The Abdus Salam International Centre for Theoretical Physics

2162-13

Advanced Workshop on Anderson Localization, Nonlinearity and Turbulence: a Cross-Fertilization

23 August - 3 September, 2010

Anderson Localization for the Nonlinear Schroedinger Equation (NLSE): Progress Report

Shmuel FISHMAN

Technion, Dept. of Physics
Haifa
Israel

Anderson Localization for the Nonlinear Schrödinger Equation (NLSE): Progress Report

Arkady Pikovsky, Avy Soffer, Alexander Iomin
Yevgeny Krivolapov, Hagar Veksler, Erez Michaely Alex Rivkind
and SF

Experimental Relevance

Nonlinear Optics
Bose Einstein Condensates (BECs)

Paradigm for competition between randomness and Nonlinearity a Fundamental Question

Outline

- Introduction
- Effective Noise Theories
- Scaling Theory
- Perturbation Theory
- Summary

The Nonlinear Schroedinger (NLS) Equation

$$i\frac{\partial}{\partial t}\psi = \mathcal{H}_o\psi + \beta|\psi|^2\psi$$

1D lattice version

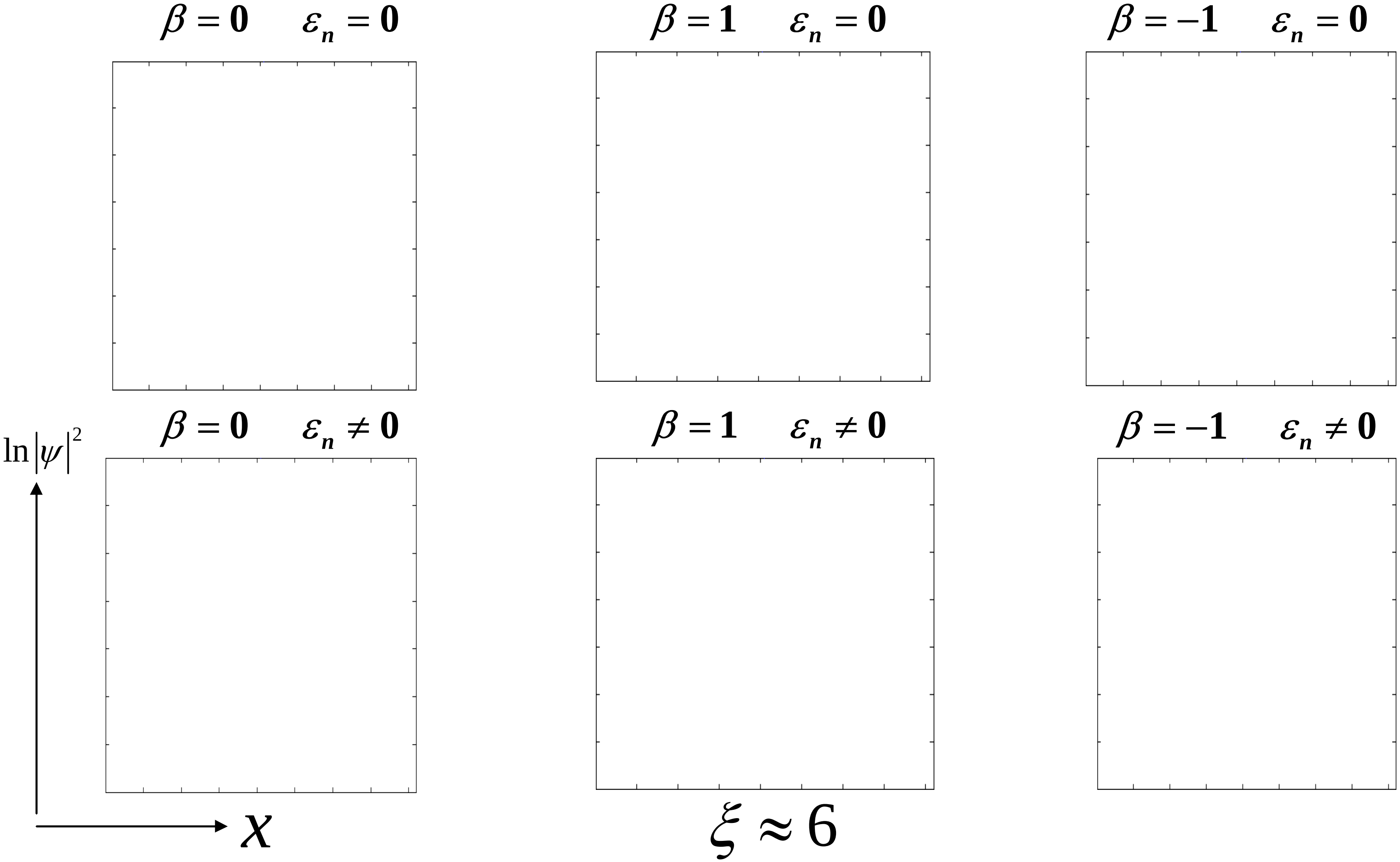
$$\mathcal{H}_0\psi(x) = -\big(\psi(x+1)+\psi(x-1)\big) + \varepsilon(x)\psi(x)$$

1D continuum version

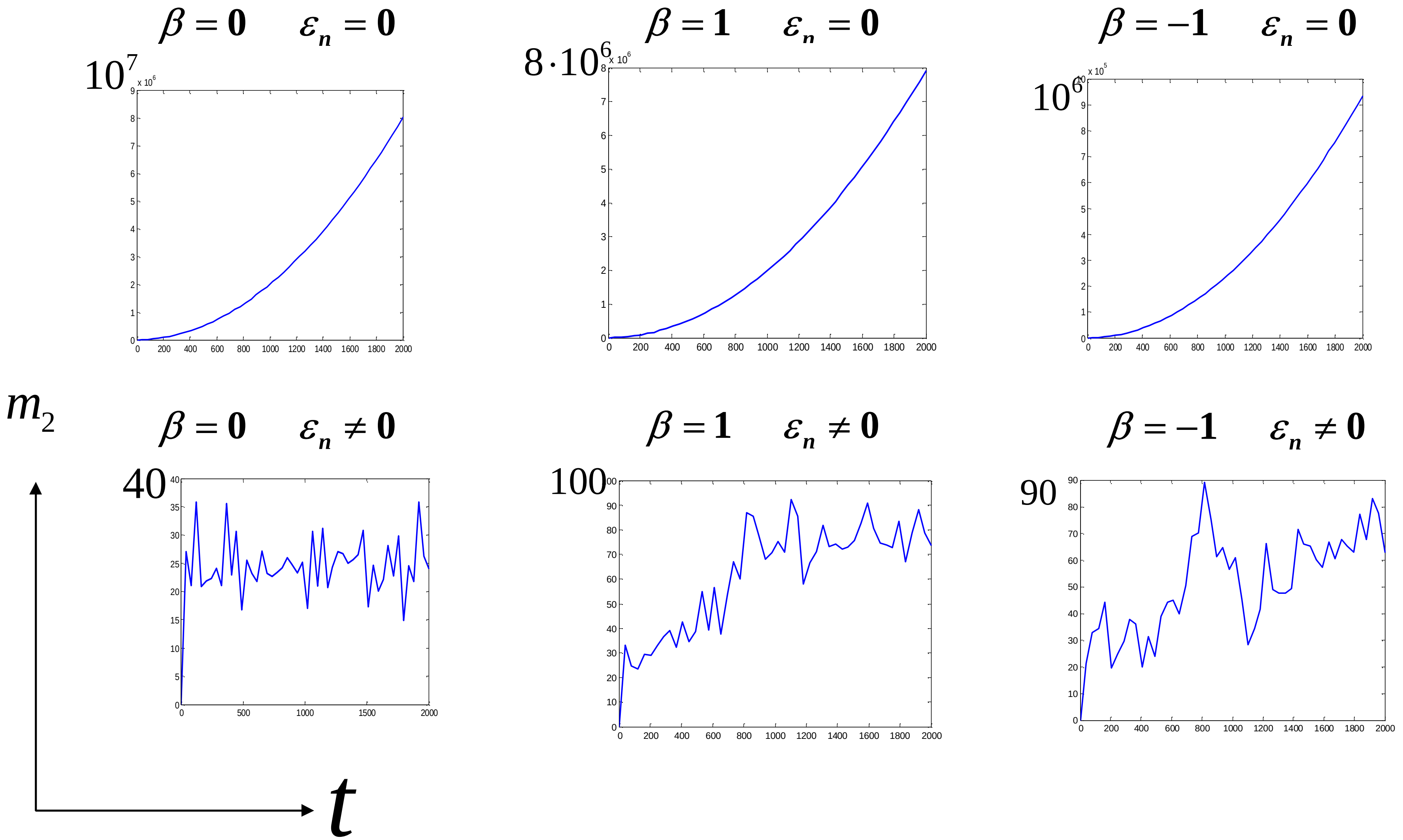
$$\mathcal{H}_0\psi(x) = -\frac{1}{2}\frac{\partial^2}{\partial x^2}\psi(x) + \varepsilon(x)\psi(x)$$

