The Abdus Salam International Centre for Theoretical Physics

2162-23

Advanced Workshop on Anderson Localization, Nonlinearity and Turbulence: a Cross-Fertilization

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Computational Studies of Nonlinear Wave Spreading in Disordered Systems

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Together with:

- **Sergej Flach, Haris Skokos, Joshua Bodyfelt, Tetyana Laptyeva**

Disordered systems

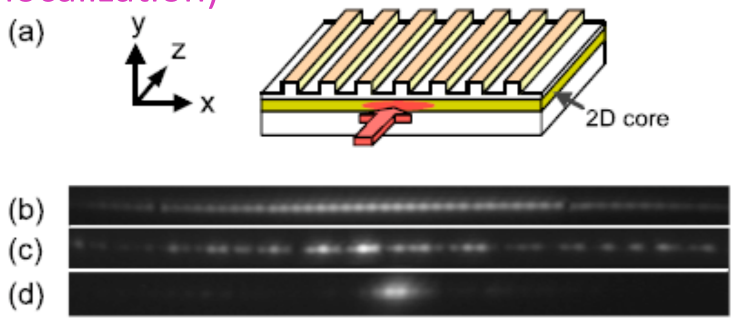
- Anderson localization of waves in disordered media

  electrons, phonons, photons, BEC, …

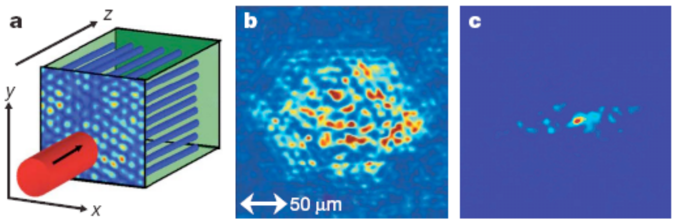
Linear propagation in disordered structures (Anderson localization)

Y. Lahini et al, PRL 100, 013906 (2008)

a) A beam incident on 1D lattice of coupled optical waveguides; randomization of the waveguide's width (disorder)



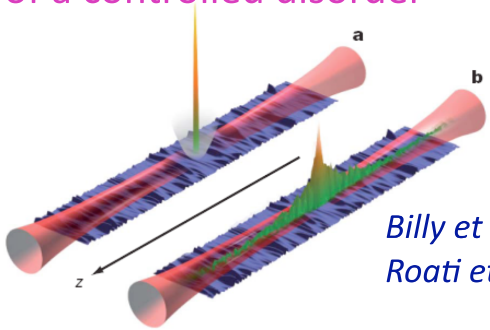
b) No disorder c) with disorder; d) localization around specific locations (single eigenmode)



T. Schwartz, et al, Nature, 446, 52 (2007)

2D periodic photonic optically-induced lattice with a speckled beam (disorder). Output: b) No disorder; c) with disorder

Direct observation of Anderson localization of BEC released into 1D waveguide in the presence of a controlled disorder



Billy et al, Nature 453, 891(2008); Roati et al, Nature 453, 895 (2008)

The BEC is produced in an opto-magnetic hybrid trap.
The longitudinal confinement is switched off at t=0

- no disorder => BEC starts to expand in the z direction;
- with disorder => BEC expansion rapidly stops

Defining the problem

- **a discrete 1D disordered medium**
- **linear equations of motion: all eigenstates are localized**
- **localization length is bounded from above**
- **add short range nonlinearity (interactions)**
- **follow the spreading of an initially localized wave packet**

Will it delocalize or stay localized?

Yes because of nonintegrability and ergodicity

No because of energy conservation –
spreading leads to small energy density,
nonlinearity can be neglected,
dynamics becomes integrable, and
Anderson localization is restored

In simulations when we observe spreading it is subdiffusive

Our strategy: numerical observation → approximate modeling/explanation → generalizations → predictions → numerical observation

Model 1: The discrete nonlinear Schrödinger lattice with disorder

$$\mathcal{H}_D = \sum_l \epsilon_l |\psi_l|^2 + \frac{\beta}{2}|\psi_l|^4 - (\psi_{l+1}\psi_l^* + \psi_{l+1}^*\psi_l)$$

ϵ_l uniformly from $\left[-\frac{W}{2}, \frac{W}{2}\right]$ W : strength of disorder Hopping $\mathrm{V}=1$

$$\dot{\psi}_l = \partial\mathcal{H}_D/\partial(i\psi_l^\star)$$

$$i\dot{\psi}_l = \epsilon_l\psi_l + \beta|\psi_l|^2\psi_l - \psi_{l+1} - \psi_{l-1}$$

Conserved quantities: energy and norm $S = \sum_l |\psi_l|^2$

Varying the norm is strictly equivalent to varying β.

Equations model light propagation and cold atom dynamics in structured media

Model 2: The Klein-Gordon chain

$$\mathcal{H}_K = \sum_l \frac{p_l^2}{2} + \frac{\tilde{\epsilon}_l}{2} u_l^2 + \frac{1}{4} u_l^4 + \frac{1}{2W} (u_{l+1} - u_l)^2$$

$$\ddot{u}_l = -\partial \mathcal{H}_K / \partial u_l$$

$\tilde{\epsilon}_l$ uniformly from $\left[\frac{1}{2}, \frac{3}{2}\right]$

$$\ddot{u}_l = -\tilde{\epsilon}_l u_l - u_l^3 + \frac{1}{W}(u_{l+1} + u_{l-1} - 2u_l)$$

Conserved quantity: energy only

Equations can be approximately mapped on model 1 for small amplitudes

Back to model 1: $i\dot{\psi}_l = \epsilon_l \psi_l + \beta|\psi_l|^2\psi_l - \psi_{l+1} - \psi_{l-1}$

The linear case: $\beta = 0 \quad \psi_l = A_l \exp(-i\lambda t)$

Eigenvalue problem: $\lambda A_l = \epsilon_l A_l - A_{l-1} - A_{l+1}$

Normal mode (NM) eigenvectors: $A_{\nu,l} \ (\sum_l A^2_{\nu,l} = 1)$

NMs are ordered according to center-of-norm coordinate: $X_\nu = \sum_l l A_l^{(\nu)2}$

Eigenvalues: $\lambda_\nu \in \left[-2 - \frac{W}{2}, 2 + \frac{W}{2}\right]$

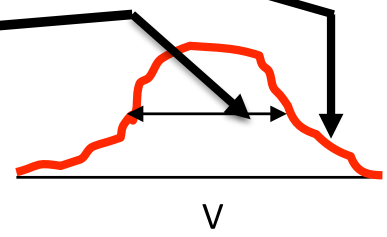
Width of EV spectrum: $\Delta_D = W + 4$

