$$



- **Explosion:**

Mean square radius

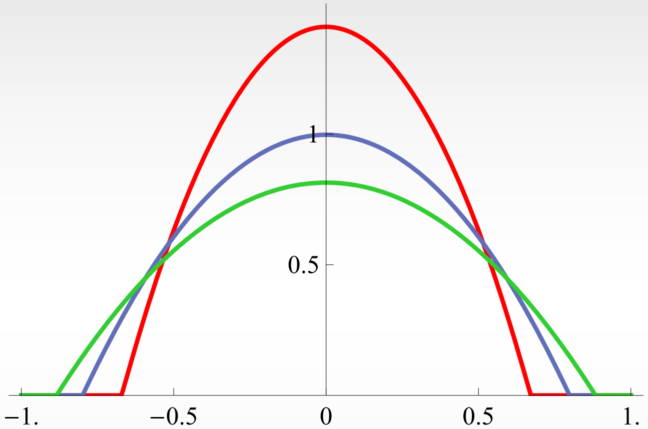
$$\langle r^2 \rangle \propto t^{1/2}$$

Decay at r=0

$$n(0,t) \propto t^{-1/2}$$

Barenblatt's solution : Explosion

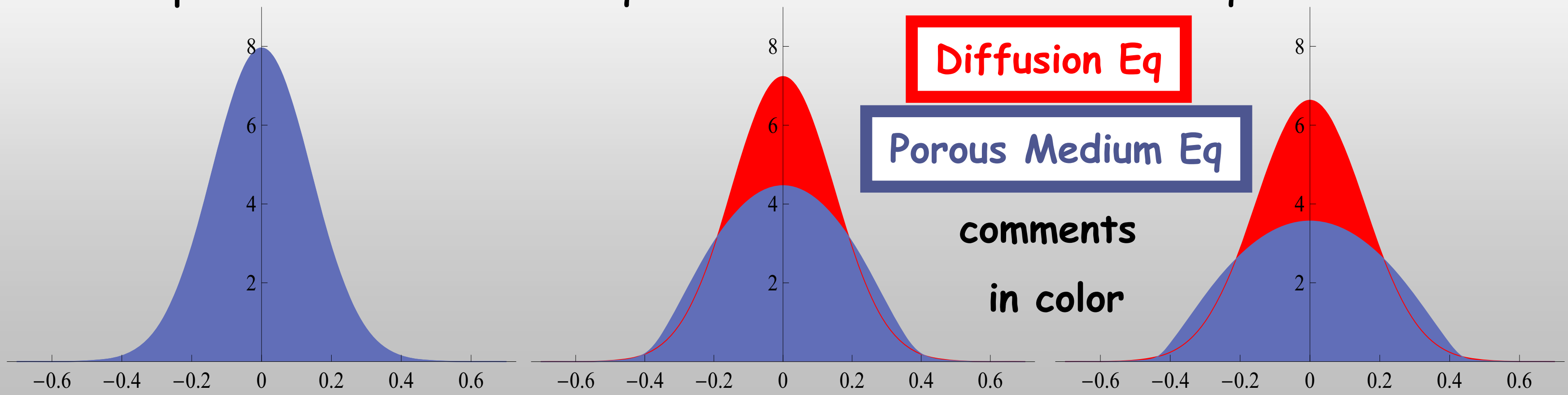
- Barenblatt's solution describes explosive behavior with a wave-front:



$$\langle r^2 \rangle \propto t^{1/2}$$

$$n(0,t) \propto t^{-1/2}$$

- Comparison: Diffusion equation vs Porous medium equation



“Explosion” for large repulsive nonlinearities?

Not exactly!

Conservation laws and the rate of diffusion ("Talanov's theorem")

- Number conservation: $$\int d^2r\, n(\mathbf{r},t) = N$$
- Energy conservation: $$E_{tot} = \int d\mathbf{r}\ \left(\overline{\varepsilon}(\mathbf{r},t) + \lambda n^2(\mathbf{r},t)\right)$$
  - Note: The integrated "quasiparticle" energy

    $$E^{qp}_{tot} = \int d\mathbf{r}\left(\overline{\varepsilon} + 2\lambda n^2(\mathbf{r},t)\right)$$ is not conserved.
- Important implication:

$$\partial_t n(\mathbf{r},t) - \frac{\tau}{m}\nabla^2\left(\overline{\varepsilon}(\mathbf{r},t) + \lambda n^2(\mathbf{r},t)\right) = \delta(t)\, n(\mathbf{r},0)$$

$$\partial_t \left\langle r^2 \right\rangle = 4D_{\varepsilon_{tot}}$$

$$\langle r^2 \rangle \equiv \int d\mathbf{r}\, r^2\, n(\mathbf{r},t)/N \qquad \varepsilon_{tot} = E_{tot}/N$$

<r^2> grows linearly in t (as in ordinary diffusion).

D is determined by the total energy per particle.

Repulsion: center and boundary

- Center of the distribution: $$\partial_t n = \frac{\tau}{m}\left[\nabla^2 \overline{\varepsilon} + 2\lambda n \nabla^2 n\right]$$
  - For large initial λn rapid decay in accordance with the Barenblatt's solution.