V random $\longrightarrow$ $\mathcal{H}_0$ Anderson Model

$$i\frac{\partial \psi(x)}{\partial t} = -\left(\psi(x+1)+\psi(x-1)\right)+\varepsilon(x)\psi(x)+\beta\left|\psi(x)^2\right|\psi(x)$$



$$m_2(t) = \sum_n x^2 \left|\psi(x)\right|^2$$



$\beta = 0 \Rightarrow$ localization

Does Localization Survive the Nonlinearity???

Does Localization Survive the Nonlinearity???

1. Yes, if there is spreading the magnitude of the nonlinear term decreases and localization takes over.
2. No, may depend on realizations or on β found in numerical calculations.
3. No, the NLSE is a chaotic dynamical system. Will it remain chaotic for all densities??
4. No?, but localization asymptotically preserved beyond some front that is logarithmic in time

Point 4, conjectured by Wang and Zhang in the limit of strong disorder:

given $$\varepsilon = hopping + \beta \quad \delta > 0, A > 0$$

tail beyond j_0 of weight < δ

there exist $C, \varepsilon(A), K > A^2$

So that for all $$t \le \left(\frac{\delta}{C}\right)\varepsilon^{-A}, \varepsilon < \varepsilon(A)$$

tail beyond $j_0 + K$ of weight < 2δ

Logarithmic front $j_0 + K$

Perturbation theory supports this conjecture for any disorder

Numerical Simulations

- In regimes relevant for experiments looks that localization takes place
- Spreading for long time (**Shepelyansky,** Pikovsky, Mulansky, Molina,

  Flach, Kopidakis, Komineas, Krimer, Laptyeva, Bodyfelt)
- We do not know the relevant space and time scales
- All results in Split-Step
- No control (but may be correct in some range)
- Supported by various heuristic arguments

Pikovsky, Sheplyansky

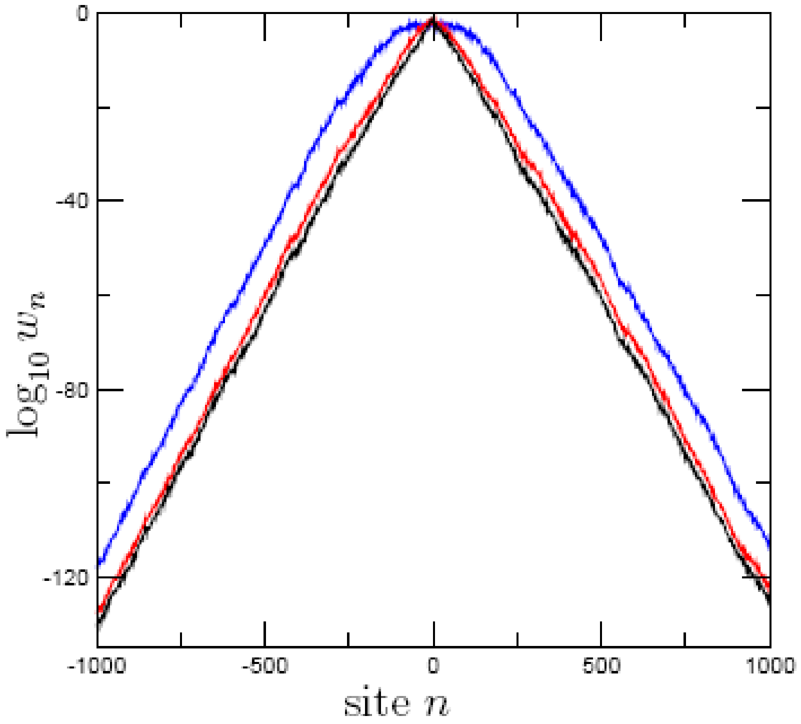


FIG. 2: (color online) Probability distribution w_n over lattice sites n at $W = 4$ for $\beta = 1$, $t = 10^8$ (top blue/solid curve) and $t = 10^5$ (middle red/gray curve); $\beta = 0, t = 10^5$ (bottom black curve; the order of the curves is given at $n = 500$). At $\beta = 0$ a fit $\ln w_n = -(\gamma|n| + \chi)$ gives $\gamma \approx 0.3$, $\chi \approx 4$. The values of $\log_{10} w_n$ are averaged over the same disorder

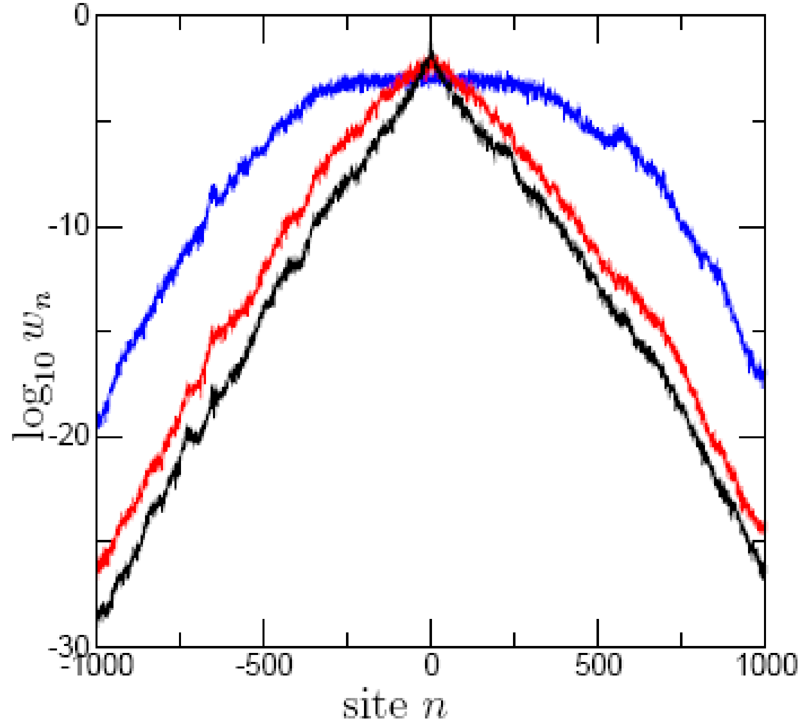
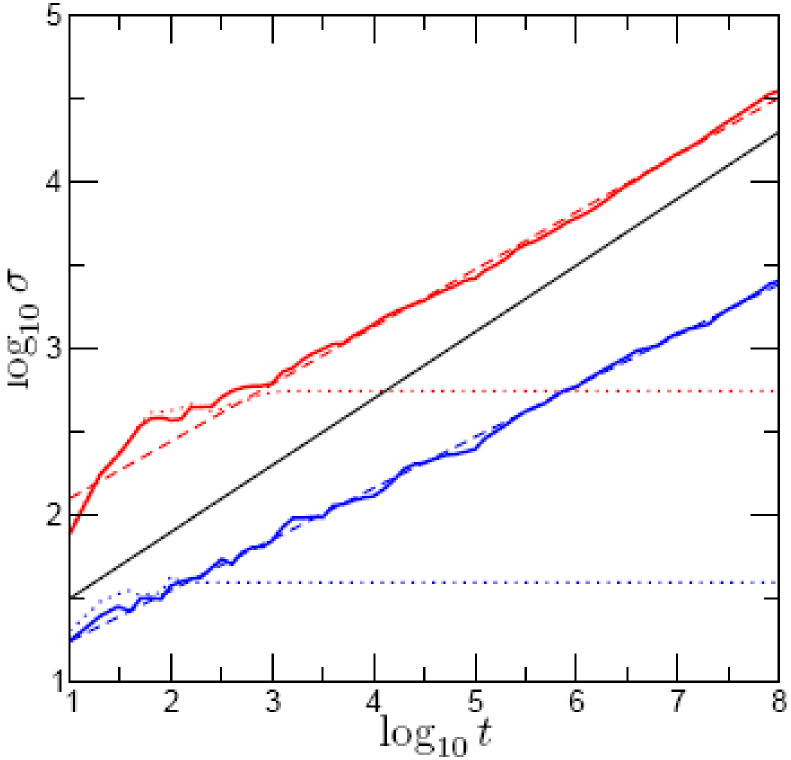


FIG. 3: (color online) Same as in Fig. 2 but with $W = 2$. At $\beta = 0$ a fit $\ln w_n = -(\gamma|n| + \chi)$ gives $\gamma \approx 0.06$, $\chi \approx -3$. The values of $\ln w_n$ are averaged over the same disorder realizations as in Fig. 1.