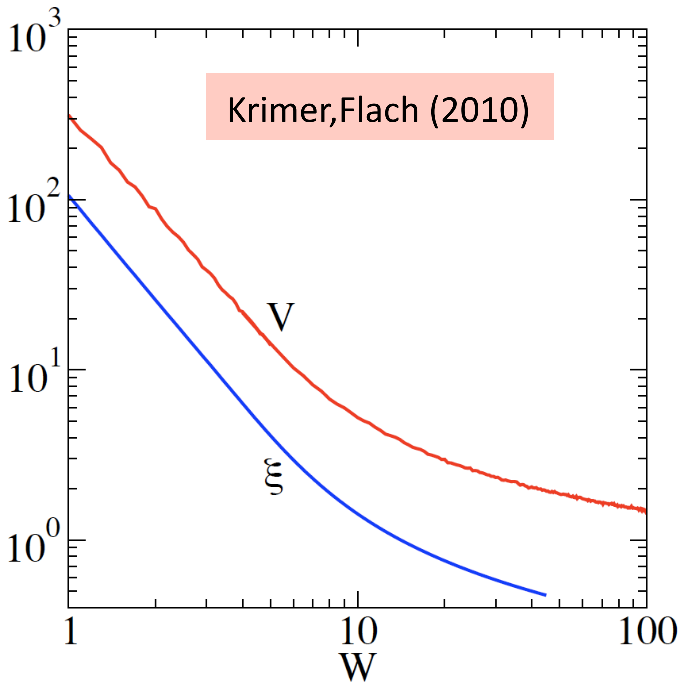
Characterization of the NMs

Asymptotic decay: $A_{\nu,l} \sim e^{-l/\xi(\lambda_\nu)}$

Localization length: $\xi(\lambda_\nu) \leq \xi(0) \approx 100/W^2$

Localization volume: V
(number of sites occupied by NM)





Red line: average localization volume ($\lambda_\nu \approx 0$);

Blue line: average localization length ($\lambda_\nu \approx 0$).

NMs with eigenvalues near bandwidth center are taken

Localization volume of NM was calculated via
the second moment of NM: $V_\nu = \sqrt{12 m_2^{(\nu)}} + 1$

$m_2^{(\nu)} = \sum_l (X_\nu - l)^2 |A_{\nu,l}|^2$ and $X_\nu = \sum_l l A_{\nu,l}^2$

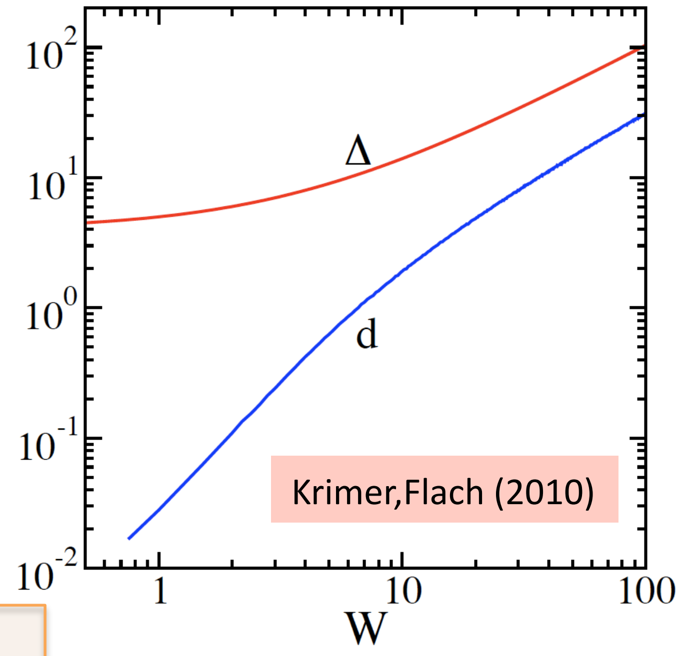
$\lambda_\nu \approx 0$: $\xi\ (W<4) \approx 100/W^2$, $V(W < 4) \approx 3.3\xi$ $V(W > 10) \approx 1$

Frequency scales set by the linear equations

Width of the eigenvalue spectrum: $\Delta = W + 4$

Average frequency spacing of NMs within the range of a localization volume: $d \approx \Delta / V$

Example. W=4: Δ=8, d=0.42

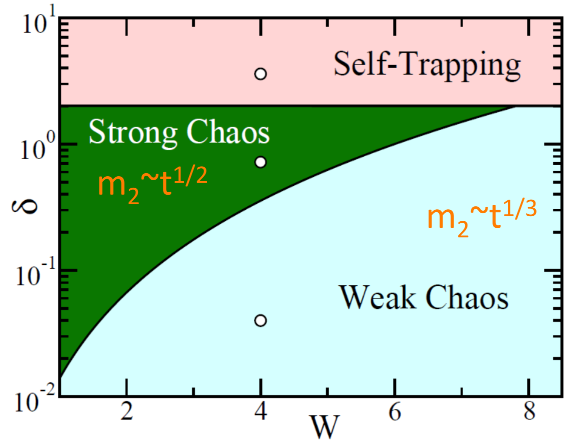


Frequencies of NMs are shifted when β≠0

Frequency shift induced by the nonlinearity: $\delta_l = \beta |\psi_l|^2$

Phase diagram

Three evolution regimes:
Weak chaos : δ < d
Strong chaos : d < δ < 2
(partial) self trapping : 2 < δ



Strong chaos is a transient (!)

More details: Joshua Bodyfelt (talk) and Tetyana Laptyeva (poster)

Transformation to NM space: $\psi_l(t) = \sum_\nu \phi_\nu(t) A_l^{(\nu)}$

Equations in normal mode space: $i\dot{\phi}_\nu = \lambda_\nu \phi_\nu + \beta \sum_{\nu_1,\nu_2,\nu_3} I_{\nu,\nu_1,\nu_2,\nu_3} \phi^*_{\nu_1} \phi_{\nu_2} \phi_{\nu_3}$

$$I_{\nu,\nu_1,\nu_2,\nu_3} = \sum_l A_{\nu,l} A_{\nu_1,l} A_{\nu_2,l} A_{\nu_3,l}$$

We analyze a normalized norm distribution $z_\nu \equiv |\phi_\nu|^2 / \sum_\mu |\phi_\mu|^2$ using

Second moment: $m_2 = \sum_\nu (\nu - \bar{\nu})^2 z_\nu$ → location of tails

$$\bar{\nu} = \sum_\nu \nu z_\nu$$

Participation number: $P = 1/\sum_\nu z_\nu^2$ → number of strongly excited modes

