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Supercritical NLS: Quasi-Periodic Solutions and Almost Global Existence

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I. Introduction.

We study nonlinear Schrödinger equations on the torus in arbitrary dimensions d:

$$-i\dot{u} = -\Delta u + |u|^{2p}u + h.o.t., \qquad p \in \mathbb{N} \text{ arbitrary},$$

where the h.o.t. are analytic and could have x dependence, which break translation invariance. As is well known the nonlinear terms correspond to interaction among particles.

The above equation can be put in the infinite dimension Hamiltonian form:

$$i\dot{u} = -\frac{\partial H}{\partial \bar{u}},$$

with u, $\bar{u}$ as canonical variables.

Unlike linear (time independent) equations, the solutions to nonlinear equations do not always exist. In fact this is the central issue in nonlinear PDE. Generally speaking for linear equations, we work in L^2. The space where quantum mechanics take place and there is global (in time) existence.

For nonlinear equations, we cannot always work in L^2. We have to work in more restricted spaces, the Sobolev spaces H^s. For example, H^1 means that we also require the function to have 1 derivative (in the appropriate sense). Since NLS is Hamiltonian, it has 2 conserved quantities, namely mass and energy. So the L^2 and H^1 norms of the solution are conserved.

It is well known that for $d \geq 3$ and p large enough, $p > \frac{2}{d-2}$, NLS is energy supercritical. For example, in $d = 5$, the cubic NLS is supercritical, i.e., the local well-posedness is in a Sobolev space H^s for $s > 1$, above the Hamiltonian H^1 topology, where there is a conservation law.

So there is no a priori global existence, not even for small data because the equations are **non** dispersive, i. e., the L^∞ norms of the eigenfunctions do not tend to 0 as the eigenvalues tend to infinity and the L^∞ norms of the solutions do not tend to 0 as $t \longrightarrow \infty$.

We want to remark that there is a correspondence between renormalizability of quantum field theory and global existence of NLS. Specializing to the cubic NLS and ϕ^4, we know that in $d = 2$ and 3, cubic NLS has a priori global existence from energy conservation; while ϕ^4 is super-renormalizable. In $d = 4$, the cubic NLS is energy critical, global existence is expected but not proven at this writing; while ϕ^4 is renormalizable. For $d \geq 5$, cubic NLS is energy supercritical, global existence unknown; ϕ^4 is not renormalizable.

The main purpose of this talk is to present a theory on supercritical NLS. In the last part of the talk we will also make connections with discrete NLS with random potential.

This nonlinear theory is constructive, i.e., existence is via explicit construction. This is in contrast with the known PDE theory, which relies on conservation laws and therefore can only deal with subcritical or critical NLS, i.e. the corresponding field theory is either super-renormalizable or renormalizable.

This new theory has both a geometric and an analytic part. It develops fine analysis of the resonance geometry given by algebraic equations to establish a spectral gap in order to construct quasi-periodic solutions (of arbitrary number of frequencies) and consequently deduce almost global existence for Cauchy problems for supercritical NLS.

We note that for subcritical (or critical) NLS, one linearizies about the Laplacian and uses L^p estimates of the eigenfunction solutions (Strichartz). Here we go one step further and linearize about appropriate approximate quasi-periodic solutions.

We note that since H^s $(s > 1)$ cannot be controlled by H^1, the general problematics here is very different from obtaining global existence in H^1 for critical NLS or for subcritical NLS in H^s for $s < 1$ (infinite energy solutions), which are locally controlled by H^1 [B].

Remark. The main new ingredient here is geometric. It is about dealing with a manifold of singularities, instead of isolated ones, which is the usual case in dynamical systems approach, cf. [BG, B,...]. The equations in these papers are essentially well posed in L^2 and therefore global existence.

The main difficulty for establishing the spectral gap is the lack of convexity due to the presence of the first order operator $i\partial/\partial t$. The work is therefore to bypass that and ensure “effective convexity” using the algebra afforded by the translation invariance over the integers $\mathbb{Z}^d$.

