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Advanced Workshop on Anderson Localization, Nonlinearity and Turbulence: a Cross-Fertilization

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INTRODUCTORY
(Nonlinear Schrodinger Equation)

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Nonlinear Schrodinger Equation

Elementary Introduction

dynamics, stability, chaos, turbulence

Gregory Falkovich

http://www.weizmann.ac.il/home/fnfal/

Trieste, August 2010

$$\mathcal{H} = \int \left[|\nabla \Psi|^2 + U(\mathbf{r})|\Psi|^2 + T|\Psi|^4 \right] \, d\mathbf{r}$$

$$i\frac{\partial \Psi}{\partial t} = \frac{\partial \mathcal{H}}{\partial \Psi^*}$$

$$\exp(-\beta \mathcal{H})$$

Nonlinear Schrödinger Equation

$$i\frac{\partial\psi}{\partial t} + \frac{\omega''}{2}\Delta\psi - T|\psi|^2\psi = 0\ .$$

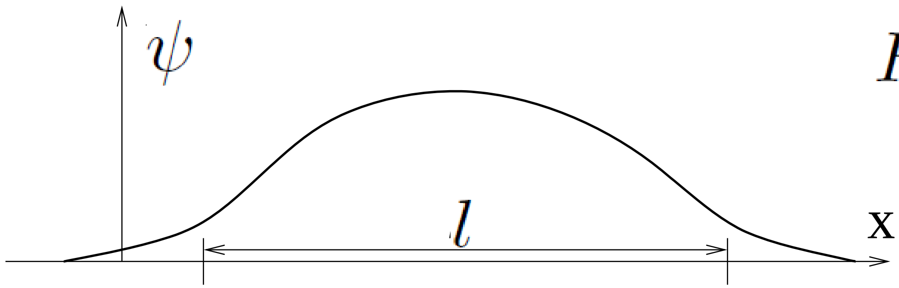
condensate $\qquad \psi_0(t) = A_0 \exp(-iTA_0^2 t)$

$$\Omega^2 = T\omega'' A_0^2 k^2 + \omega''^2 k^4/4$$

instability when $T\omega'' < 0$

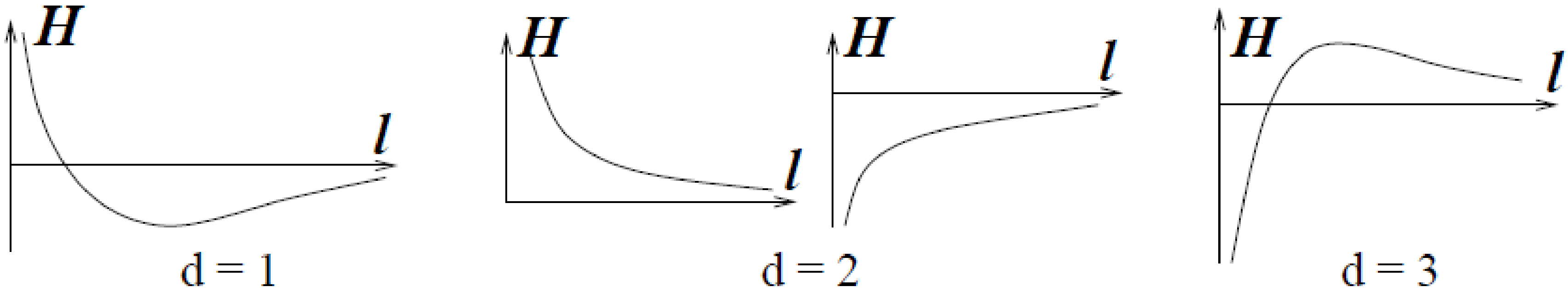
$$H = \frac{1}{2}\int\left(\omega''|\nabla\psi|^2 + T|\psi|^4\right)d\mathbf{r}$$

$$N = \int |\psi|^2$$



$$H \simeq \omega'' N l^{-2} + T N^2 l^{-d}$$

$$N \simeq |\psi|^2 l^d$$



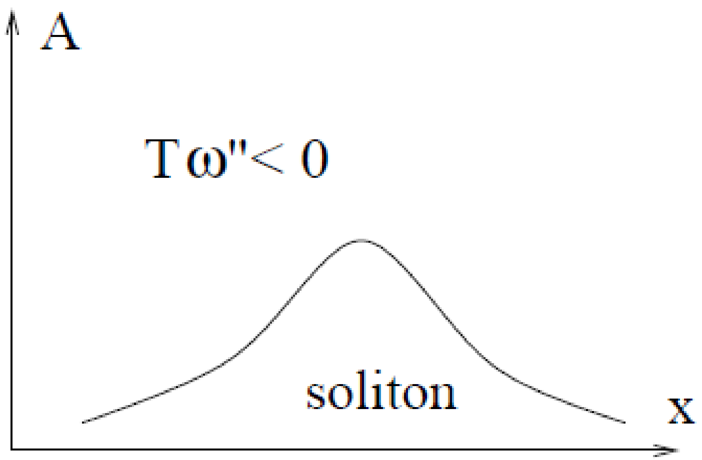
$$l^2(t) = \int |\psi|^2 r^2 \, d\mathbf{r}$$

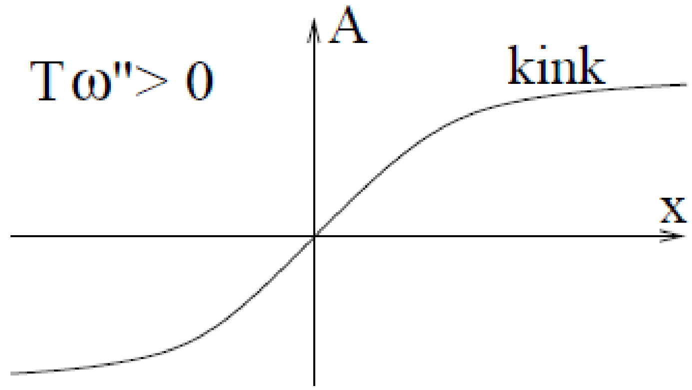
$$\begin{aligned} \frac{d^2 l^2}{dt^2} &= \frac{i\omega''}{2} \partial_t \int r^2 \nabla (\psi^* \nabla \psi - \psi \nabla \psi^*) \, d\mathbf{r} \\ &= 4H + 2(d-2)\omega'' T \int |\psi|^4 \, d\mathbf{r} \end{aligned}$$

One-dimensional case - integrable

$$i\psi_t + \psi_{xx} - T|\psi|^2\psi = 0$$

$$i\partial_t\psi_1 + \psi\psi_2 = E\psi_1$$
$$-i\partial_t\psi_2 - \psi^*\psi_1 = E\psi_2$$





Integrability can be broken by
1) discretization in either real space or fourier space,
2) additional terms e.g. pumping and dissipation.