Compactness index: $\zeta = \frac{P^2}{m_2}$

→ K adjacent sites equally excited: $\zeta = 12$

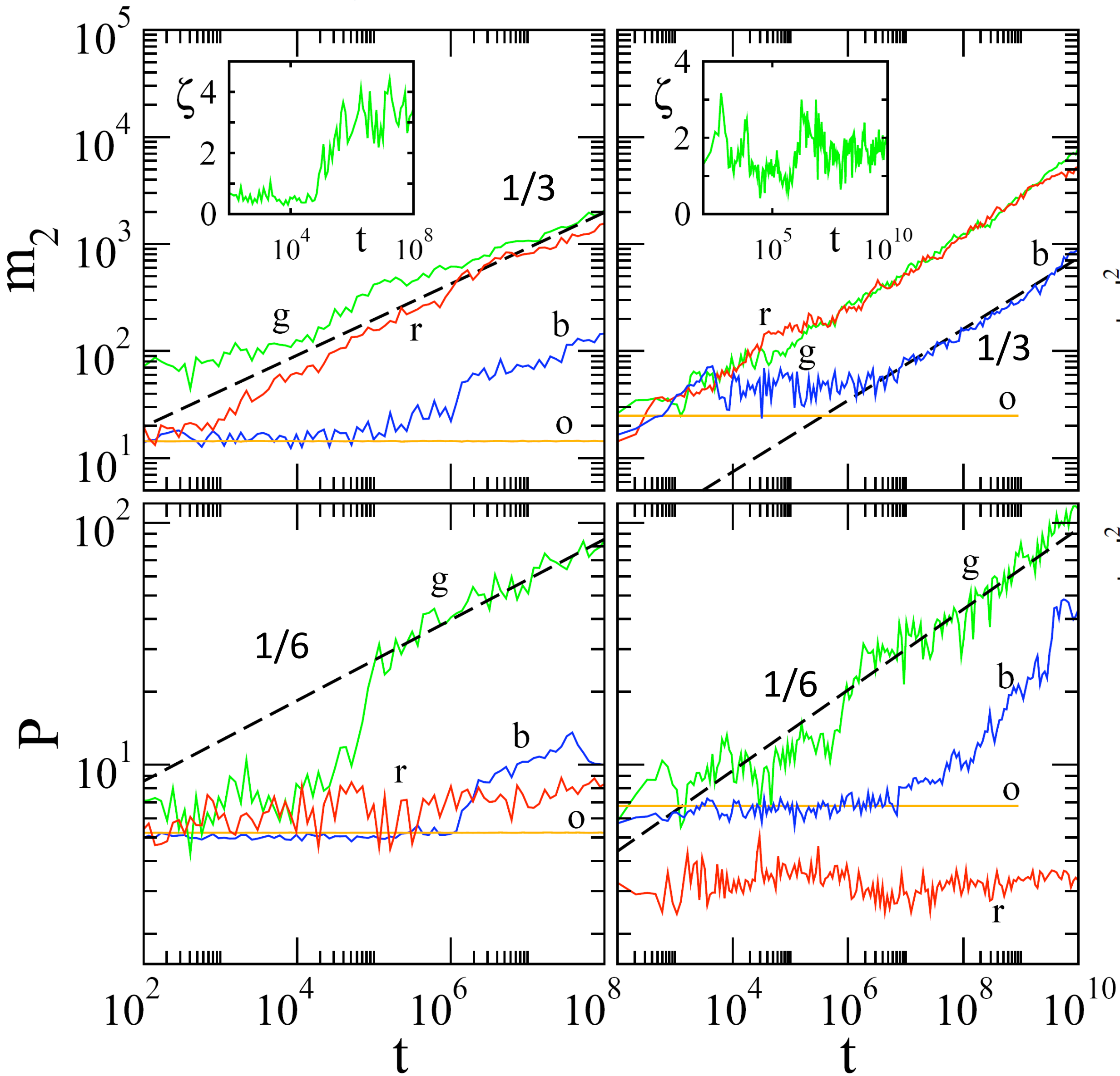
→ K adjacent sites, every second empty or equipartition: $\zeta = 3$

Results for single site excitations
(weak chaos and self trapping)

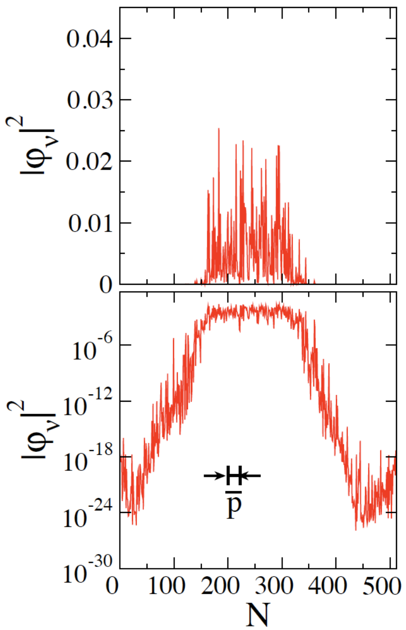
$$\psi_l = \delta_{l,l_0} \quad \epsilon_{l_0} = 0$$

Flach,Krimer,Skokos (2009)
Skokos,Krimer,Komineas,Flach (2009)

DNLS W=4, β= 0, 0.1, 1, 4.5   KG W=4, E= 0, 0.05, 0.4, 1.5



Wavepacket spreads far beyond localization volume.



We averaged the measured exponent in weak chaos over 20 realizations:

$\alpha = 0.33 \pm 0.02$ (DNLS)

$\alpha = 0.33 \pm 0.05$ (KG)