II. Quasi-Periodic Solutions and Almost Global Existence for Supercritical NLS

The solutions to the linear Schrödinger equation on the d-torus:

$$-i\dot{u} = -\Delta u,$$

are provided by spectral theory. They are linear combinations of the eigenfunction solutions:

$$u = e^{-i\omega_j t} e^{ij\cdot x}.$$

Since $\omega_j = j^2$, e. v. of the Laplacian, they are time periodic. In general, they are time quasi-periodic with several frequencies. When the amplitude of u is small, it is therefore natural to ask about the persistence of such solutions under nonlinear perturbations (stability issue). These solutions play a related role in the nonlinear setting by providing a basis for the (smooth) flow.

We seek q-p solutions with b frequencies to

$$-i\dot{u} = -\Delta u + |u|^{2p}u + h.o.t.,\ p \in \mathcal{N} \text{ arbitrary},$$

in the form

$$u = \sum \hat{u}(n,j)e^{in\cdot\omega t}e^{ij\cdot x}, \quad (n,j) \in \mathbb{Z}^{b+d},$$

where h.o.t. is analytic and depends on both u and x and ω is to be determined. This is the so called amplitude-frequency modulation, fundamental to nonlinear equations. For linear equations, ω are the eigenvalues, they are fixed once and for all. In this language, a solution $u^{(0)}$ to the linear equation can be written as

$$u^{(0)} = \sum_j \hat{u}(e_j,j)e^{-i\omega_j^{(0)}t}e^{ij\cdot x}, \quad (n,j) \in \mathbb{Z}^{b+d},$$

where e_j is a basis vector in $\mathbb{Z}^b$ and $\omega_j^{(0)} = e_j \cdot \omega^{(0)} = j^2$.

Theorem 1. [W] For any b, there exists a set $\Omega \subset (\mathbb{R}^d)^b$ of codimension 1. Assume $j = \{j_k\}_{k=1}^b \in (\mathbb{R}^d)^b \backslash \Omega$ and $u^{(0)} = \sum_{k=1}^b a_k e^{-ij_k^2 t} e^{ij_k \cdot x}$ a solution to the linear equation with b frequencies and $a = \{a_k\} \in (0, \delta]^b$. There exist C, $c > 0$, such that for all $\epsilon \in (0,1)$, there exists $\delta_0 > 0$ and for all $\delta \in (0, \delta_0)$ a Cantor set $\mathcal{G}$ with

$$\text{meas } \{\mathcal{G} \cap \mathcal{B}(0, \delta)\}/\delta^b \geq 1 - C\epsilon^c.$$

For all $a \in \mathcal{G}$, there is a quasi-periodic solution of b frequencies to the nonlinear Schrödinger equation

$$u(t,x) = \sum a_k e^{-i\omega_k t} e^{ij_k \cdot x} + \mathcal{O}(\delta^3),$$

with basic frequencies $\omega = \{\omega_k\}$ satisfying

$$\omega_k = j_k^2 + \mathcal{O}(\delta^{2p}).$$

The remainder $\mathcal{O}(\delta^3)$ is in an appropriate analytic norm on $\mathbb{T}_b^{+d}$.

To state the almost global existence result for Cauchy problems, it is convenient to write the nonlinear equation as

$$-i\dot{u} = -\Delta u + \delta|u|^{2p}u, \quad p \in \mathbb{N} \text{ arbitrary}, \qquad (*)$$

on $\mathbb{T}^d$ and consider initial data of size 1. (The same construction works with higher order terms which can have explicit x dependence.) We say an entire function is generic if its Fourier support is in Ω^c. We note that it is the same good geometry as in Theorem 1.