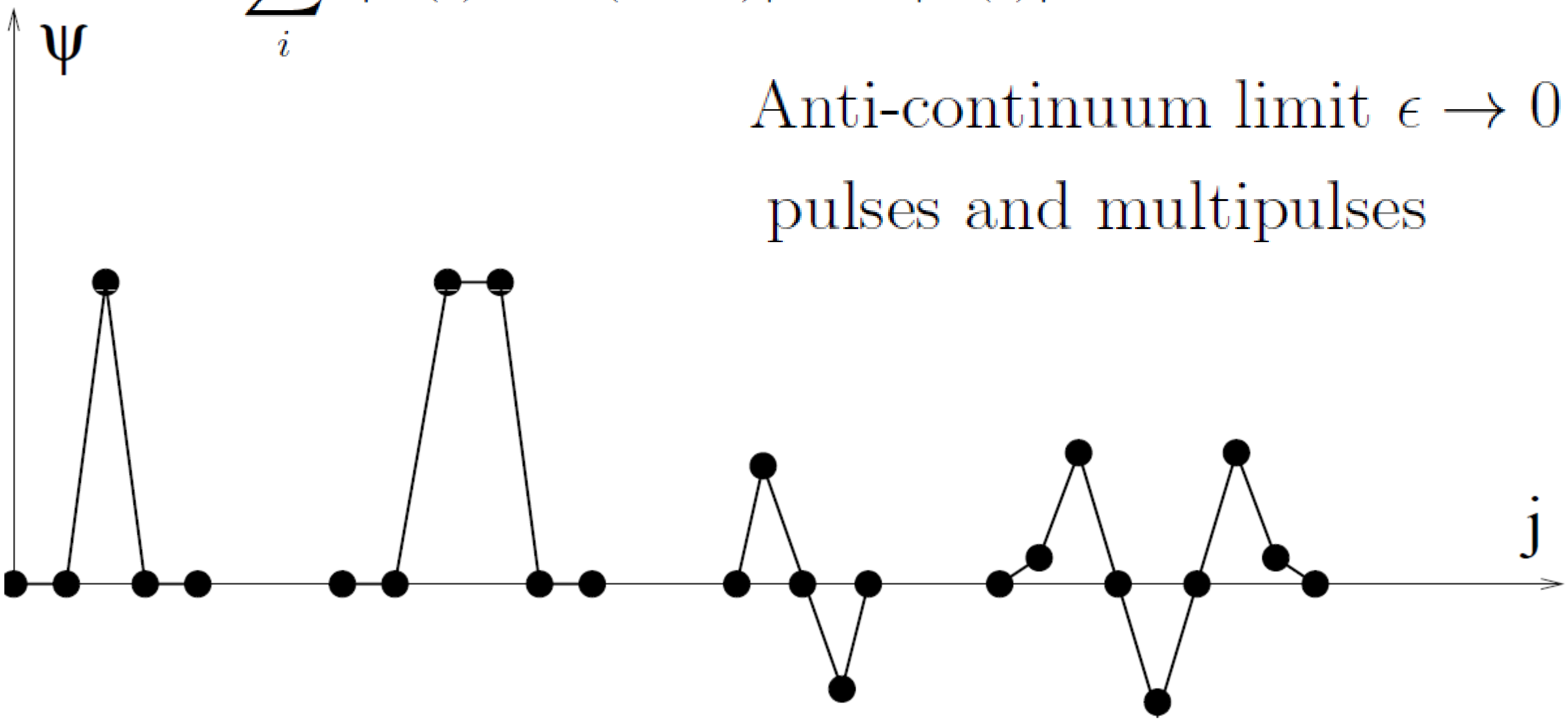
Discretization in real space - solid state physics and waveguide optics

$$i\frac{\partial \Psi(i,t)}{\partial t} = \frac{\partial \mathcal{H}}{\partial \Psi^*(i,t)},$$

$$\mathcal{H} = \sum_i \epsilon|\Psi(i) - \Psi(i+1)|^2 + T|\Psi(i)|^4$$

Anti-continuum limit $\epsilon \to 0$

pulses and multipulses



Superconductor – isolator phase transition

Spectral discretization – finite system length

$$\frac{dE_m}{dt} = -i\beta m^2 E_m - i\gamma \sum_{ik} E_i E_k E^*_{i+k-m} = -i\frac{\partial H}{\partial E^*_m}$$

$$H = \sum_m \beta m^2 |E_m|^2 + (\gamma/2) \sum_{ikm} E_i E_k E^*_{i+k-m} E^*_m$$

$$P = \sum |E_m|^2 \qquad\qquad M = \sum m|E_m|^2$$

$$i\frac{dE_0}{dt} = \gamma\left(|E_0|^2 + 2|E_1|^2 + 2|E_{-1}|^2\right) E_0 + 2\gamma E_1 E_{-1} E_0^* \ ,$$

$$i\frac{dE_1}{dt} = \beta E_1 + \gamma\left(|E_1|^2 + 2|E_0|^2 + 2|E_{-1}|^2\right) E_1 + \gamma E^*_{-1} E_0^2 \ ,$$

$$i\frac{dE_{-1}}{dt} = \beta E_{-1} + \gamma\left(|E_{-1}|^2 + 2|E_0|^2 + 2|E_1|^2\right) E_0 + \gamma E_1^* E_0^2$$