Results for multiple site excitations

W=4

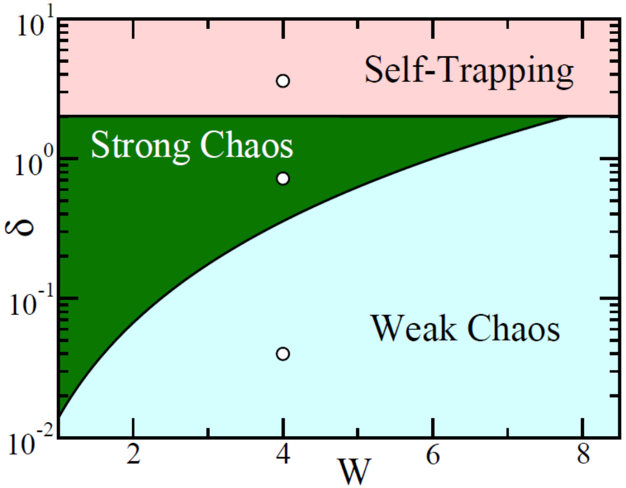
Wave packet with L=V=20 sites

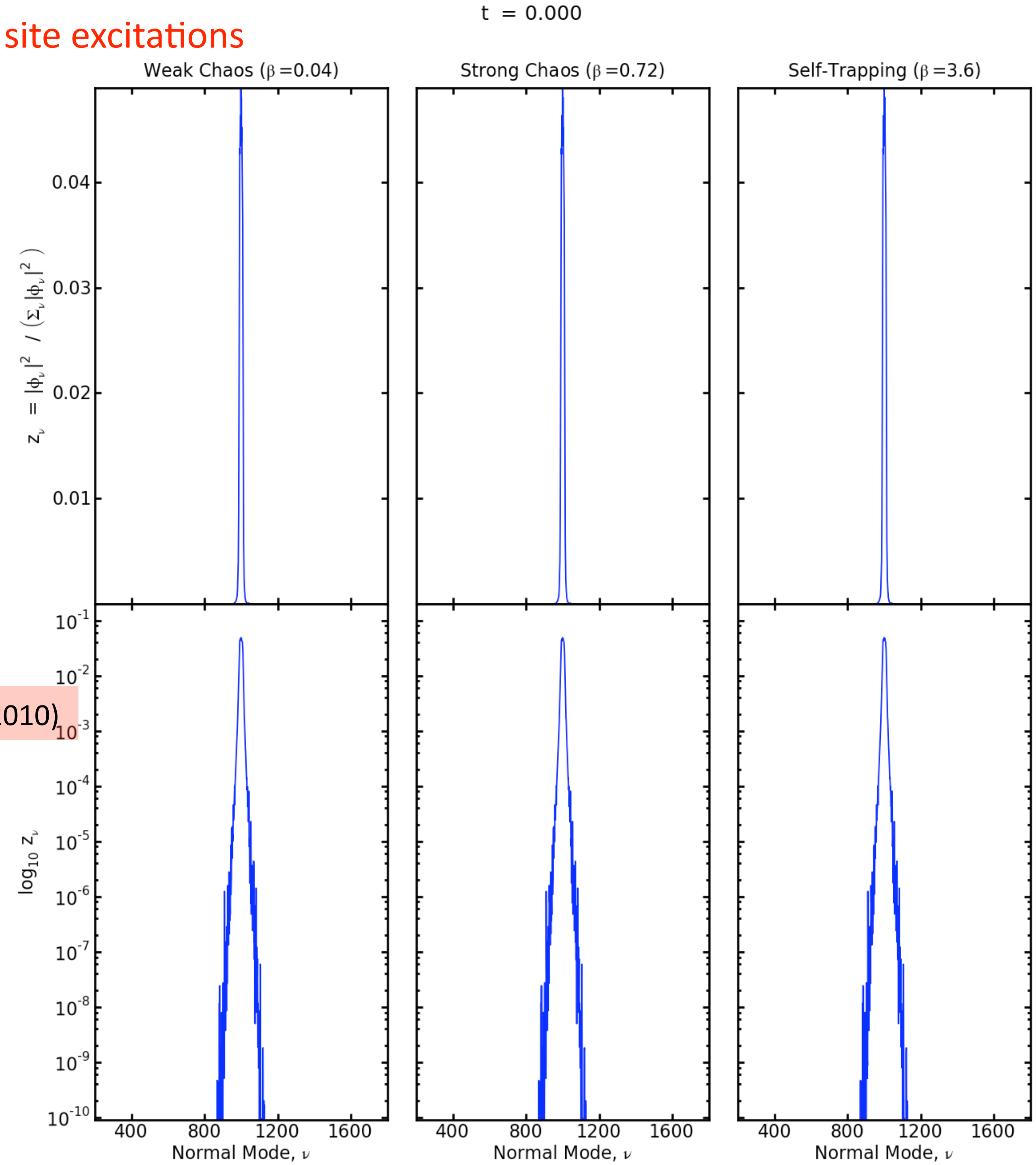
Norm density n=1

Random initial phases

Averaging over
1000 realizations

Laptyeva, Bodyfelt,Krimer,Skokos,Flach (2010)





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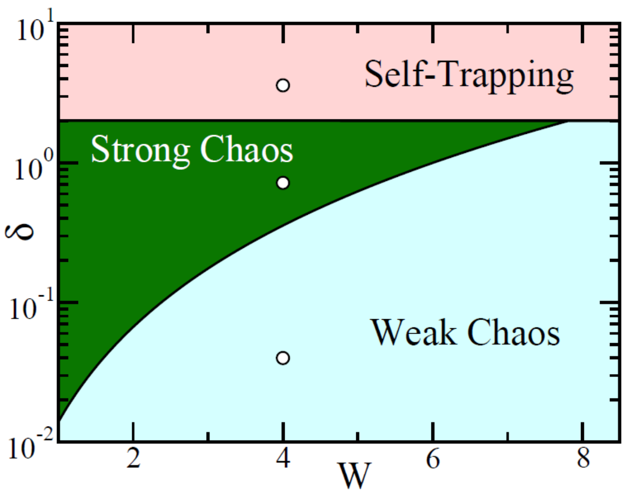
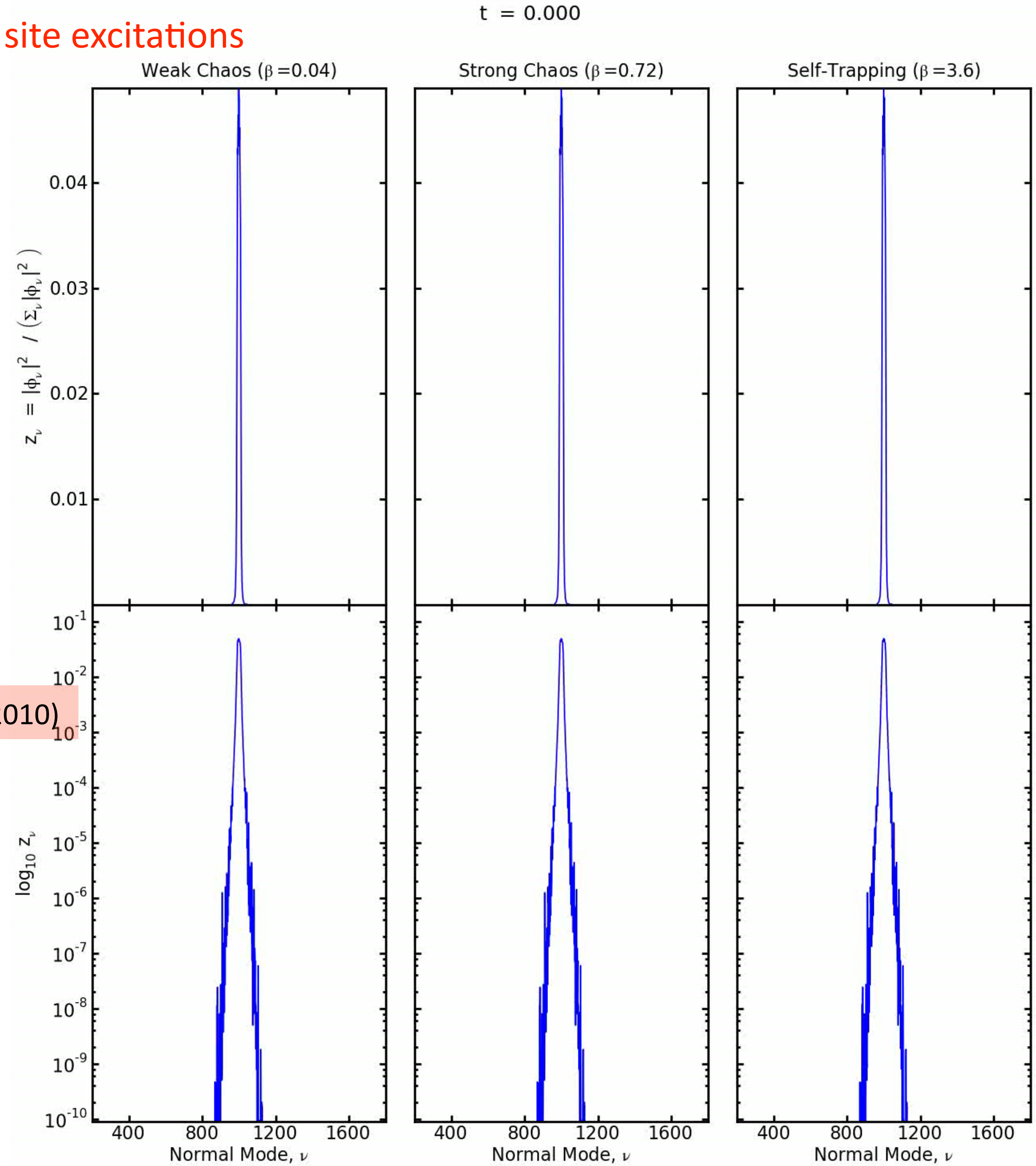
Laptyeva, Bodyfelt,Krimer,Skokos,Flach (2010)