Theorem 2. [W] Let $u_0 = u_1 + u_2$. Assume u_1 is generic and $\|u_2\| = \mathcal{O}(\delta)$, where $\|\cdot\|$ is an analytic norm on $\mathbb{T}^d$. Then there exists an open set $\mathcal{A} \subset \mathcal{B}(0,1)$ of positive measure, such that for all $A > 1$, there exists δ_0, such that for all $\delta \in (0, \delta_0)$, if $\{\hat{u}_1\} \in \mathcal{A}$, then (*) has a unique solution $u(t)$ for $|t| \leq \delta^{-A}$ satisfying $u(t=0) = u_0$ and $\|u(t)\| \leq \|u_0\| + \mathcal{O}(\delta)$, *meas* $\mathcal{A} \to 1$ as $\delta \to 0$.

We note that the above result holds in arbitrary dimension d and for arbitrary nonlinearity p. Recall that for $d \geq 3$ and $p > \frac{2}{d-2}$, the nonlinear Schrödinger is energy supercritical and there is no a priori global existence. Theorem 1. is global, although the solutions are not for *fixed* initial data. Theorem 2. is the Cauchy consequence of Theorem 1. This is reasonable as the known invariant measure for smooth flow is supported by KAM tori.

III. Semi-Classical Corollaries

We have the following semi-classical analogues, which reveal further the geometric nature of this construction.

Corollary 1. [W] Assume

$$u^{(0)}(t,x)=\sum_{k=1}^{b} a_k e^{-ij_k^2 t} e^{ij_k\cdot x},$$

a solution to the linear equation is generic with $\{j_k\}_{k=1}^{b}\in[K\mathbb{Z}^d]^b$, $K\in\mathbb{N}^+$ and $a=\{a_k\}\in(0,1]^b=\mathcal{B}(0,1)$. There exist C, $c>0$, such that for all $\epsilon\in(0,1)$, there exists $K_0>0$ and for all $K>K_0$ a Cantor set $\mathcal{G}$ with

$$\text{meas}\,\{\mathcal{G}\cap\mathcal{B}(0,1)\}\geq 1-C\epsilon^c.$$

For all $a \in \mathcal{G}$, there is a quasi-periodic solution of b frequencies to the nonlinear Schrödinger equation:

$$u(t,x) = \sum a_k e^{-i\omega_k t} e^{ij_k \cdot x} + \mathcal{O}(1/K^2),$$

with basic frequencies $\omega = \{\omega_k\}$ satisfying

$$\omega_k = j_k^2 + \mathcal{O}(1).$$

The remainder $\mathcal{O}(1/K^2)$ is in an analytic norm about a strip of width $\mathcal{O}(1)$ in t and $\mathcal{O}(1/K)$ in x on $\mathbb{T}^{b+d}$.

Corollary 2. [W] Assume u_0 is *generic* with frequencies $\{j_k\}_{k=1}^{b} \in [K\mathbb{Z}^d]^b$, $K \in \mathbb{N}^+$. Let $\mathcal{B}(0,1) = (0,1]^b$. Then there exists an open set $\mathcal{A} \subset \mathcal{B}(0,1)$ of positive measure, such that for all $A > 1$, there exists $K_0 > 0$, such that for all $K > K_0$, if $\{|\hat{u}_0|\} \in \mathcal{A}$, then the nonlinear Schrödinger equation has a unique solution $u(t)$ for $|t| \leq K^{-A}$ satisfying $u(t=0) = u_0$ and $\|u(t)\| \leq \|u_0\| + \mathcal{O}(1/K^2)$, where $\|\cdot\|$ is an analytic norm (about a strip of width $\mathcal{O}(1/K)$) on $\mathbb{T}^d$, moreover meas $\mathcal{A} \to 1$ as $K \to \infty$.

Remark. In the above two corollaries, the small parameter is extracted from the geometry of the bi-characteristics. Corollary 1 is new to the KAM context.

Corollaries 1 and 2 construct quantitative *large* kinetic energy solutions, which are of relevance to the quantum Euler equations (BE condensation) after transformation to action, angle (hydrodynamic) representation of NLS. They could provide global or almost global singular solutions.