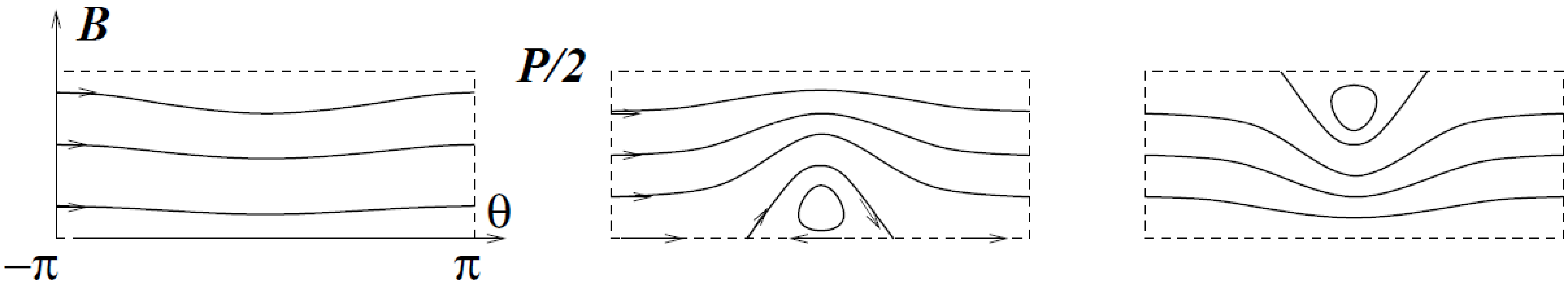
3-mode

$$E_1 = a_1 \exp(i\theta_1) \text{ and } E_{-1} = a_1 \exp(i\theta_{-1}$$

$$B = a_1^2 \qquad\qquad \theta = 2\theta_0 - \theta_1 - \theta_{-1}$$

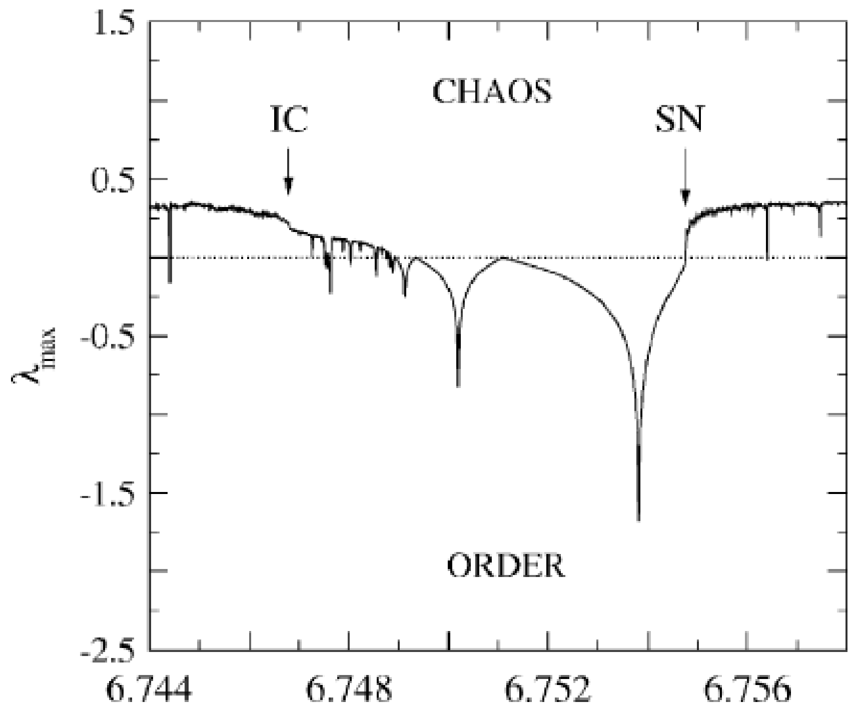
$$\frac{dB}{dt} = 2\gamma B(P-2B)\sin\theta = -\frac{\partial H}{\partial \theta} \ ,$$

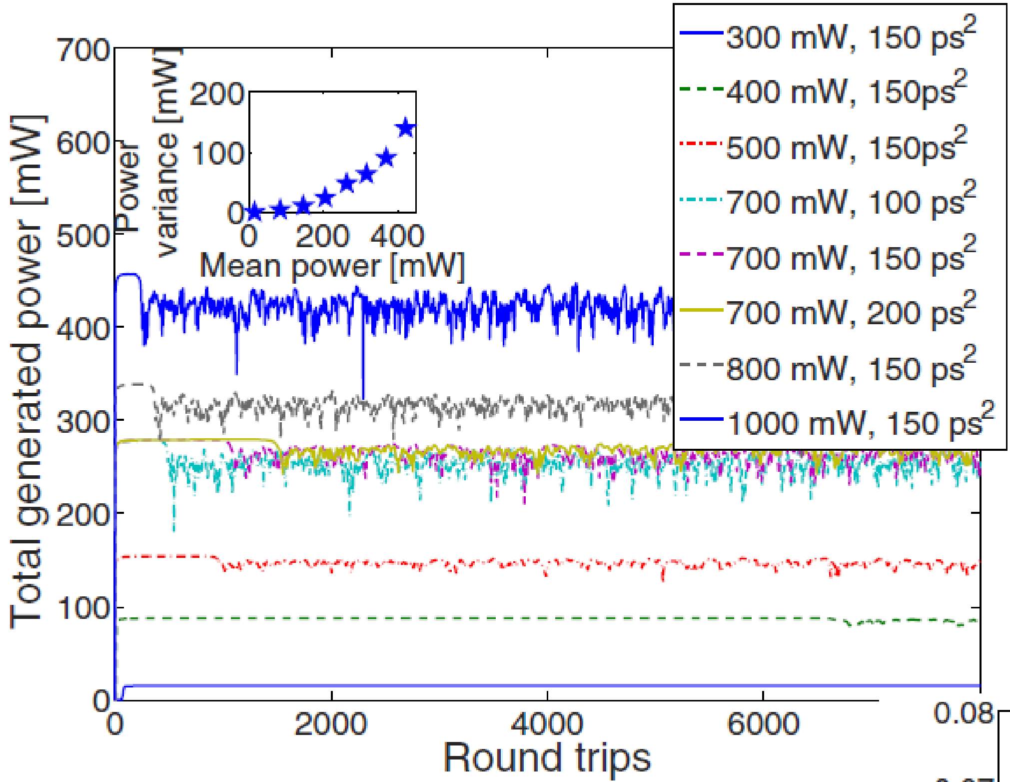
$$\frac{d\theta}{dt} = 2\beta + 2\gamma(P-3B) + 2\gamma(P-4B)\cos\theta = \frac{\partial H}{\partial B}$$