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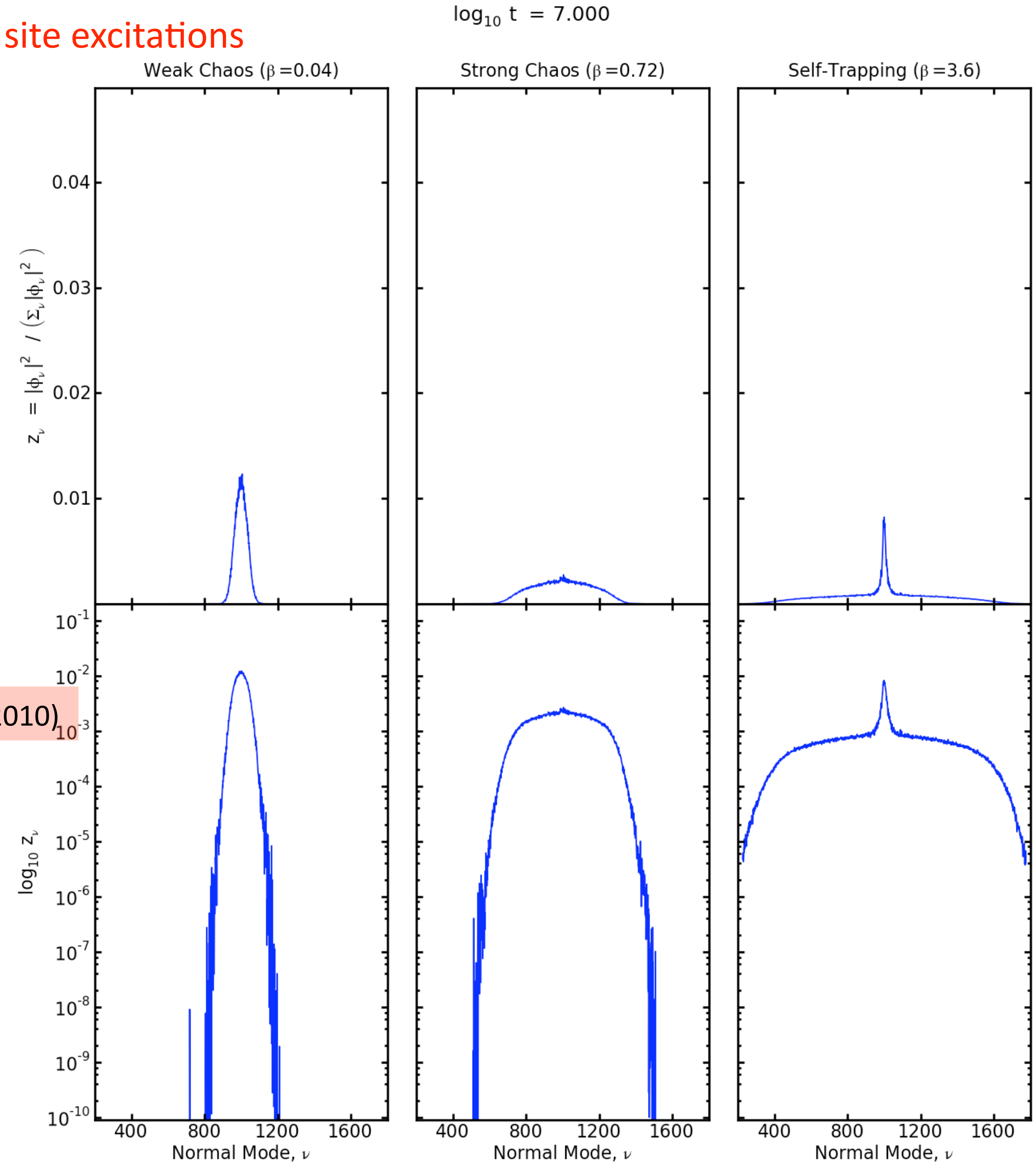
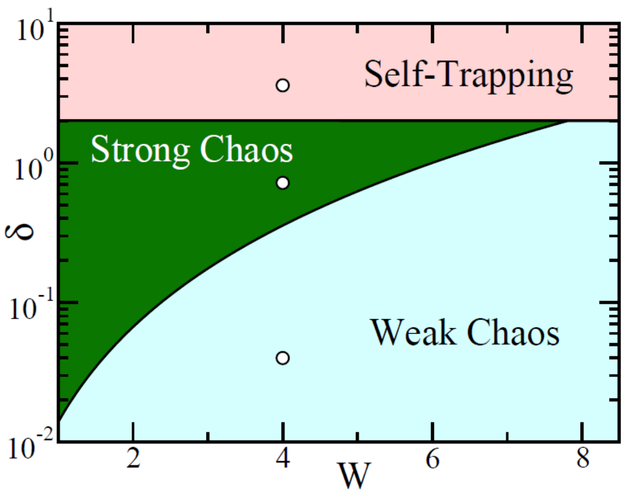
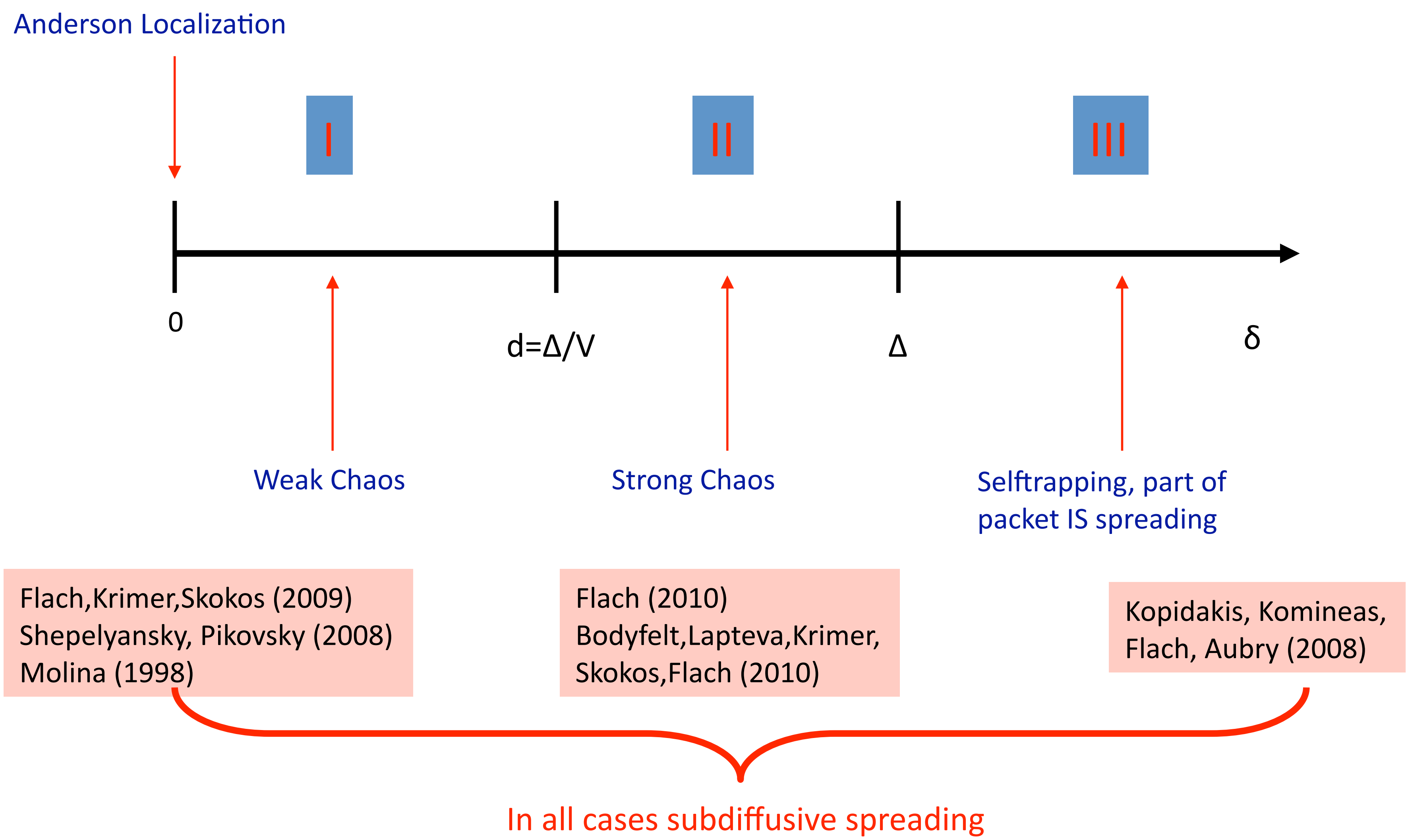