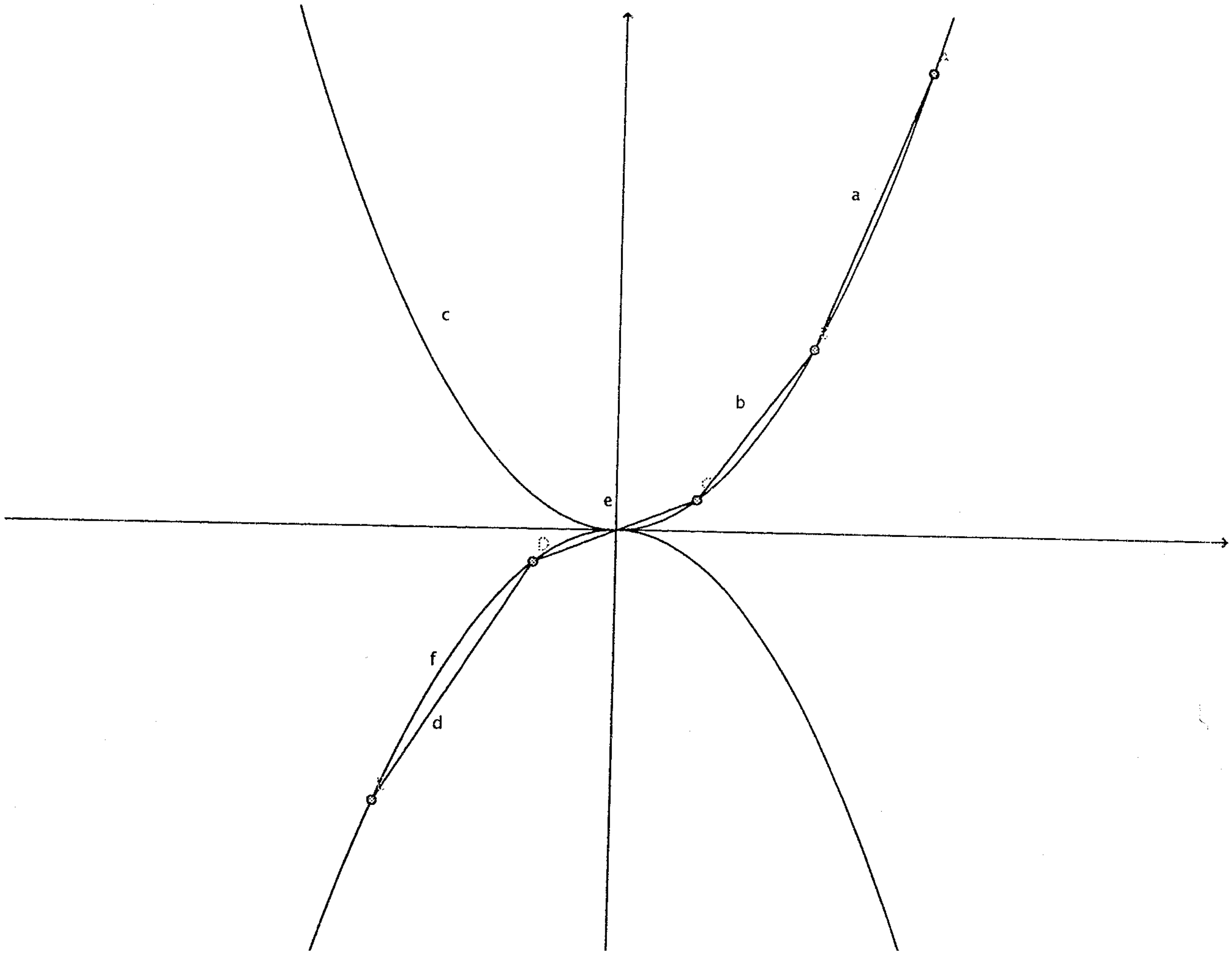
IV. Proof of Theorems 1 and 2.