$$-2\gamma P < \beta < 0$$

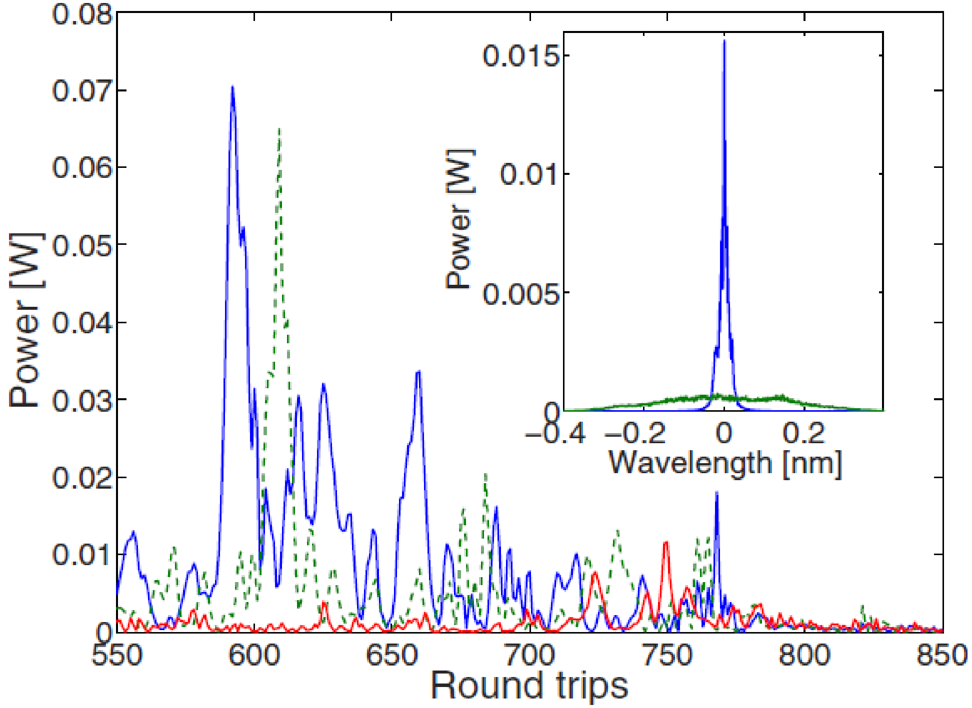
$$
\begin{aligned}
i\frac{dE_0}{dt} &= \gamma\big(|E_0|^2 + 2|E_1|^2 + 2|E_{-1}|^2\big)E_0 + 2\gamma E_1 E_{-1} E_0^* + g(P)E_0 \ , \\
i\frac{dE_1}{dt} &= \beta E_1 + \gamma\big(|E_1|^2 + 2|E_0|^2 + 2|E_{-1}|^2\big)E_1 + \gamma E_{-1}^* E_0^2 - E_1\delta \ , \\
i\frac{dE_{-1}}{dt} &= \beta E_{-1} + \gamma\big(|E_{-1}|^2 + 2|E_0|^2 + 2|E_1|^2\big)E_0 + \gamma E_1^* E_0^2 - E_{-1}\delta
\end{aligned}
$$





Fiber laser
with normal dispersion

Condensate destruction
by chaotic dynamics



2d dynamics

Unstable case – collapse or spreading depending on the sign of H.

Stable case – vortices and grey solitons exist

Condensate phase change by 2mπ around a vortex.
Vortices of different “charges”.

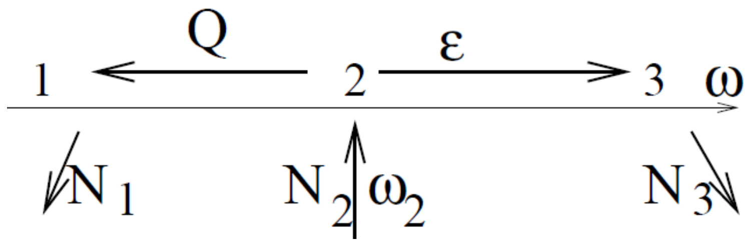
Thermal equilibrium

1d
Integrable systems do not thermalize.
Allowing interaction of solitons leads to condensation – one soliton survives which is in equilibrium with waves.
Negative temperature states.

2d
Complex field with short-range interaction –
BKT phase transition from low-temperature state of paired vortices with power-law decay of correlations to high-temperature Debye plasma with exponential decay of correlations.