Laptyeva, Bodyfelt,Krimer,Skokos,Flach (2010)

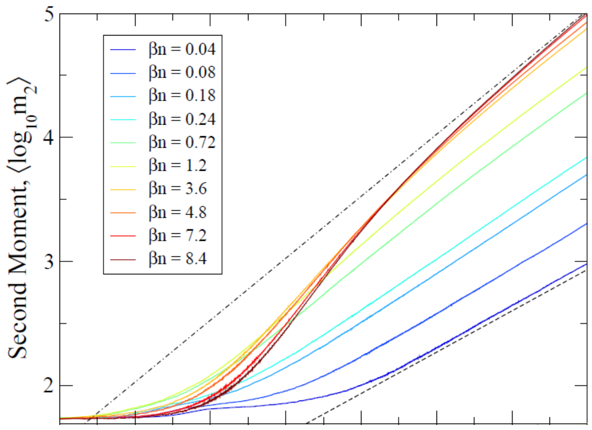


Sequence of dynamical regimes at fixed strength of disorder

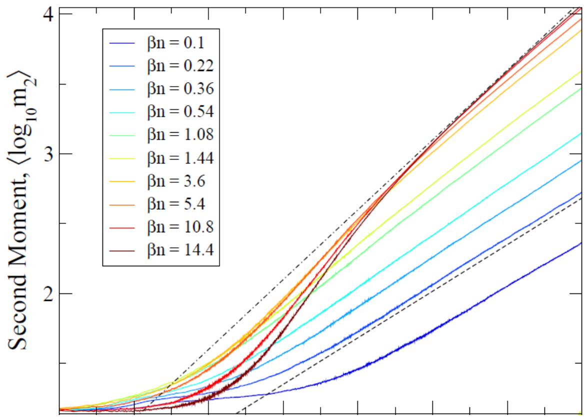


Dynamics at moderate and strong disorder strengths

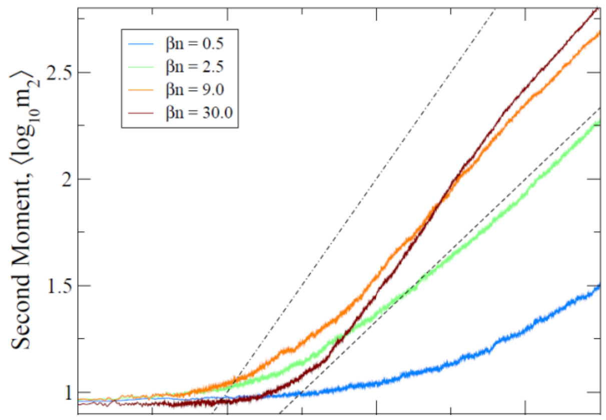
Question: do we find a stop or slowing down of the spreading?



W=4, L=20, n=1

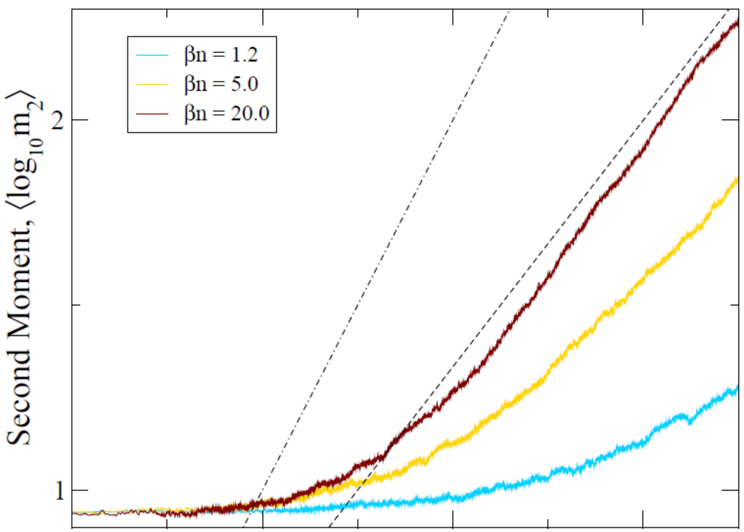


W=6, L=10, n=1

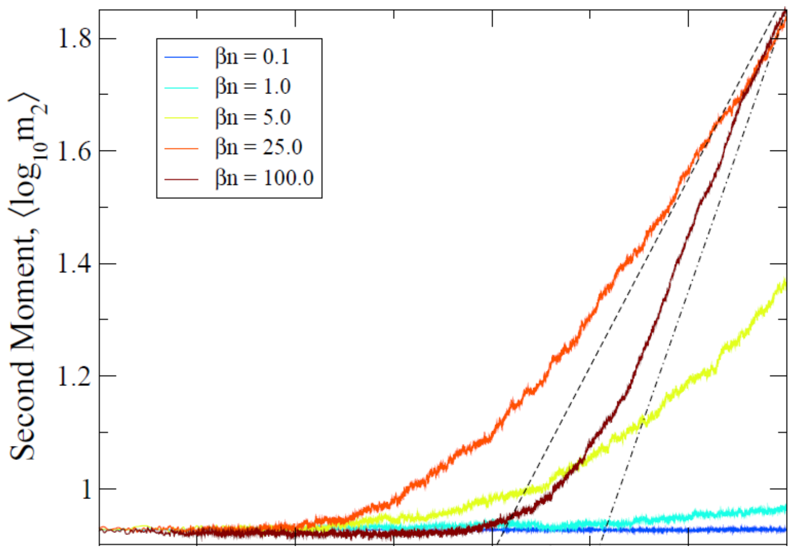


W=15, L=10, n=1

Averaging over 1000 realizations



W=25, L=10, n=1



W=40, L=10, n=1

Figs. from J. Bodyfelt

Model which explains spreading

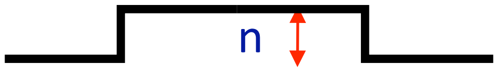
Flach, Krimer, Skokos (2009)