- Near the "boundary" of the distribution:

$$\begin{aligned} \overline{\varepsilon}(\mathbf{r},t) &= \int (d\varepsilon)\, \varepsilon n(\mathbf{r},t,\varepsilon) \\ &\equiv \varepsilon_{kin}(\mathbf{r},t)\, n(\mathbf{r},t) \end{aligned}$$

$$\partial_t n \approx \frac{\tau}{m}\left[\nabla^2 \overline{\varepsilon} + 2\lambda (\nabla n)^2\right]$$

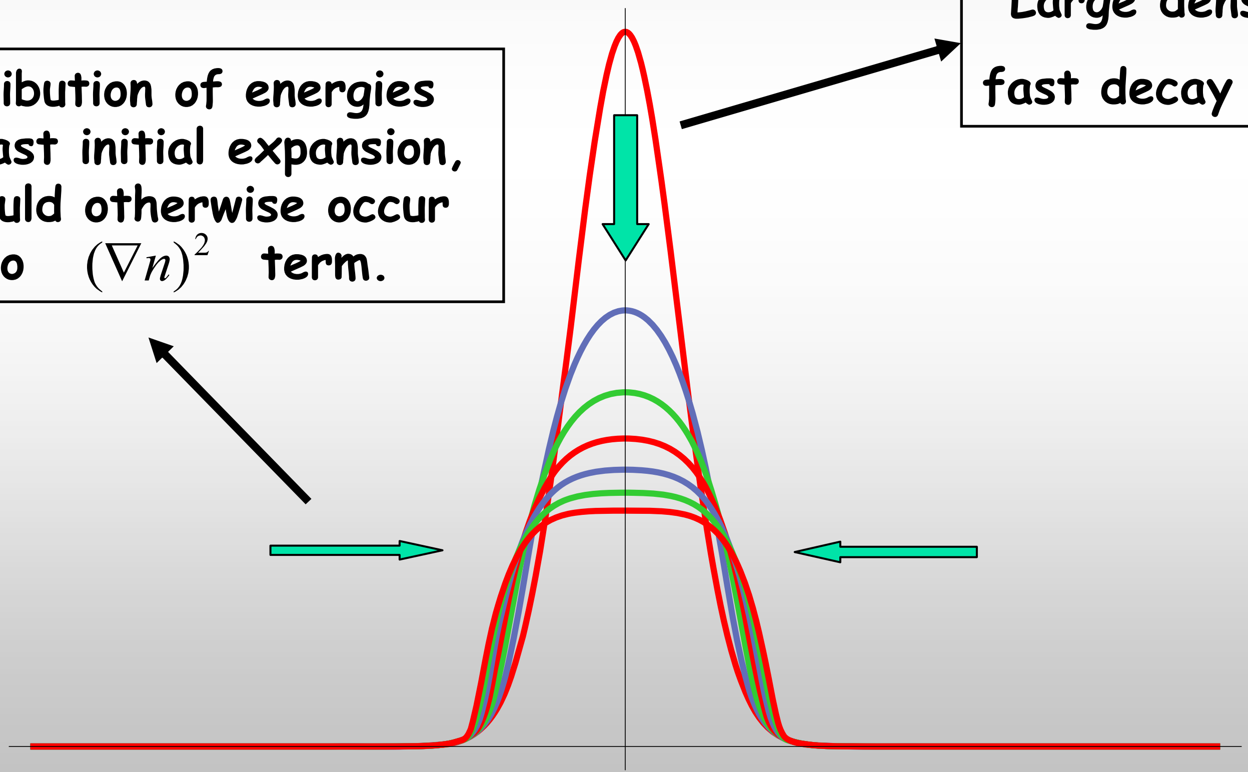
  - The second term is responsible for the propagation of the wave-front in Barenblatt's solution.
  - In our case: Internal redistribution of energy brings "fast particles" to the boundary and $\nabla^2 \overline{\varepsilon} < 0$ blocks the "explosive" expansion.

Repulsion: "Locked explosion"

Qualitative picture, repulsive nonlinearity:

Redistribution of energies blocks fast initial expansion, that would otherwise occur due to $(\nabla n)^2$ term.

Large density leads to fast decay in the center.