Weak turbulence $\langle \psi_k(t)\psi_{k'}(t)\rangle = n_k(t)\delta(k+k')$

$$N = \int n_k \, d\boldsymbol{k} \qquad E \approx \int \omega_k n_k \, d\boldsymbol{k} \qquad \omega_k = \beta k^2$$



$$N_1 + N_3 = N_2 \qquad \omega_1 N_1 + \omega_3 N_3 = \omega_2 N_2$$

$$N_1 = N_2 \frac{\omega_3 - \omega_2}{\omega_3 - \omega_1}, \qquad N_3 = N_2 \frac{\omega_2 - \omega_1}{\omega_3 - \omega_1}$$

$$\omega_1 \ll \omega_2 < \omega_3 \longrightarrow \omega_2 N_2 \approx \omega_3 N_3$$

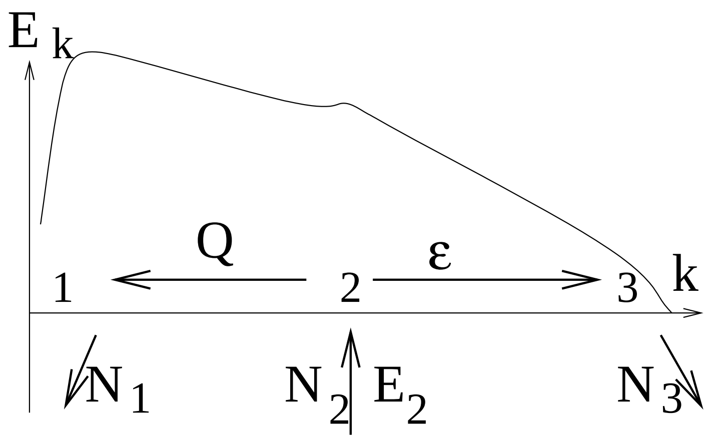
$$\omega_1 < \omega_2 \ll \omega_3 \longrightarrow N_2 \approx N_1$$

$$\frac{\partial n_k}{\partial t} = I_k^{(4)} = div_k \mathbf{Q} \qquad \frac{\partial E_k}{\partial t} = div_k \epsilon$$

$$I_k^{(4)} = \frac{\pi}{2} \int |T_{k123}|^2 \left[n_2 n_3 (n_1 + n_k) - n_1 n_k (n_2 + n_3) \right] \delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3)$$

$$\times \delta(\omega_k + \omega_1 - \omega_2 - \omega_2)\, d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 \qquad T_{kpqs} = const$$

Thermal equilibrium $n_k \propto (\mu + \beta k^2)^{-1}$



Direct energy cascade

$$\epsilon = \text{const} \;\rightarrow\; n_k \propto \epsilon^{1/3} k^{-d}$$

Inverse action cascade

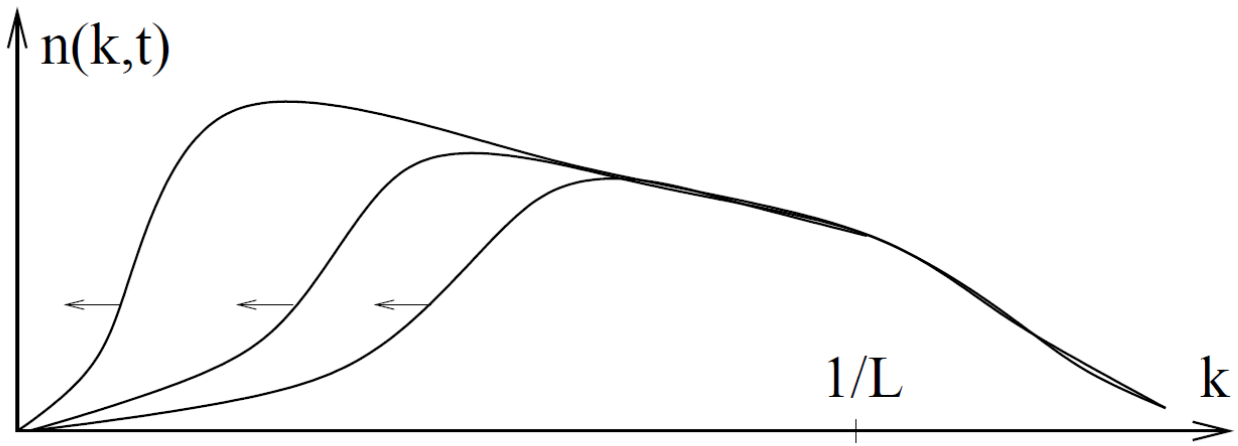
$$Q = \text{const} \;\rightarrow\; n_k \propto Q^{1/3} k^{-d+2/3}$$

3d

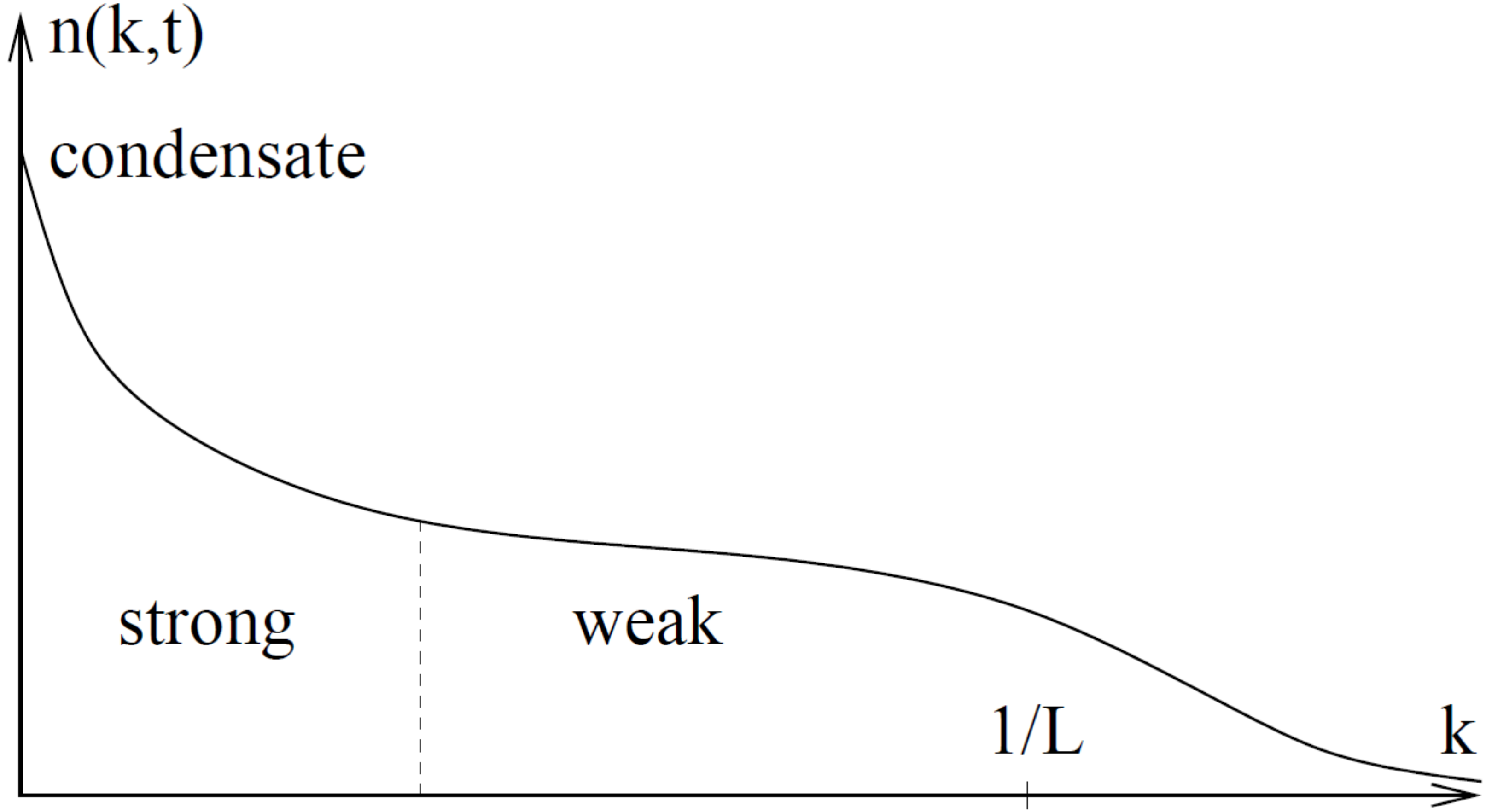
$$\epsilon = \text{const} \;\rightarrow\quad n_k \propto \epsilon^{1/3} k^{-3} / \ln(kL)$$