The proof of the theorems uses a Newton scheme and relies on analyzing connected sets on the bi-characteristics

$$\mathcal{C} = \{(n, j) | \pm n \cdot \omega^{(0)} + j^2 = 0\},$$

in order to achieve amplitude-frequency modulation. The main difficulty is that the Hamiltonian is in infinite dimensions and the Kolmogorov non-degeneracy condition (or its weaker versions) are completely violated. This is why we first need to do a geometric selection (excluding Ω) to ensure a spectral gap in order to have convergence of the Newton scheme.

1d periodic, manifold of singularities:



The new ingredient is a fine analysis of the resonances. This is made possible by studying systems of polynomial equations and obtaining separation property by avoiding flat pieces so that the connected (by convolution) sets on $\mathcal{C}$ are of sizes $\leq$ max (d+2, 2b). Algebraically this is a dimension reduction argument reducing to polynomials in 1 variable, which generically have separated zeroes. The spectral gap here is brought on by algebraically avoiding the non-convex directions.

Remark. In fact, requiring the resultants to be non-zero gives the main part of the genericity conditions on $u^{(0)}$ or u_1.

We conclude the talk by making a few remarks on the discrete NLS with disorder and putting it under the same roof as NLS on torus.

V. NLS with Random Potential.

We consider the lattice nonlinear random Schrödinger equation in $d = 1$:

$$i\dot{q}_j = v_j q_j + \epsilon_1 (\Delta q)_j + \epsilon_2 |q_j|^2 q_j, \quad j \in \mathbb{Z}^d,$$

where $V = \{v_j\}$ is a family of independent identically distributed (i.i.d.) random variables in $[0,1]$ with uniform distribution, $0 < \epsilon_1, \epsilon_2 \ll 1$.

Theorem 3. (with Zhifei Zhang, 2008) Given $A > 1$ and $\delta > 0$, for all initial datum $q(0) \in \ell^2$, let j_0 be such that

$$\sum_{|j|>j_0} |q_j(0)|^2 < \delta,$$

(which is always possible by choosing j_0 large enough), then for all $t < \epsilon^{-A}$ and $0 < \epsilon < \epsilon(A)$,

$$\sum_{|j|>j_0+A^2} |q_j(t)|^2 < 2\delta,$$

with probability $\geq 1 - exp(-\frac{j_0}{A^2} e^{-2A^2 \epsilon^{\frac{1}{CA}}})$.

So there is Anderson localization for arbitrary long time for arbitrary ℓ^2 datum. If one views the linear Schrödinger equation as an approximation, then Theorem 3 validates the utility of (linear) Anderson localization theory.

There has been quite a bit of reaction to the above result since. One of the persistent questions regards the small parameter ϵ_1, which was not needed for the linear theory in $d = 1$ due to the Furstenburg theorem. Here I want to make a comment on this. It is in fact related to the last remark about the resultant.

Furstenburg theorem or more generally linear Anderson localization requires that appropriate pairs of eigenvalues are not too close. It is a condition on linear combination of 2 eigenvalues only, which can be expressed in terms of the resultant of 2 polynomials. Therefore this is essentially an algebraic property and does not require information on eigenfunctions.

On the other hand, nonlinear localization needs bounds on more general linear combinations of eigenvalues, which requires detailed information on the support of the eigenfunctions. Furstenburg or linear Anderson localization only gives that the eigenfunctions are localized, but the localization is **not** uniform with respect to energy. This does not suffice to have the necessary eigenvalue bounds to start nonlinear localization. Hence the raison d'être of small ϵ_1.

Finally we note that NLS on the torus is elliptic, while lattice NLS with random potential is not. The randomness replaces ellipticity, but there is lack of uniformilty, which is also present for elliptic problems in $d \geq 2$ due to degeneracy. In the elliptic case, we overcame the difficulty by appealing to geometry and algebra. In the non-elliptic case, we believe the difficulty lies in dealing with highly (singular) dependent random variables. Afterwards the mechanism is akin to the elliptic case in Theorems 1 and 2.