Slope does not change (contrary to Fermi-Ulam-Pasta)

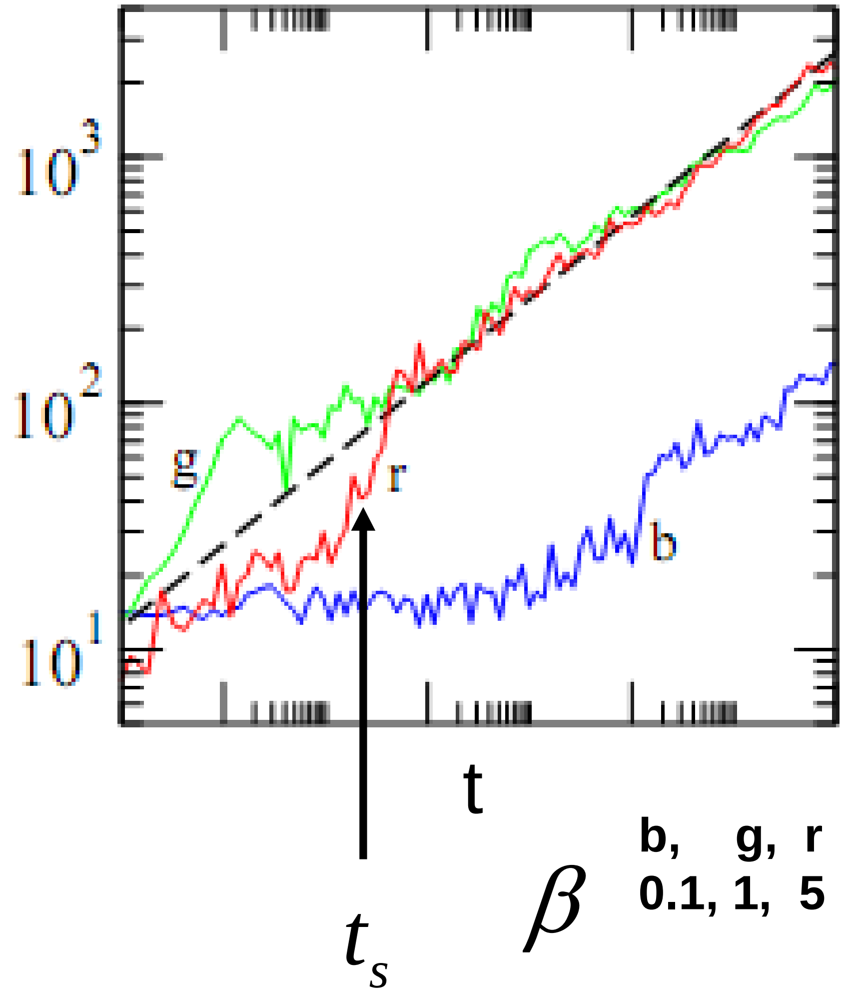
Pikovsky, Shepelyansky

$$\log < x^2 >$$



S.Flach, D.Krimer and S.Skokos

$$\log < x^2 >$$



Effective Noise Theories

- D. Shepeyansky and A. Pikovsky
- S. Flach, Ch. Skokos, D.O. Krimer, S. Komineas

$$\psi\left(x,t\right)=\sum_{m} c_m\left(t\right)e^{-iE_m t}u_m\left(x\right)$$

$$i\frac{\partial}{\partial t}c_n=\beta\sum_{m_1,m_2,m_3} V_n^{m_1,m_2,m_3}c_{m_1}{}^{*}c_{m_2}c_{m_3}e^{i(E_n+E_{m_1}-E_{m_2}-E_{m_3})t}$$

Overlap $$V_n^{m_1,m_2,m_3}=\sum_{x} u_n(x)u_{m_1}(x)u_{m_2}(x)u_{m_3}(x)$$

$$\left|V_n^{m_1m_2m_3}\right|\leq[const]e^{-\frac{1}{3}\gamma\left(\left|x_n-x_{m_1}\right|+\left|x_n-x_{m_2}\right|+\left|x_n-x_{m_3}\right|\right)}$$

of the range of the localization length ξ

$$i\frac{\partial}{\partial t}c_n = \beta \sum_{m_1,m_2,m_3} V_n^{m_1,m_2,m_3} c_{m_1}^{\ *} c_{m_2} c_{m_3} e^{i(E_n+E_{m_1}-E_{m_2}-E_{m_3})t}$$

Assume $\left|c_{m_1}^2\right| \approx \left|c_{m_2}^2\right| \approx \left|c_{m_3}^2\right| \approx \rho$ **initially** $\left|c_n^2\right| \ll \rho$

$$i\frac{\partial}{\partial t}c_n \approx (\ldots) f(t)$$

$f(t)$ **Random uncorrelated**

leading to

$$\left\langle x^2 \right\rangle \sim t^{1/3}$$

Details …..

$$i\frac{\partial}{\partial t}c_n = \beta \sum_{m_1,m_2,m_3} V_n^{m_1,m_2,m_3} c_{m_1}{}^{*} c_{m_2} c_{m_3} e^{i(E_n+E_{m_1}-E_{m_2}-E_{m_3})t}$$

Assume $\left|c_{m_1}^2\right| \approx \left|c_{m_2}^2\right| \approx \left|c_{m_3}^2\right| \approx \rho$ **initially** $\left|c_n^2\right| \ll \rho$

$$i\frac{\partial}{\partial t}c_n \approx P\beta\rho^{3/2} f(t)$$

$f(t)$ **Random uncorrelated**

Assume $P = A\,\beta\,\rho$ **!!!**

$$\left\langle \left|c_n^2\right| \right\rangle = P^2\beta^2\rho^3 t$$

Equilibration time

Equilibrium $\left\langle \left|c_n^2\right| \right\rangle = \rho$

$$T_{eq} = \frac{1}{A^2\beta^4\rho^4}$$

Typical size of $V_n^{m_1,m_2,m_3}$?

$$D = \frac{1}{T_{eq}} = A^2 \beta^4 \rho^4$$

$$\frac{1}{\rho^2} \sim m_2 = Dt = A^2 \beta^4 \rho^4 t$$

$$\frac{1}{\rho^2} \sim m_2 = \left[A^2 \beta^4 t \right]^{1/3}$$

Consistent $T_{eq} \sim t^{2/3} \ll t$

Can it go on forever?

What happens when nearly no weight in in localization volume?

Questions

1. Is $f(t)$ really random?
2. What is $\left\langle f\left(t^{'}\right) f\left(t^{'}+t\right)\right\rangle$?
3. What is the dependence of A on parameters ξ, β ?
4. Can the process go for ever?

Competition

Spreading → **effective number of degrees of freedom increases**

chaos enhanced

Spreading → **amplitude decreases** → **regularity enhanced**

Who wins??

$$i\frac{d}{dt}\psi(x) = -J\left(\psi(x+1)+\psi(x-1)\right)+\varepsilon(x)\psi(x)+\left|\psi(x)\right|^2\psi(x)$$

x integer $1 \le x \le L$

$\psi(x)$ Are dynamical variables

Initial data, nearly homogeneous spreading in space

Growth of deviations

$$\delta\psi(t) \sim \delta\psi(t=0)e^{\lambda t}$$

λ Largest Lyapunov exponent

$\lambda > 0 \longrightarrow$ Chaos

Is it possible that chaos disappears?