$$Q = \text{const} \;\rightarrow\quad n_k \propto Q^{1/3} k^{-7/3}$$

$$n_k(t) = (t_0 - t)^{7/2} f[k(t_0 - t)^{-3/2}]$$



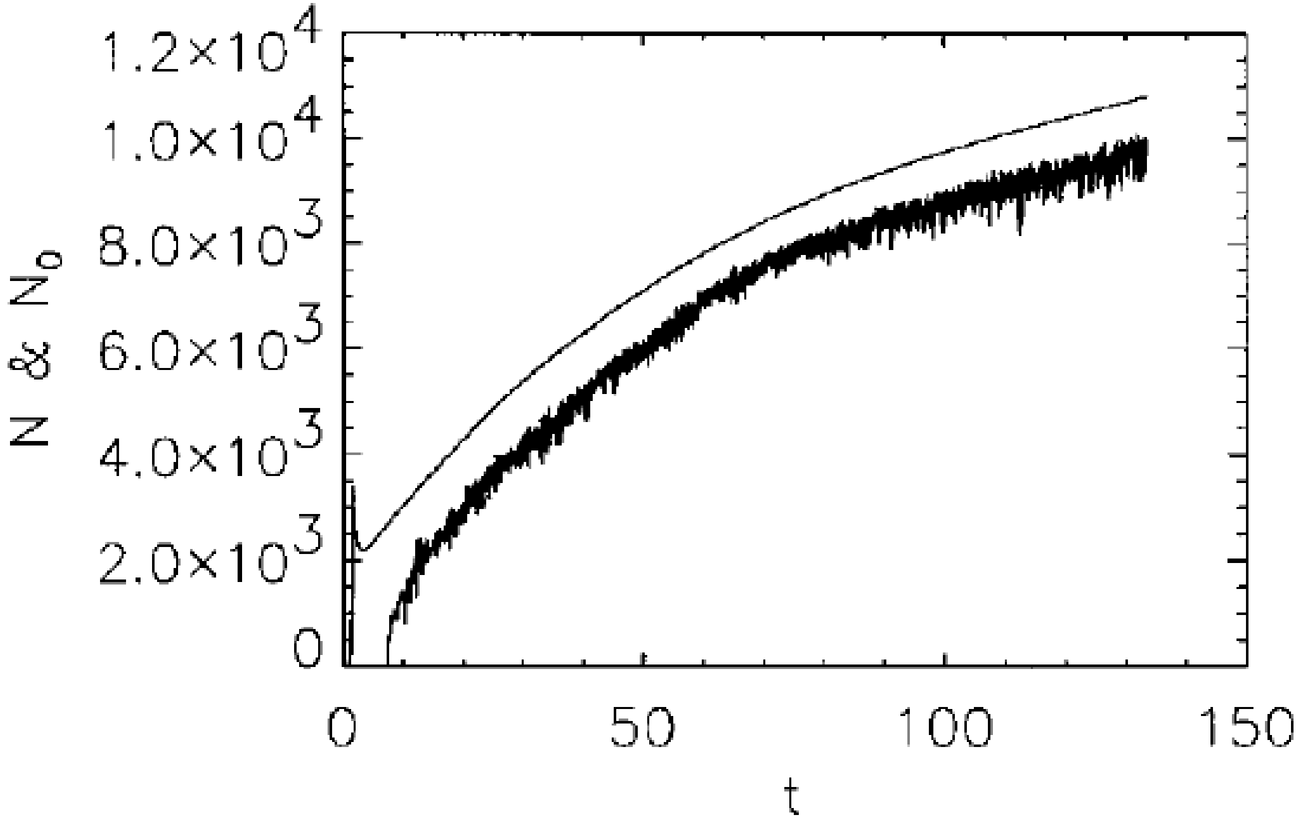
Problem: whether condensate is formed in a finite time?



n(k,t)
condensate
strong
weak
1/L
k

Growth of stable condensate in the defocusing case,
The problem: how correlation scale grows?