Flach (2010)

Conjectures:

- some modes in packet interact resonantly and therefore evolve chaotic
- Probability of resonance (or fraction of resonant modes) inside the packet: P(βn)
- the packet modes induce a stochastic force with amplitude proportional to P(βn)
- spreading = heating of cold exterior

Exterior mode: $i\dot{\phi}_\mu \approx \lambda_\mu \phi_\mu + \beta n^{3/2} \mathcal{P}(\beta n) f(t)$, $\langle f(t) f(t') \rangle = \delta(t - t')$



average norm density

fraction of resonant modes

stochastic force

The amplitude of exterior mode grows as: $|\phi_\mu|^2 \sim \beta^2 n^3 (\mathcal{P}(\beta n))^2 t$

The momentary diffusion rate of packet equals the inverse time the exterior mode needs to heat up to the packet level: $D = 1/T \sim \beta^2 n^2 (\mathcal{P}(\beta n))^2$

with the probability: $\mathcal{P}(\beta n) \approx 1 - e^{-\beta n/d} = \begin{cases} 1, & \beta n/d > 1 \text{ (strong chaos)} \\ \beta n/d, & \beta n/d < 1 \text{ (weak chaos)} \end{cases}$

The diffusion equation $m_2 \sim Dt$ yields $m_2 \sim 1/n^2 \sim \beta(1 - e^{-\beta n/d}) t^{1/2}$

$$m_2 \sim \begin{cases} \beta t^{1/2}, & \beta n/d > 1 \text{ (strong chaos)} \\ d^{-2/3}\beta^{4/3}t^{1/3}, & \beta n/d < 1 \text{ (weak chaos)} \end{cases}$$

Flach (2010)

strong chaos is a transient → a crossover to the asymptotic law of weak chaos at larger t

Generalization to larger dimensions D and different nonlinearity powers σ:

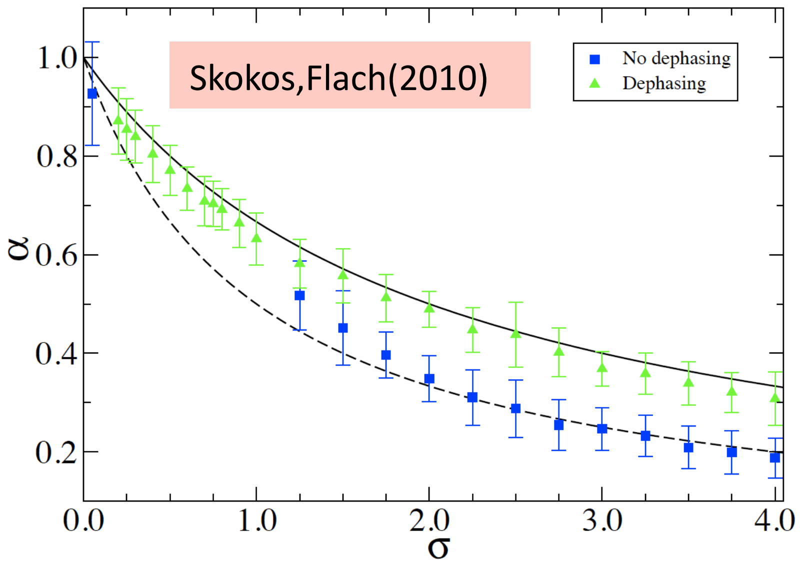
$$i\dot{\psi}_{\boldsymbol{l}} = \epsilon_{\boldsymbol{l}}\psi_{\boldsymbol{l}} - \beta|\psi_{\boldsymbol{l}}|^{\sigma}\psi_{\boldsymbol{l}} - \sum_{\boldsymbol{m}\in D(\boldsymbol{l})}\psi_{\boldsymbol{m}}$$

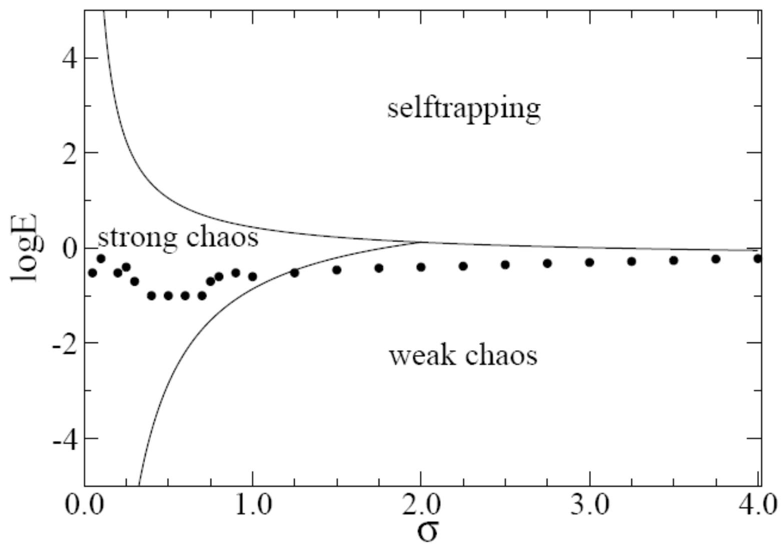
$$m_2 \sim (\beta^2 t)^{\frac{2}{2+\sigma D}}, \text{ strong chaos}$$