Divide chain into intervals of length L_0 Number of intervals $\frac{L}{L_0}$

Assuming independence, if intervals large enough $L_0 \gg \xi$

The probability to be regular:

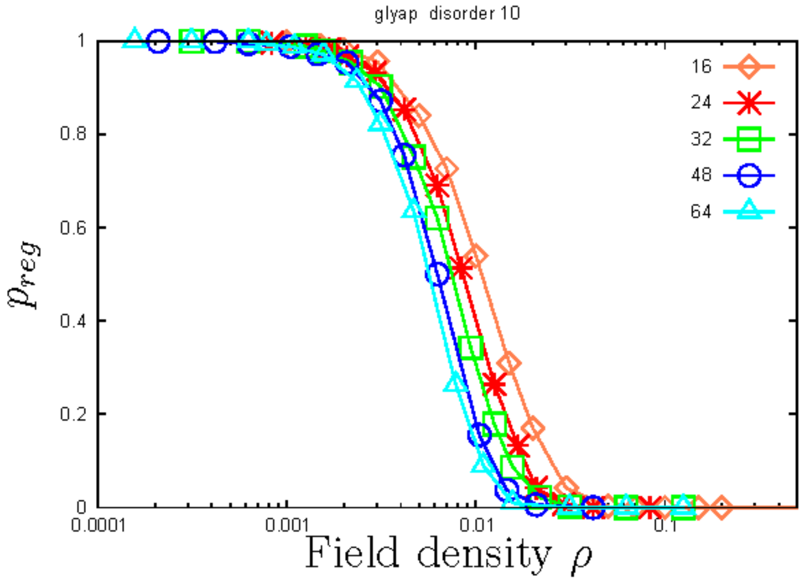
$$p_{reg}(W,\rho,L) = p_{reg}(W,\rho,L_0)^{L/L_0}$$

Regularity=all orbits regular

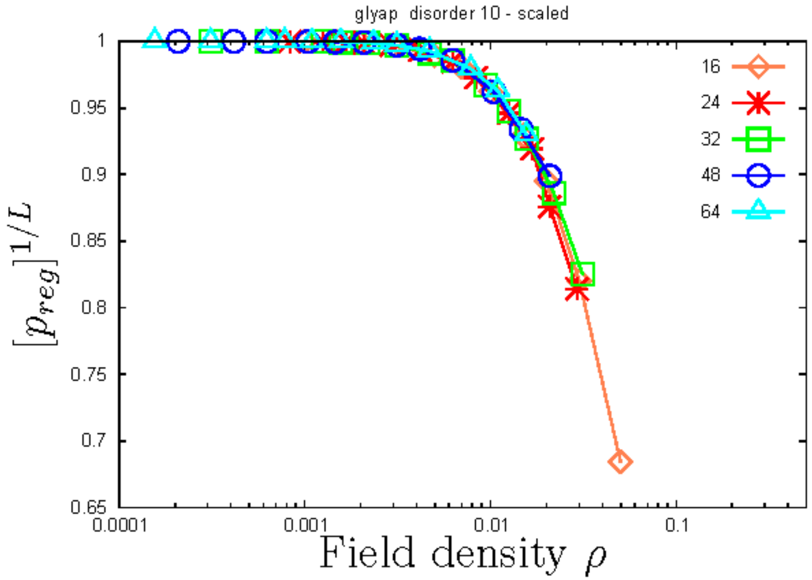
density $\rho = \frac{1}{L}\sum_{x=1}^{L}\left|\psi(x)\right|^2$

$$\overline{p}_{reg}(W,\rho) \equiv p_{reg}(W,\rho,L)^{1/L} = p_{reg}(W,\rho,L_0)^{1/L_0}$$

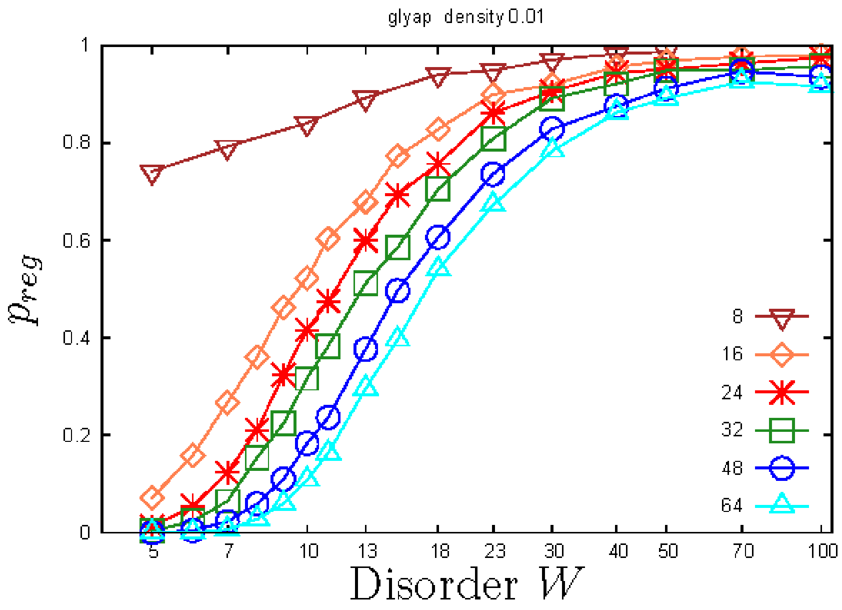
independent of L

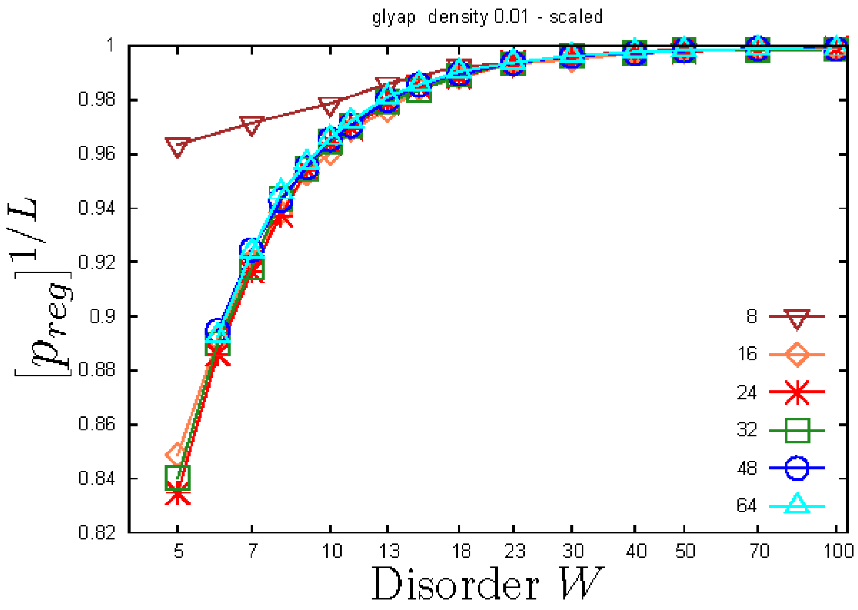


glyap disorder 10
16
24
32
48
64
p_{reg}
Field density ρ



glyap disorder 10 - scaled
16
24
32
48
64
$[p_{reg}]^{1/L}$
Field density ρ


glyap density 0.01
p_{reg}
Disorder W
8
16
24
32
48
64


glyap density 0.01 - scaled
$[p_{reg}]^{1/L}$
Disorder W
8
16
24
32
48
64

Scaling

Define $$Q \equiv \frac{\overline{p}_{reg}}{1-\overline{p}_{reg}} \rightarrow \overline{p}_{reg} = \frac{1}{1+1/Q}$$