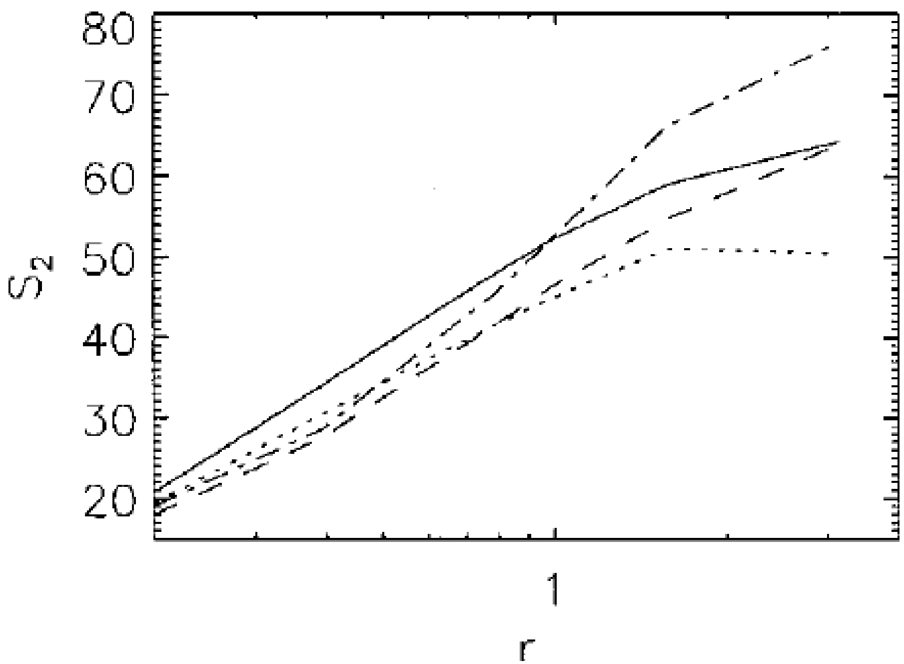
$$R^2 \simeq Ht^2 \qquad 1/[1 - \log(t/t_0)]$$

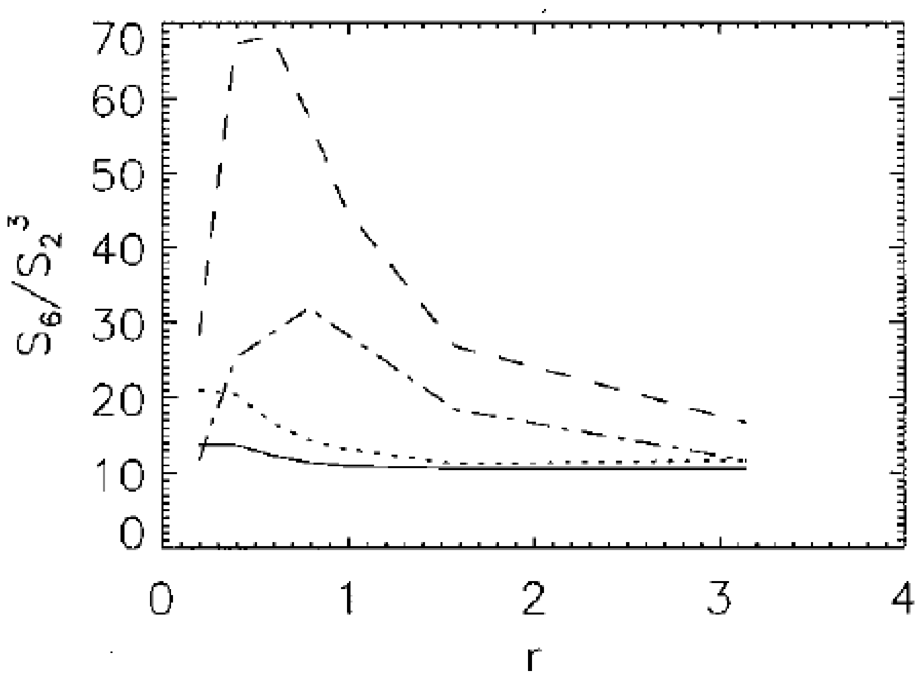


2d

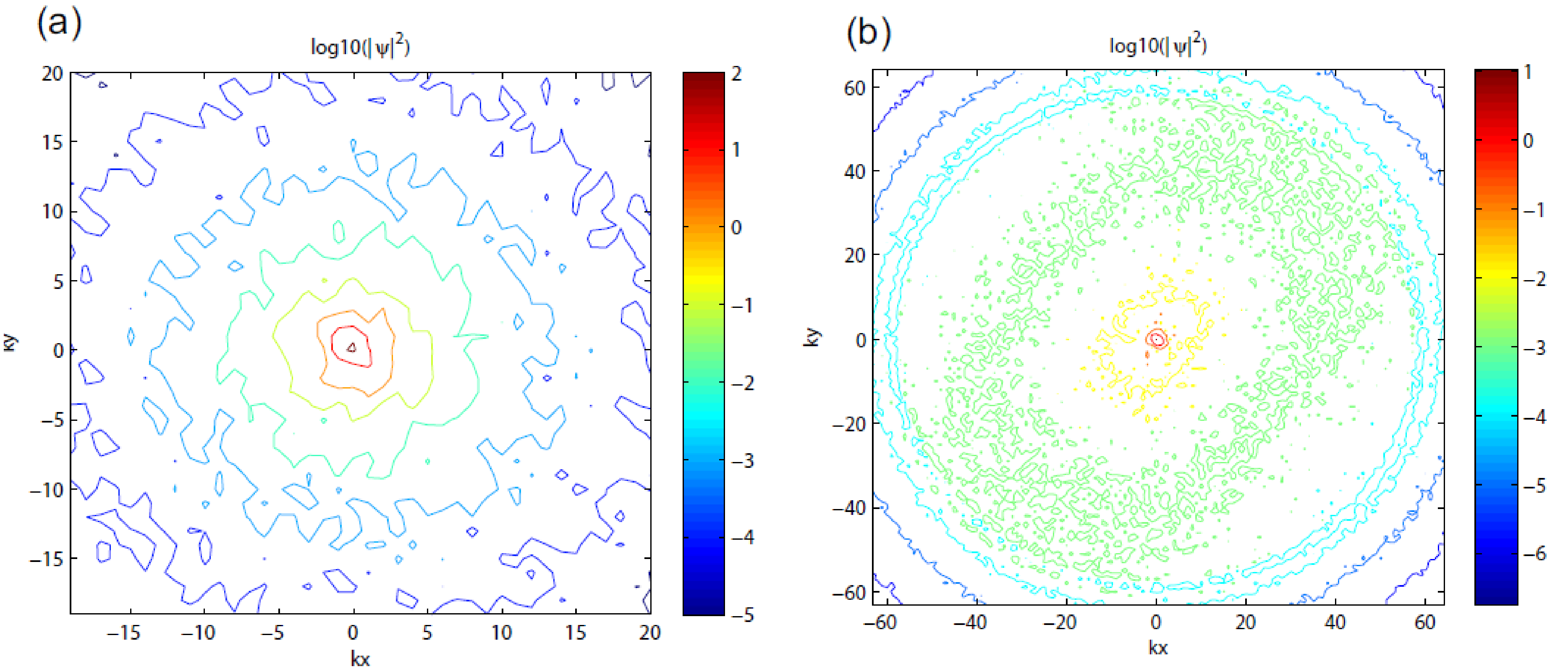
$$\epsilon = \text{const} \quad \rightarrow \quad n_k \propto \epsilon^{1/3} k^{-2} f[\ln(kL)] \quad ?$$