$$m_2 \sim (\beta^4 t)^{\frac{1}{1+\sigma D}}, \text{ weak chaos}$$

Previous considerations correspond to σ=2 (cubic nonlinearity) and D=1

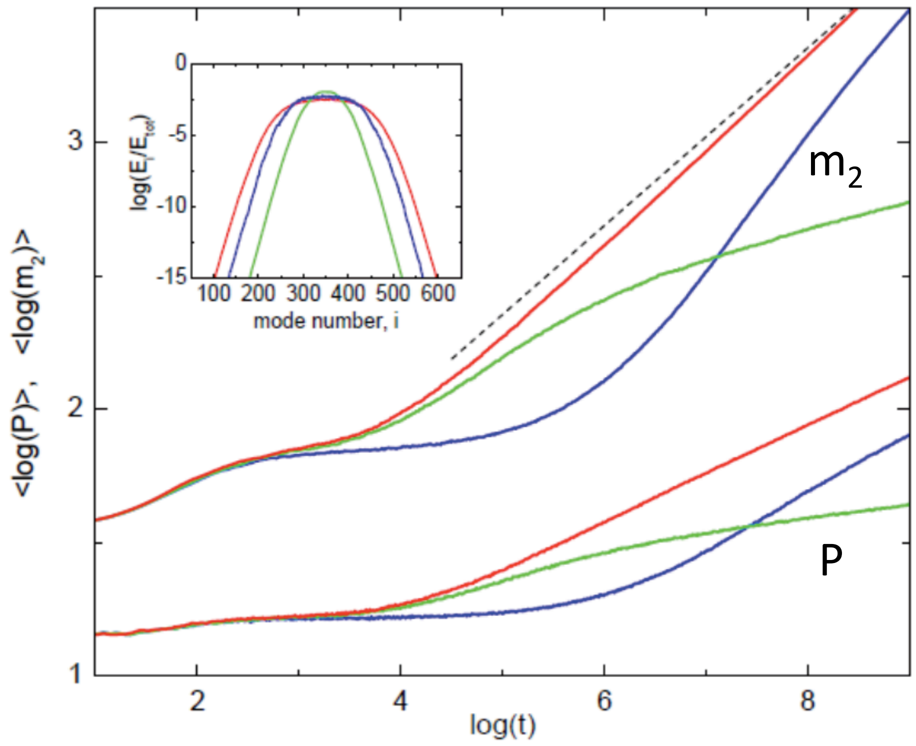
Numerical evaluation for KG, D=1, W=4, single site excitations





For σ<1 we enter to the region of strong chaos without dephasing

Full nonlinear system versus linear inside and linear outside (KG)



Average <log(m_2)> [upper part] and <log(P)> [lower part] vs. time for W=4, L=21, ε=0.02. Dashed line: $t^{1/3}$

Averaging over 1000 realizations

Fig. from T. Laptyeva

Red line: system is fully nonlinear

Blue line: system is linear inside the packet and nonlinear outside

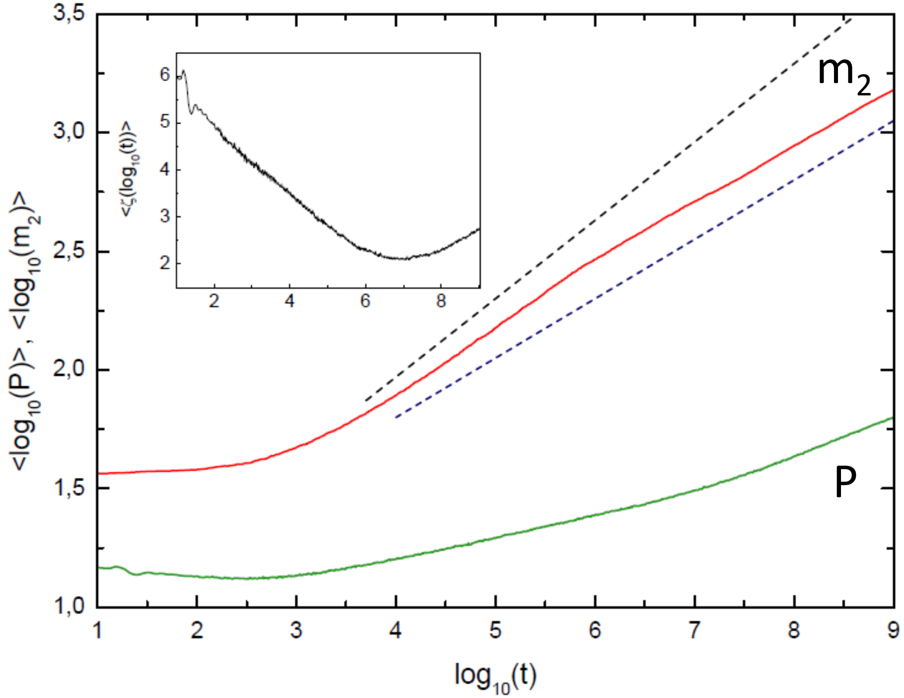
Green line: system is nonlinear inside the packet and linear outside

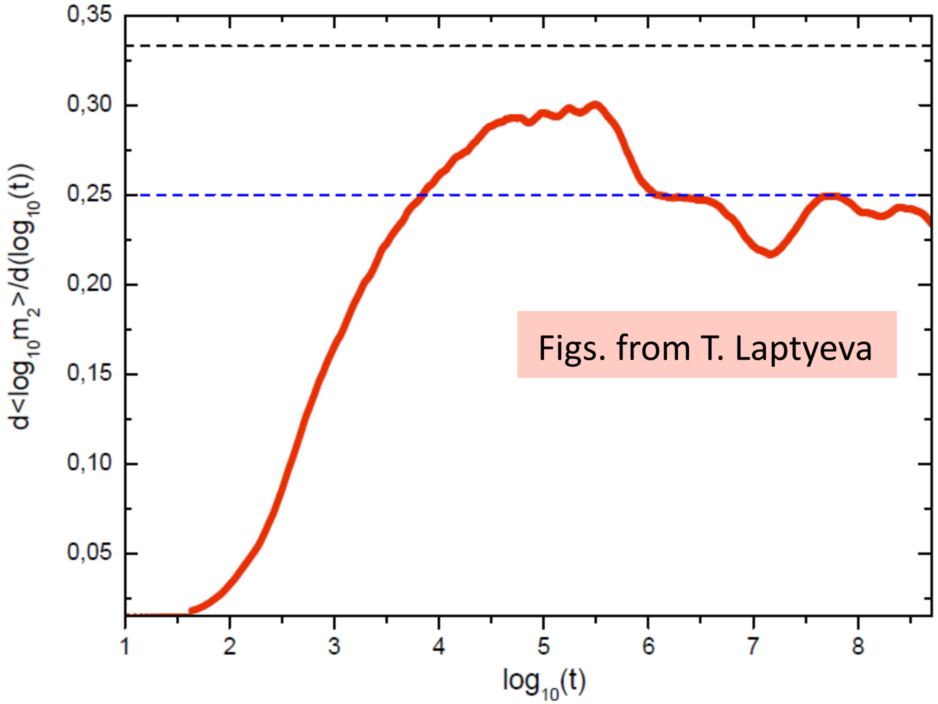
Inset: energy distributions at final time

Fröhlich Spencer Wayne Model (KG)

$$\mathcal{H} = \sum_i \frac{p_i^2}{2} + \epsilon_i \frac{u_i^2}{2} + \frac{u_i^4}{4} + \frac{C}{4}(u_{i+1} - u_i)^4$$

ε from [1/2 , 3/2]





Left figure: average $\langle \log(m_2) \rangle$ and $\langle \log(P) \rangle$ vs. time for C=1, L=21, ε=0.05.

Inset: the average compactness index vs. time. Dashed lines: $t^{1/3}$, $t^{1/4}$.

Right figure: derivatives d<log(m2)>/dlog(t) vs. time.

Averaging over 1000 realizations

Conclusions

- In simulations when we observe spreading, it is always subdiffusive
  we do not see any crossover to the localization at later times

- for DNLS and KG we found similar subdiffusive spreading behavior
  which is due to a finite number of resonant chaotic modes

- spreading is universal due to nonintegrability

- strong nonlinearity: partial localization due to selftrapping
  but part of wavepacket spreads

- weak nonlinearity: Anderson localization on finite times:
  After that – detrapping, and wavepacket delocalizes

- intermediate nonlinearity: wavepacket delocalizes without transients

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