Scaling function $$Q = \frac{1}{W^a} q\left(\frac{\rho}{W^b}\right)$$

$$\rho = \frac{1}{L}\sum_{x=1}^{L} \left|\psi(x)\right|^2$$

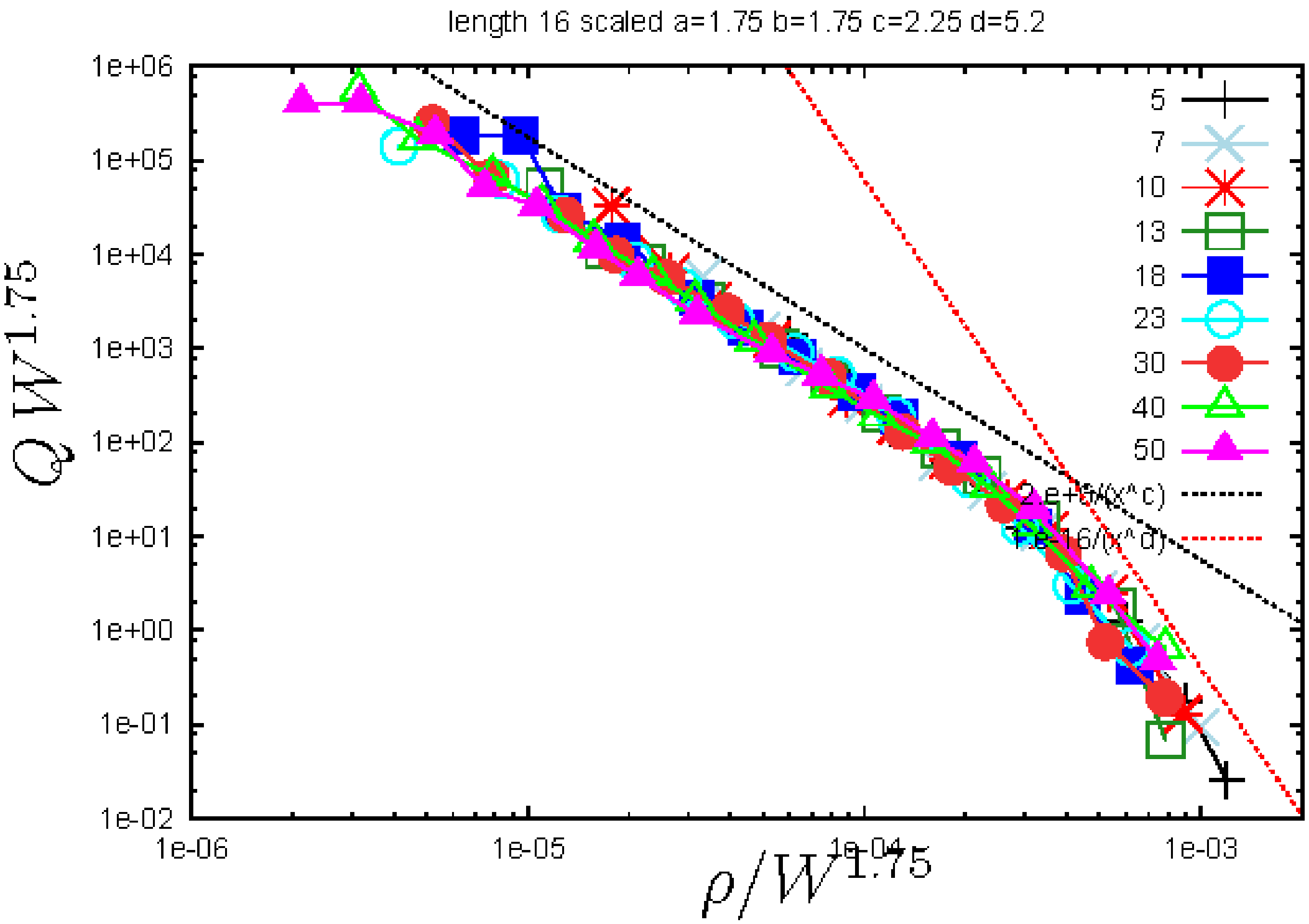
$\sum_{x=1}^{L} \left|\psi(x)\right|^2$ conserved

$Q(x)$ **singular**

$$\overline{p}_{reg} = \frac{1}{1+1/Q} \approx 1-\frac{1}{Q} \approx 1-\rho^{2.25}$$

$$p_{reg} = \overline{p}_{reg}^{L} \rightarrow \ln p_{reg} = L \ln \overline{p}_{reg} \sim L\rho^{2.25} \sim L^{-1.25}$$

In the limit $L \rightarrow \infty$ $\ln p_{reg} \rightarrow 0 \rightarrow p_{reg} = 1$



length 16 scaled a=1.75 b=1.75 c=2.25 d=5.2
$Q\,W^{1.75}$
$\rho/W^{1.75}$
1e+06
1e+05
1e+04
1e+03
1e+02
1e+01
1e+00
1e-01
1e-02
1e-06
1e-05
1e-04
1e-03
5
7
10
13
18
23
30
40
50
2.e+6/(x^c)
1.e-16/(x^d)

If no additional singularity in Q

Spreading

↓

No chaos

↓

Localization ?

Perturbation Theory

The nonlinear Schroedinger Equation on a Lattice in 1D

$$i\frac{\partial}{\partial t}\psi = \mathcal{H}_o\psi + \beta\left|\psi\right|^2\psi$$

$$\mathcal{H}_0\psi(x) = -\big(\psi(x+1)+\psi(x-1)\big)+\varepsilon(x)\psi(x)$$

$\varepsilon\left(x\right)$ random $\longrightarrow$ $\mathcal{H}_0$ Anderson Model

Eigenstates $\mathcal{H}_0 u_m(x) = E_m u_m(x)$

$$\psi\left(x,t\right)=\sum_m c_m\left(t\right)e^{-iE_mt}u_m\left(x\right)$$

Perturbation theory steps

- Expansion in nonlinearity
- Removal of secular terms
- Control of denominators
- Probabilistic bound on general term
- Control of remainder
- **Use perturbation theory to obtain a numerical solution that is controlled a posteriori**

$$i\frac{\partial}{\partial \mathrm{t}}c_n = \beta \sum_{m_1,m_2,m_3} V_n^{m_1,m_2,m_3} c_{m_1}{}^* c_{m_2} c_{m_3} e^{i(E_n+E_{m_1}-E_{m_2}-E_{m_3})t}$$

Overlap $$V_n^{m_1,m_2,m_3} = \sum_x u_n(x)u_{m_1}(x)u_{m_2}(x)u_{m_3}(x)$$

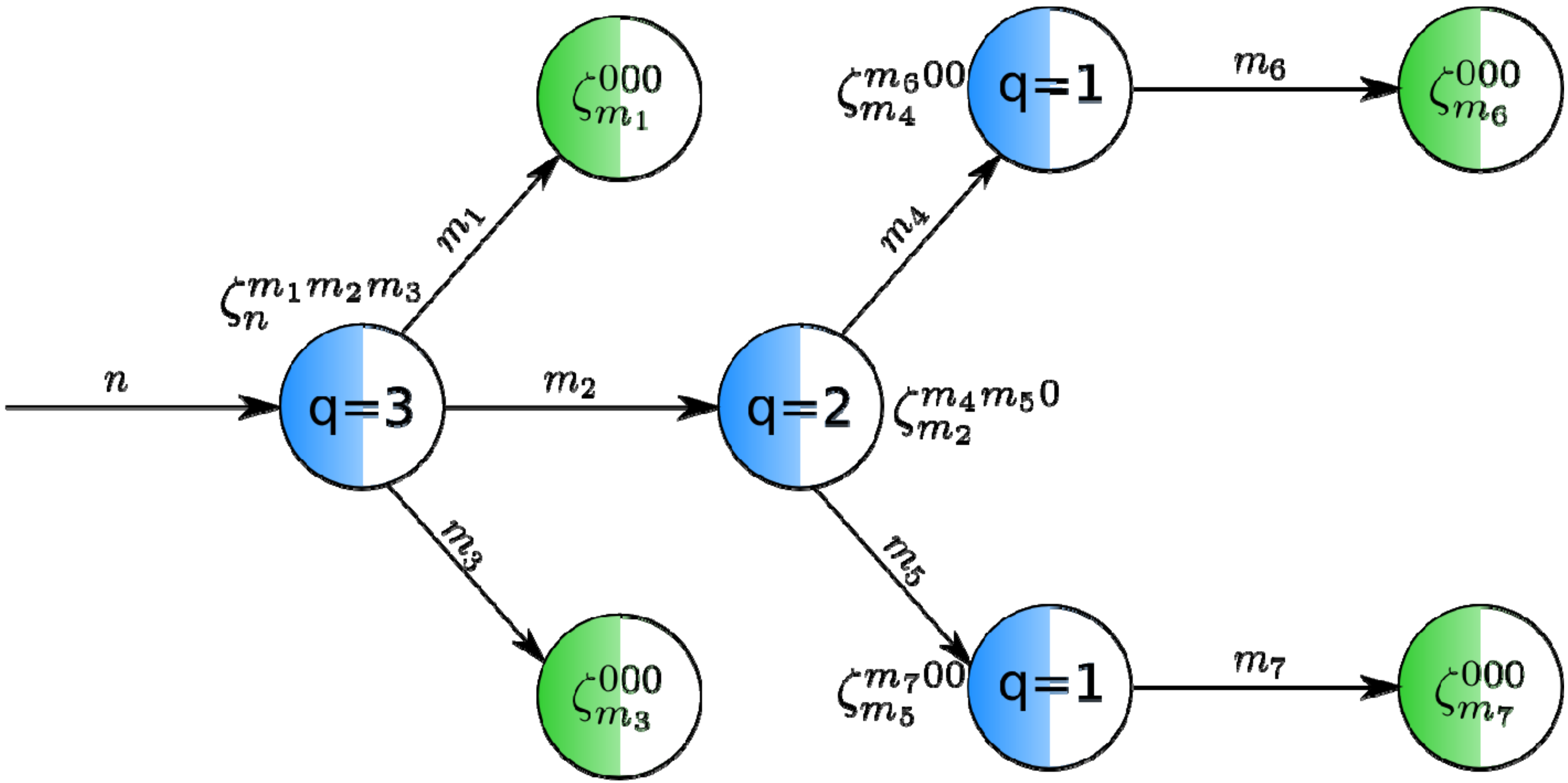
of the range of the localization length ξ