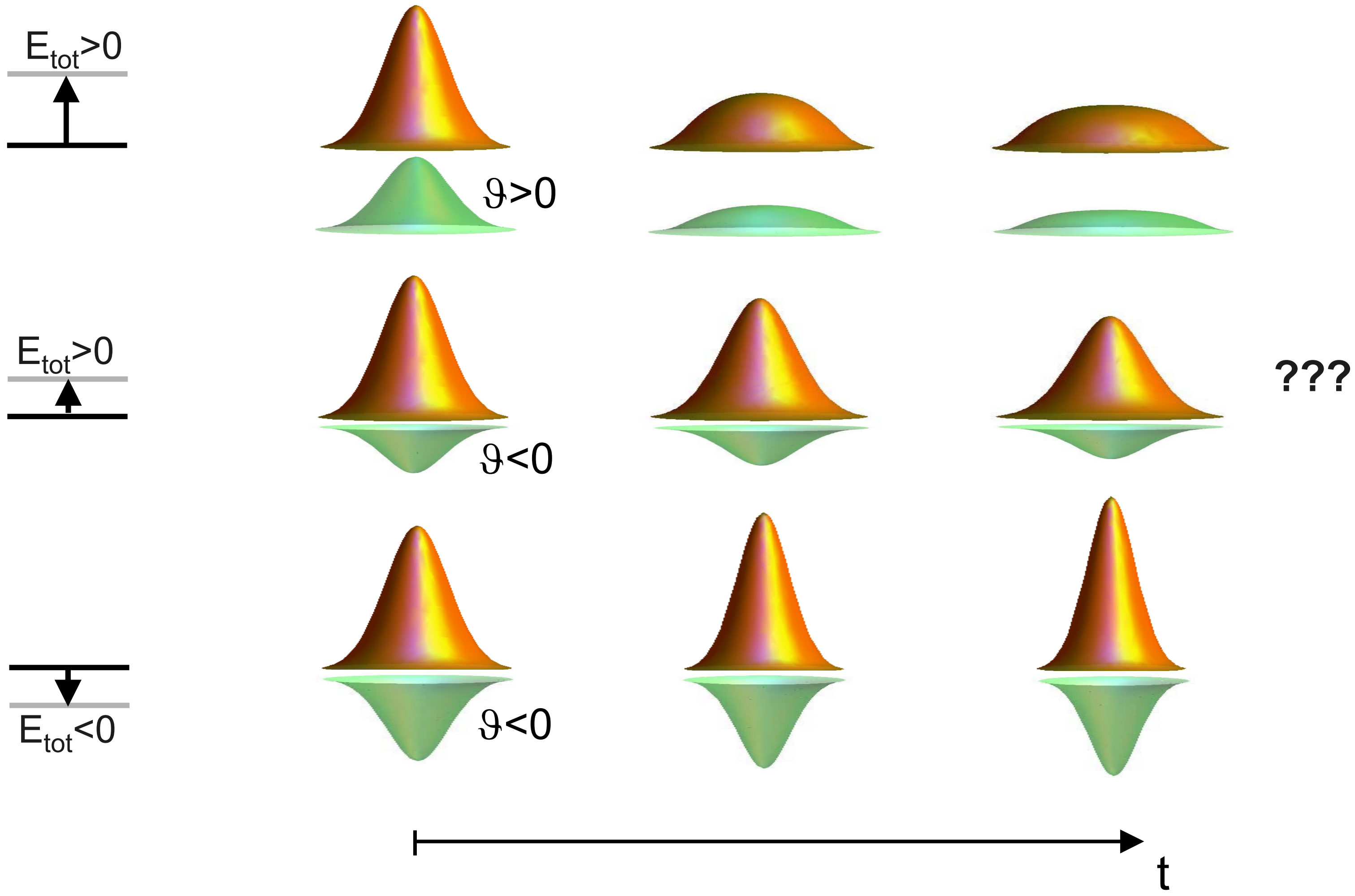


Attraction: "Diffusive collapse" and fragmentation

- In the case of attraction one may have $E_{tot}<0$ or $E_{tot}>0$.
- For $E_{tot}<0$ the mean square radius $\langle r^2 \rangle$ would become negative after a finite time, leading to a collapse (diffusive collapse).
- Even when $E_{tot}>0$ the collapse can play a role for attractive nonlinearity, if part of the cloud has a negative energy, while the remaining part expands. As a result one may expect a fragmentation of the cloud.

(Nonlinear central limit theorem for the porous medium equation.)

Summary: illustration



Conclusion

- A set of equations has been derived that describes the evolution of an injected wave-packet in the presence of random scatterers and nonlinearity.

- A repulsive nonlinearity increases the effective diffusion coefficient, an attractive nonlinearity makes it smaller.
  - This is in line with the experimental observation that an attractive nonlinearity supports localization. Still, whether true localization has been observed, or just a slowing down of a diffusing cloud (or its fragmentation), should be reanalyzed.

- For a repulsive nonlinearity, the explosive stage is "locked" by redistribution of energies.

- For an attractive nonlinearity diffusive collapse is expected for $E_{tot}<0$ and fragmentation for $E_{tot}>0$.

Remarks on localization effects

- A repulsive nonlinearity increases the diffusion coefficient, an attractive nonlinearity makes it smaller.
  - This is in line with the experimental observation that a repulsive nonlinearity increases the localization length, while an attractive nonlinearity supports localization.

- Weak localization due to interference:
  - Our analysis did not account for interference effects. Such effects lead to logarithmic corrections to the diffusion coefficient in two dimensions, which favor localization. For an attractive nonlinearity three effects seem important.
    - During the expansion the time variation of the potential leads to dephasing.
    - As the cloud expands, the potential energy increases (becomes less negative), and the kinetic energy decreases.
    - Additionally longer paths become available for interference.
  - The first effect weakens localization, the last two support it.