$$Q = \text{const} \quad \rightarrow \quad n_k \propto Q^{1/3} k^{-4/3} \quad ?$$



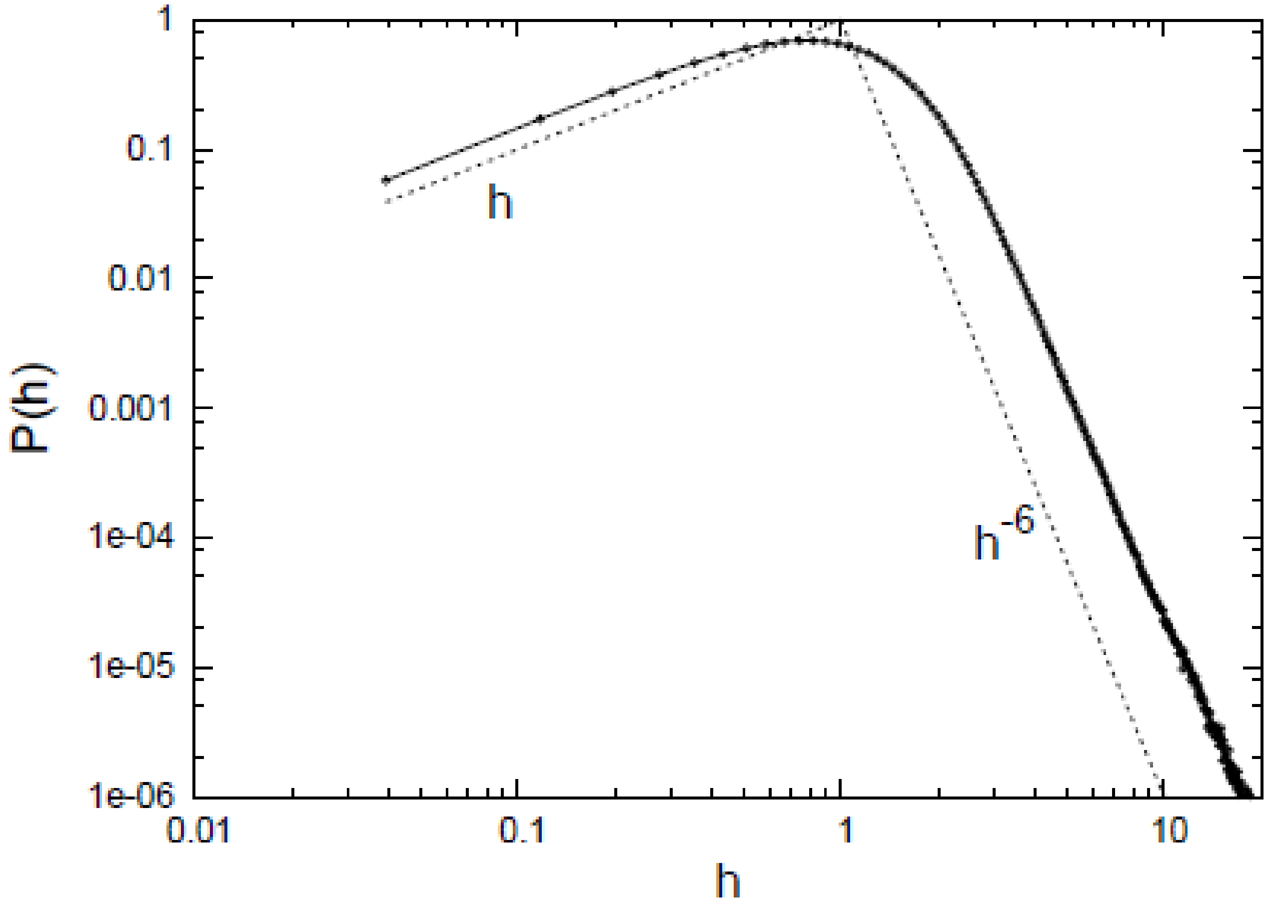


Breakdown of isotropy



The time averaged spectrum of the condensate (a) and the fluctuations (b)

Strong turbulence in 2d focusing case: non-Gaussian PDF



Open problems

- Precise form of weakly turbulent direct cascades.
- Inverse cascades in strong turbulence, difference between focusing and defocusing cases.
- Kinetics of condensate growth and disappearance of vortices.
- Statistics of over-condensate fluctuations, conformal invariance in 2d?
- Anomalous scaling and non-Gaussianity due to collapses in the focusing case.