perturbation expansion

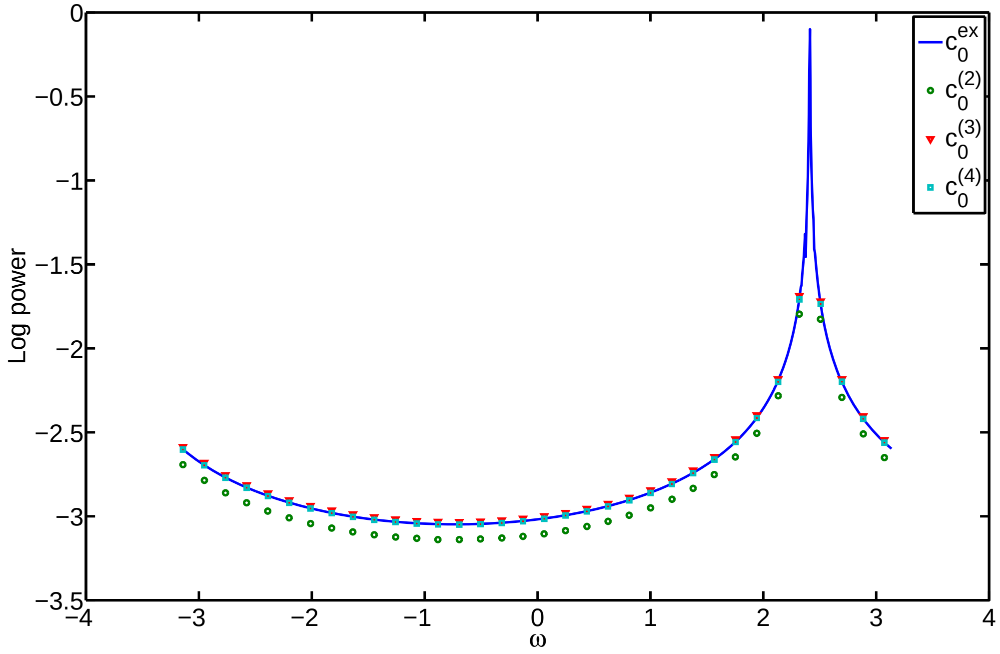
$$c_n(t) = c_n{}^{(o)} + \beta c_n{}^{(1)} + \beta^2 c_n{}^{(2)} + \ldots\ldots + \beta^{N-1} c_n{}^{(N-1)} + \beta^N Q_N(n)$$

Iterative calculation of $c_n{}^{(l)}$

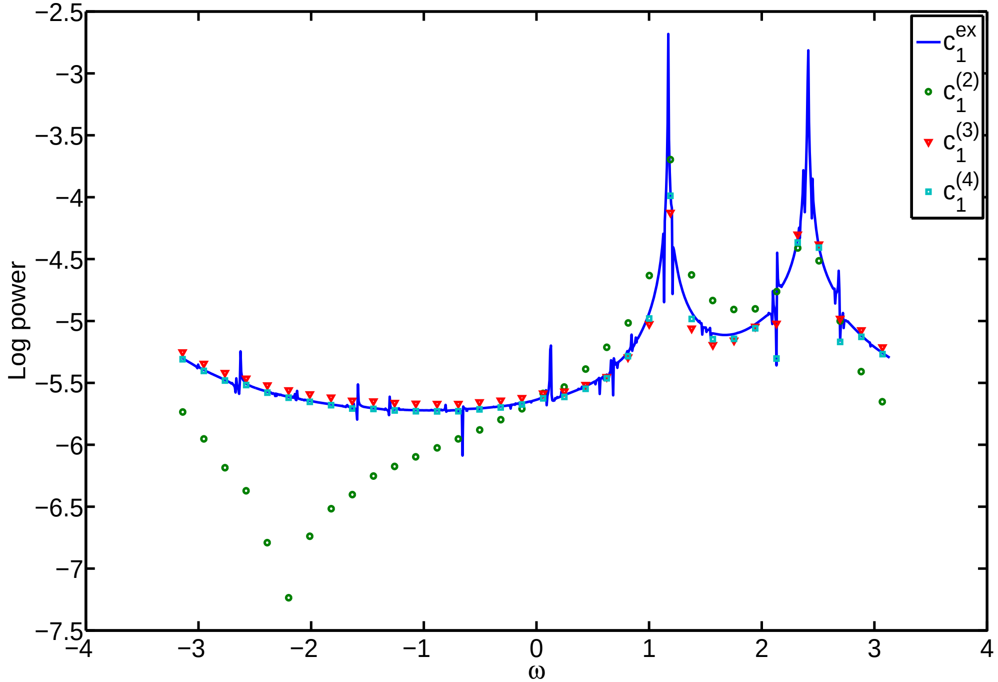
start at $$c_n^{(0)} = c_n(t=0) = \delta_{n0}$$



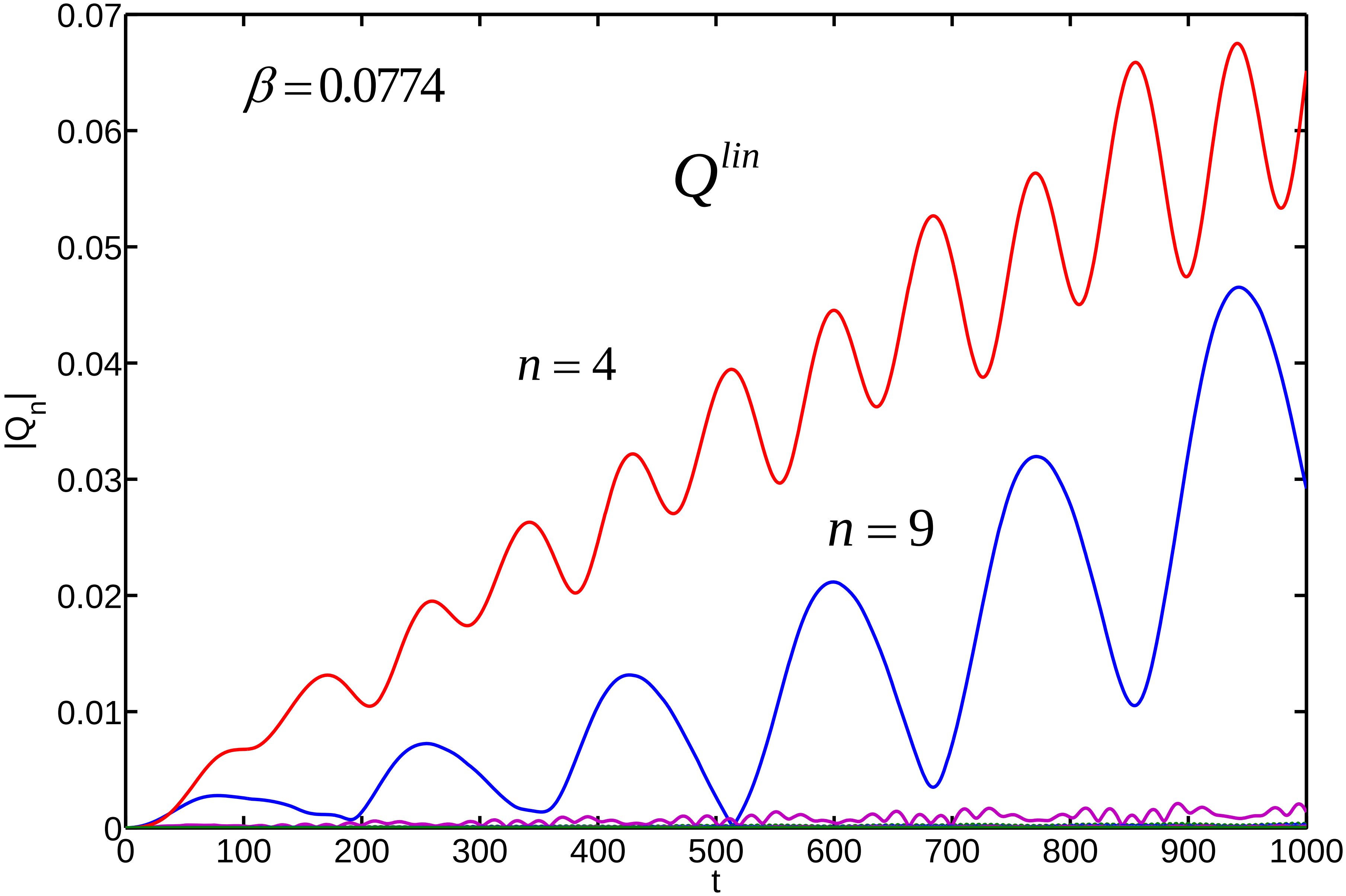
An example of a graph that is used to construct the general term. The graph describes an 8-th order term, $\zeta_n^{m_1 m_2 m_3} \zeta_{m_1}^{000} \zeta_{m_3}^{000} \zeta_{m_2}^{m_4 m_5 0} \zeta_{m_4}^{m_6 00} \zeta_{m_5}^{m_7 00} \zeta_{m_6}^{000} \zeta_{m_7}^{000}$.



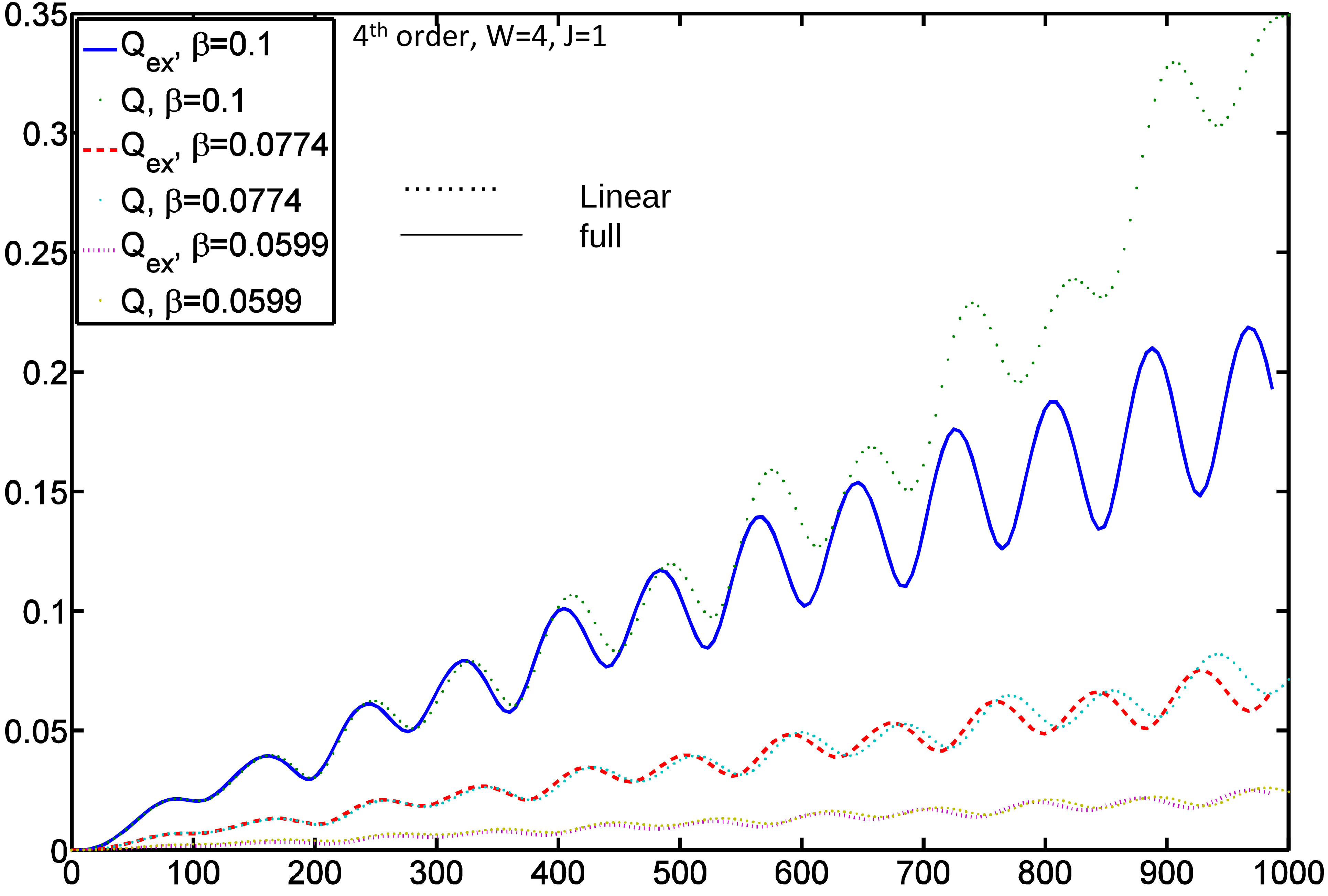
c_0^{ex}
$c_0^{(2)}$
$c_0^{(3)}$
$c_0^{(4)}$
Log power
ω



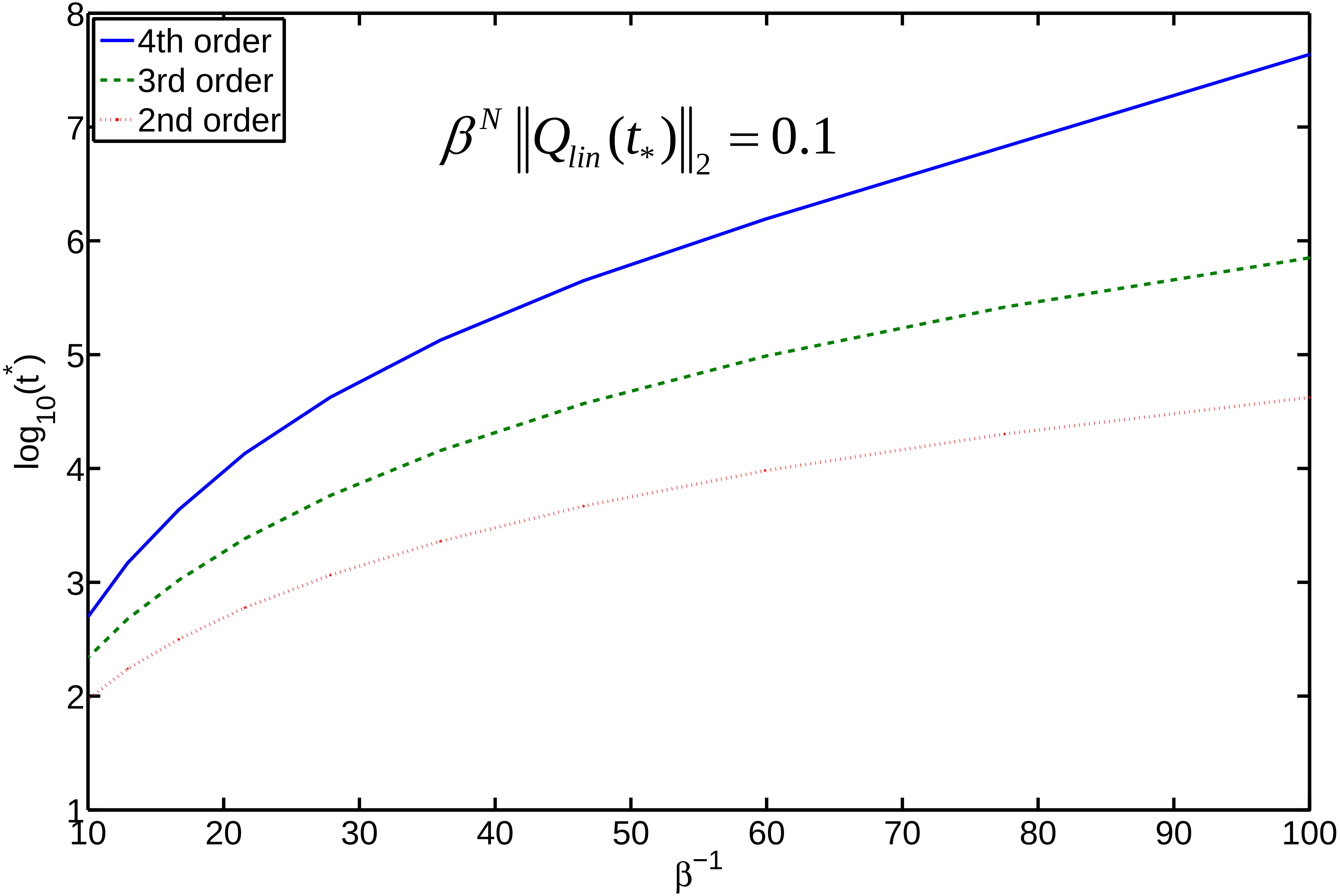
c_1^{ex}
$c_1^{(2)}$
$c_1^{(3)}$
$c_1^{(4)}$
Log power
ω



Nearly resonant modes



4th order, W=4, J=1
Q_ex, β=0.1
Q, β=0.1
Q_ex, β=0.0774
Q, β=0.0774
Q_ex, β=0.0599
Q, β=0.0599
Linear
full
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0
100
200
300
400
500
600
700
800
900
1000



Empirical criterion for bootstrap to hold

The Bound on the remainder

$$\left|\beta^N Q_N(n)\right| \leq A(N,\gamma)\beta^N t e^{-\gamma|x_n|} = Ae^{(\ln t - N|\ln\beta| - \gamma|x_n|)}$$

x_n Localization center of state n

For fixed order and time

$$\lim_{\beta\to 0}\frac{\beta^N Q_N(n)}{\beta^{N-1}} = 0$$

Expansion Asymptotic

One can show that for strong disorder $A(N,\gamma)\xrightarrow[\gamma\to\infty]{} 0$

Looks that $A \sim \exp(-\gamma)$

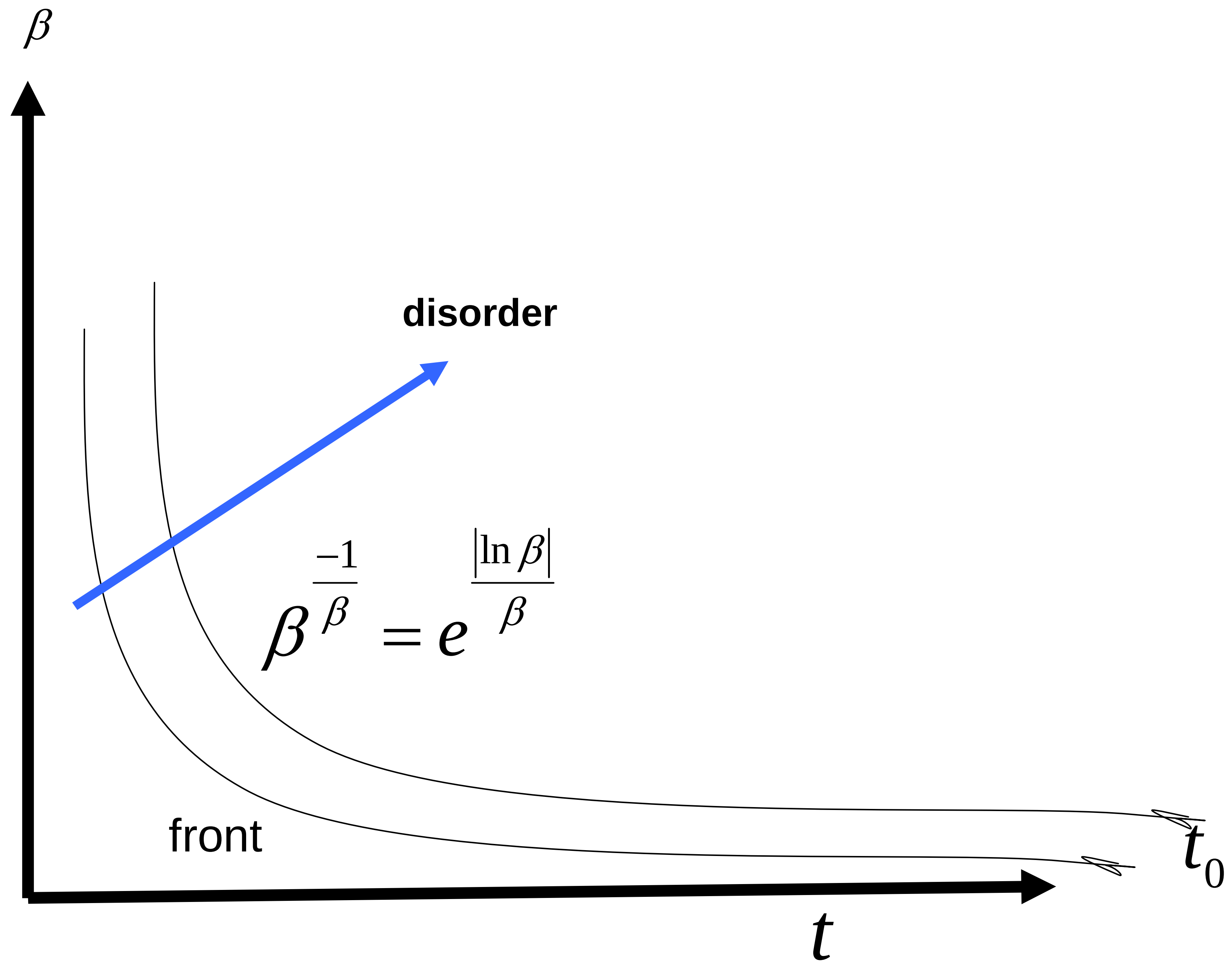
Difficulties in the calculation of A

Front logarithmic in time $\bar{x} \propto \frac{1}{\gamma}\ln t$ For limited time

Localization for $|x| > \bar{x}$

Bound on error

- Solve linear equation for the remainder of order N
- If bounded to time t_0 perturbation theory accurate to that time.
- Order of magnitude estimate $\beta^N t_0 \sim 1$ if asymptotic $\beta^N N! \sim 1$ hence $t_0 \sim N!$ for optimal order (up to constants).
- $t_0 = \beta^{-1/\beta}$ validity time of perturbation theory



β
disorder
$\beta^{\frac{-1}{\beta}} = e^{\frac{|\ln\beta|}{\beta}}$
front
t_0
t

Summary Perturbation Theory

1. A perturbation expansion in β was developed
2. Secular terms were removed
3. A bound on the general term was derived
4. Perturbation theory was used to obtain a controlled numerical solution
5. A bound on the remainder was obtained, indicating that the series is asymptotic.
6. For limited time tending to infinity for small nonlinearity, front logarithmic in time $\overline{x} \propto \ln t$
7. Improved for strong disorder

Emerging Picture

- For small nonlinearity initially no spreading
- For strong nonlinearity some part does not spread
- For some nonlinearity wide regime of sub-diffusion
- Asymptotic spreading at most logarithmic (shown for limited time):

a. perturbation theory

b. rigorous results in the limit of strong disorder

- Unlikely that sub-diffusion continues forever:

a. scaling theory showing that as result of spreading system becomes regular

b. Effective noise “theories” indicate that as a result of spreading noise decays.

Coherent picture for various regimes?